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Infinity and large cardinals in set theory

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1 Introduction

First of all, I would like to thank Paolo Mancosu for inviting me to take part in this meeting and I would like to say how much I found his seminars scientifically fascinating and technically remarkable. It was truly a delight. A big thank-you.

In Paolo's seminars, many aspects of infinity were discussed, in particular *alternatives* to the infinitary properties of ZFC set theory (Zermelo-Frankel with axiom of choice).

For my part, I would like to come back to some of these particularly remarkable and unexpected properties by focusing on the work of Hugh Woodin, who is one of the great explorers of this "vast and mysterious place" that is a universe V of ZFC. I have long followed the astonishing discoveries of Hugh Woodin and have written several studies about him. I would like to indicate here how one arrives at his so-called *Ultimate-L conjecture*. It is an intellectual journey of a very high technicality that I will try to map. For me, creative mathematicians are like explorers and I think that one of the roles of the philosophy of mathematics is to elaborate the *cartography* of their great discoveries.

My purpose is to explain this claim of Woodin in his contribution to the 2010 International Congress of Mathematians :

> "The situation has now (2010) changed dramatically and there is for the first time a genuine prospect for the construction of an ultimate enlargement of L. This arises not from the identification of a strongest large cardinal axiom but from

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> the unexpected discovery that at a specific critical stage in the hierarchy of large cardinal axioms, the construction of an enlargement of L compatible with this large cardinal axiom must yield the ultimate enlargement of L. More precisely this construction must yield an enlargement which is compatible with all stronger large cardinal axioms." (p.505)

On the philosophical level I will not say much since, being Kantian, the problems of ontological commitment and noumenal reality do not interest me much. Knowing whether to adopt a "universe" view (there must exist a single "true" V) or a "multiverse" view (there can exist several "true" V) but with an ontological commitment making such a foundational pluralism problematic) does not bother me too much. I rather adopt a *formalistic* multiverse view à la Joel Hamkins.

I refer you, for example, to Mark Balaguer for the analysis of the Platonist and anti-Platonist theses between truth and fiction. The large cardinals axioms (LCA) I will discuss are mainly related to *reverse mathematics* : what axioms do we need to demonstrate this or that property that we consider interesting for some "good" reasons. But they are ontologically inflationary and not deflationary, i.e. existentially parsimonious. Epistemologically, I therefore refer to MarcoPanza's fine analyzes of *epistemic parsimony*, which may well not be existentially parsimonious and is well adapted to reverse mathematics..

2 Preliminaries

I assume you know the basics of ZF and ZFC set theory. I will say nothing about alternative axioms like those of Bernays (sets and classes) or those of Morse-Kelley (the theory admits quantifications on classes and is non-conservative on ZFC) or those admitting unfounded sets (Peter Aczel , 1988), or Grothendieck's universes restricting to *set* models of ZFC (as V_κ for κ a strongly inaccessible cardinal) in order to avoid classes and make a link with category theory.

2.1 Axioms and cumulative hierarchy

Among the classical ZFC axioms, two are the source of many difficulties:

Power set: the set $\mathcal{P}(X) = \{u \subseteq X\})$ of subsets of every set X exists in V,

Choice: every family of sets X_s, $s \in S$, has a choice function f associating to each $s \in S$ an element $f(s) \in X_s$ (this axiom of existence doesn't define any specific f and is highly non-constructive).[1]

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[1] The other axioms are (see e.g. Jech [1978]): *Extensionality*: sets are determined

$\mathcal{P}(X)$ is basic and the universe V is naturally *stratified* by the cumulative von Neumann hierarchy of successive levels of V indexed by the class On of ordinals: $V_0 = \emptyset$, $V_{\alpha+1} = \{X : X \subset V_\alpha\}$ for a successor ordinal, and $V_\lambda = \bigcup_{\alpha<\lambda} V_\alpha$ for a limit ordinal λ.

2.2 Ordinals and cardinals

Two classes of sets are of fondamental importance for enumerate and characterize the size of sets. They are of a completely different nature.

On the one hand, *ordinals* are the sets that are $\in$-transitive ($y \in X$ implies $y \subset X$, that is $\bigcup X \subseteq X$ or $X \subset \mathcal{P}(X)$) and well-ordered by $\in$. All well-ordered sets are order-isomorphic to a single ordinal and if the AC is valid, every set can be well-ordered and become order-isomorphic to an ordinal. Every ordinal is a successor: $\alpha = \beta + 1$ or a limit ordinal $\alpha = \mathrm{Sup}\,(\beta : \beta < \alpha)$ (and then $\forall \beta < \alpha,\ \beta + 1 < \alpha$). A limit ordinal is like an "horizon" for enumeration: it is impossible to reach its limit in a finite number of steps. The smallest limit ordinal is $\omega \simeq \mathbb{N}$.

> The sum of two ordinals is their concatenation (non-commutativity: $1+\omega = \omega \neq \omega+1$). The product of two ordinals $\alpha.\beta$ is β-times the concatenation of α (lexicographic order) (non-commutativity: $2.\omega = \omega \neq \omega.2 = \omega + \omega$). An ordinal α is a limit ordinal iff there exists β s.t. $\alpha = \omega.\beta$.

On the other hand, *cardinals* $|X|$ are equivalence classes for the equivalence relation of equipotence: $X \,\mathrm{eq}\, Y$ if there exists a bijective (i.e. one-to-one onto) map $f : X \to Y$. They highly depend upon the functions existing in the ZFC-universe under consideration.

Cantor theorem. $|X| < |\mathcal{P}\,(X)|$. □

> Indeed, let $f : X \to \mathcal{P}\,(X)$. Then $Y = \{x \in X : x \notin f(x)\}$ exists (Comprehension axiom). But $Y \notin f(X)$ for if it would exist $z \in X$ with $f(z) = Y$, then $z \in Y \Leftrightarrow z \notin Y$. Contradiction. □

If $|A| = \kappa$, then $|\mathcal{P}\,(A)| = 2^\kappa$ since a subset $X \subseteq A$ is equivalent to its characteristic function $\chi_X : A \to \{0, 1\}$ and $\chi_X \in 2^\kappa$. Cantor theorem implies therefore $\kappa < 2^\kappa$.

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by their elements; *Pairing*: the pair $\{a, b\}$ exists for every sets a and b; *Union*: the union $\bigcup X = \{u \in x \in X\}$ of every set X exists; *Comprehension* or *Separation* (axiom schema): if $\varphi(x)$ is a formula, the subset $\{x \in X : \varphi(x)\}$ exists for every set X; *Replacement* (axiom schema): if $y = f(x)$ is function (i.e. a relation $\varphi(x, y)$ s.t. $\varphi(x, y)$ and $\varphi(x, z)$ imply $y = z$), then the image $\{f(x) : x \in X\}$ exists for every set X; *Infinity*: there exists an inductive set, that is a set I s.t. $\emptyset \in I$ and if $x \in I$ then $x \cup \{x\} \in I$; *Regularity* (well-foundedness): all sets have minimal $\in$-elements.

For ordinals, cardinals *numbers* are the α s.t. $|\beta| < |\alpha|$ for every $\beta < \alpha$. They are the minimal elements in the equivalence classes of equipotent ordinals. So every *infinite* cardinal number is a *limit* ordinal and a natural (but extremely hard) challenge is, for every infinite cardinal number κ with succesor κ^+ to find bijections between κ and the ordinals $\kappa < \alpha < \kappa^+$. Every well-ordered set has a cardinal number for cardinal. These cardinal numbers are the *alephs* $\aleph_\alpha$. Each $\aleph_\alpha$ has a successor, namely $\aleph_\alpha^+ = \aleph_{\alpha+1}$. If α is a limit ordinal, $\aleph_\alpha = \omega_\alpha = \mathrm{Sup}_{\beta<\alpha}(\omega_\beta)$.

Theorem. *For alephs, the sum and product operations are trivial* (the bigger takes all): $\aleph_\alpha + \aleph_\beta = \aleph_\alpha.\aleph_\beta = \max(\aleph_\alpha, \aleph_\beta)$. □

> A consequence is that if $\alpha \leq \beta$ then $\aleph_\alpha^{\aleph_\beta} = 2^{\aleph_\beta}$. Indeed, $2^{\aleph_\beta} \leq \aleph_\alpha^{\aleph_\beta} \leq \left(2^{\aleph_\alpha}\right)^{\aleph_\beta} = 2^{\aleph_\alpha.\aleph_\beta} = 2^{\aleph_\beta}$. But other *exponentiations* raise fundamental problems.

Under the axiom of choice AC, *every* set can be well-ordered and therefore all cardinals are alephs. In particular $2^{\aleph_0}$ is an aleph $\aleph_\alpha$.

3 The underdetermination of cardinal arithmetic in *ZFC*

Let V be a ZFC-universe. We work in $\mathbb{R}$ or in the isomorphic Baire space $\mathcal{N} = \omega^\omega$. The first limit we meet is that the axioms of ZFC are radically insufficient for determining the *cardinal arithmetic* of V as is clearly shown by the following celebrated result.[2] Cardinal aritmetic is essentially computable from the powers 2^κ and the cofinalities $\mathrm{cf}(\kappa)$ ($\kappa^{\mathrm{cf}(\kappa)}$ is the Gimel function).

The concept of *cofinality* $\mathrm{cf}(\alpha)$ of an ordinal α was introduced by Felix Hausdorff. It is defined as the smallest cardinality χ of a cofinal proper subset X of α (i.e. $\mathrm{Sup}\, X = \alpha$). $\mathrm{cf}(\alpha)$ is a limit ordinal and $\mathrm{cf}(\alpha) \leq \alpha$.

> For instance, $\mathrm{cf}(\omega + \omega) = \mathrm{cf}(\aleph_{\alpha+\omega}) = \omega$.

The cardinal κ is called *regular* if $\mathrm{cf}(\kappa) = \kappa$ i.e. if, as far as we start with $\alpha < \kappa$, it is impossible to reach the horizon of κ in less than κ steps. In some sense the length of κ is equal to its “asymptotic” length and cannot be exhausted before reaching the horizon. As $\mathrm{cf}(\mathrm{cf}(\kappa)) = \mathrm{cf}(\kappa)$, $\mathrm{cf}(\kappa)$ is always regular. As $\mathrm{cf}(\omega + \omega) = \mathrm{cf}(\aleph_{\alpha+\omega}) = \omega$, $\omega + \omega$ and $\aleph_{\alpha+\omega}$ are singular singular. A cardinal κ is singular iff $\kappa = \bigcup_{i\in I} \alpha_i$ with $|I| < \kappa$ and all $|\alpha_i| < \kappa$.

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[2] See e.g. Jacques Stern [1976] for a presentation.

3.1 (William) Easton's theorem

For every ordinal α let $F(\alpha)$ be the power function defined by $2^{\aleph_\alpha} = \aleph_{F(\alpha)}$.[3] One can show that:

1. F is a monotone increasing function: if $\alpha \leq \beta$ then $F(\alpha) \leq F(\beta)$;
2. *König's law*: $\mathrm{cf}\left(\aleph_{F(\alpha)}\right) > \aleph_\alpha$.

The function $2^{\aleph_\alpha} = \aleph_{F(\alpha)}$ is of the utmost importance for understanding the global structure of V.

> König's law is a consequence of a generalization of Cantor theorem which says that $1 + 1 + \ldots$ (κ times) $< 2.2.\ldots$ (κ times): if $\kappa_i < \lambda_i\ \forall i \in I$, then $\sum_{i \in I} \kappa_i < \prod_{i \in I} \lambda_i$. We want to prove that with a family of $\kappa_i < 2^{\aleph_\alpha}$ of length ω_α we cannot reach $\mathrm{Sup}\,(\kappa_i) = \aleph_{F(\alpha)} = 2^{\aleph_\alpha}$. But for infinite cardinals $\mathrm{Sup} = \sum$. So, let $\lambda_i = 2^{\aleph_\alpha}$ for $i < \omega_\alpha$. Then $\sum_{i<\omega_\alpha} \kappa_i < \prod_{i<\omega_\alpha} \lambda_i = \left(2^{\aleph_\alpha}\right)^{\aleph_\alpha} = 2^{\aleph_\alpha}$. Therefore with an ω_α sequence of $\kappa_i < 2^{\aleph_\alpha}$ it is impossible to get $\mathrm{Sup}\,(\kappa_i) = 2^{\aleph_\alpha}$

An immediate corollary of König's law is

Theorem. If κ is an infinite cardinal, $\kappa < \kappa^{\mathrm{cf}(\kappa)}$ (compare with Cantor: $\kappa < 2^\kappa$). □

Indeed, by definition of cofinality, there exist $\kappa_i < \kappa$ for $i < \mathrm{cf}(\kappa)$ and $\kappa = \mathrm{Sup}\,(\kappa_i) = \sum_{i<\mathrm{cf}(\kappa)} \kappa_i$ But then $\kappa = \sum_{i<\mathrm{cf}(\kappa)} \kappa_i < \prod_{i<\mathrm{cf}(\kappa)} \kappa = \kappa^{\mathrm{cf}(\kappa)}$. □

In fact one can prove that the essential cardinals for cardinal arithmetic are the 2^κ and the $\kappa^{\mathrm{cf}(\kappa)}$ (Gimel function). They enable to compute all the $\aleph_\alpha^{\aleph_\beta}$:

Theorem.[4]

1. If $\alpha \leq \beta$, then $\aleph_\alpha^{\aleph_\beta} = 2^{\aleph_\beta}$.
2. If $\alpha > \beta$ and $\exists \gamma < \alpha$ s.t. $\aleph_\gamma^{\aleph_\beta} \geq \aleph_\alpha$, then $\aleph_\alpha^{\aleph_\beta} = \aleph_\gamma^{\aleph_\beta}$.
3. If $\alpha > \beta$ and $\forall \gamma < \alpha$ we have $\aleph_\gamma^{\aleph_\beta} < \aleph_\alpha$ then
   (a) if $\aleph_\alpha$ is regular or $\mathrm{cf}(\aleph_\alpha) > \aleph_\beta$ then $\aleph_\alpha^{\aleph_\beta} = \aleph_\alpha$;
   (b) if $\mathrm{cf}(\aleph_\alpha) \leq \aleph_\beta < \aleph_\alpha$ then $\aleph_\alpha^{\aleph_\beta} = \aleph_\alpha^{\mathrm{cf}(\aleph_\alpha)}$. □

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[3] More generally one can consider the power function $(\lambda, \kappa) \mapsto \lambda^\kappa$ for each pair (λ, κ) of cardinals.

[4] See Jech [1978], p. 49.

If the generalized continuum hypothesis (GCH) holds, König's law is trivial because $F(\alpha) = \alpha+1$, every cardinal $\aleph_{\alpha+1}$ is regular and therefore $\mathrm{cf}\,(\aleph_{\alpha+1}) = \aleph_{\alpha+1} > \aleph_\alpha$.

The fact that ZFC radically underdetermines cardinal arithmetic is particularly evident in Easton's striking result:

Easton's theorem. *For* regular *cardinals* $\aleph_\alpha$*, one can impose via forcing in* ZFC *the power function* $2^{\aleph_\alpha} = \aleph_{F(\alpha)}$ *for quite every function* F *satisfying (i) and (ii).* $\square$

> For regular cardinals κ, we have $\kappa^{\mathrm{cf}(\kappa)} = \kappa^\kappa = 2^\kappa$ but for singular cardinals σ we have $\sigma^{\mathrm{cf}(\sigma)} = \begin{pmatrix} \sigma \\ \mathrm{cf}\,(\sigma) \end{pmatrix} 2^{\mathrm{cf}(\sigma)}$ where $\begin{pmatrix} \kappa \\ \lambda \end{pmatrix}$ for $\kappa > \lambda$ is a sophisticated generalization of the binomial formula.

Regular cardinals are widely underdetermined but singular cardinal are more constrained.

The *Singular Cardinal Hypothesis* (SCH) says that if κ is singular, then 2^κ is as small as possible with respect to the 2^μ for $\mu < \kappa$. In particular is κ is singular and a strong limit, then $2^\kappa = \kappa^+$ (κ satisfies the continuum hypothesis. Warning : ω is regular and CH can fail.). The SCH is motivated by Silver's theorem of "preservation" : if κ is singular with $\mathrm{cf}\,(\kappa) > \omega$ and if $2^\mu = \mu^+$ for $\mu < \kappa$ then $2^\kappa = \kappa^+$. Jensen called Silver's theorem a "schocking discovery" and Kanamori (2011, p.29) claimed

> "this was a dramatic event and would stimulate dramatic developments."

3.2 Cohen forcing

The proof of Easton's theorem uses iterated Cohen forcing.

Cohen forcing (1963)[5] allows to construct in a very systematic way "generic" extensions N of *inner models* M of ZF or ZFC (that is $\in$-transitive submodels $M \subset V$ of ZF or ZFC with $On \subseteq M$) where some desired properties become valid. The idea is to add new sets in the model and is very reminiscent of what happens in number theory when we add to $\mathbb{Q}$ some solutions of algebraic equations with coefficients in $\mathbb{Q}$ (that is *describable* in the ground field $\mathbb{Q}$ even if solutions *do not* exist in $\mathbb{Q}$) and look at algebraic extensions as $\mathbb{Q}(\sqrt{-2})$. There is indeed a very

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[5] Awarded a Fields Medal in 1966, Paul Cohen died on March 23, 2007.

deep analogy between the analysis of ZFC models and that of number fields.

Suppose for instance that, starting with a ground inner model M of ZFC in V, we want to construct another inner model N where ω_1^M (that is the cardinal $\omega_1 = \aleph_1$ *as defined in* M) *collapses* and becomes *countable*. We need to have at our disposal *in* N a surjection $f : \omega \to \omega_1^M$ that, by definition of ω_1^M, cannot belong to M. Suppose nevertheless that such an f exists. Then for every n the restriction $f\ |_n = (f(0), \ldots, f(n-1))$ exists and is an element of the ground model M. Such a set of $f\ |_n$ would be a sort of "blueprint" of a possible f. Let us therefore consider the set $P = \{p\}$ of *all* finite sequences $p = (\alpha_0, \ldots, \alpha_{n-1})$ of countable ordinals $\alpha_i < \omega_1^M$ of M. Such p are called *forcing conditions* and must be interpreted as forcing $f\ |_n = p$. The set P exists, is well defined in M, and is endowed with a natural partial order "$q \leq p$ iff $p \subseteq q$"[6]. If f exists, we can consider $G = \{f\ |_n\}_{n \in \mathbb{N}}$ which is a subset of P in V s.t. $\cup G = f$. But as $f \notin M$, *G cannot be a subset of P in M.*

If f exists, it is trivial to verify that G satisfies the following *coherence* properties:

1. *Gluing and restriction conditions* (as for sheaves): if $p, q \in G$, then p and q are initial segments of f and are compatible in the sense that $p \leq q$ or $q \leq p$ and therefore there exists a common smaller element $r \in G$ satisfying $r \leq p$, $r \leq q$.

2. for every $n \in \omega$, there exists $p \in G$ s.t. $n \in \mathrm{dom}(p)$ (i.e. $\mathrm{dom}(f) = \omega$).

3. for every M-countable ordinal $\alpha < \omega_1^M$, there exists $p \in G$ s.t. $\alpha \in \mathrm{range}(p)$ (i.e. $\mathrm{range}(f) = \omega_1^M$, it is the fundamental condition of surjectivity for the collapsing of ω_1^M in N).

Cohen's idea is to construct sets G in V satisfying these properties and to show that extending the ground inner model M by such a G yields an appropriate inner model $N = M[G]$ which is the smaller inner model of V containing M and G.

So, one supposes that a partially ordered set of forcing conditions P is given. A subset of conditions $D \subseteq P$ is called *dense* if for every $p \in P$ there is a smaller $d \leq p$ belonging to D. One then defines *generic* classes $G \subseteq P$ of conditions. A subset of P, $G \notin M$, is generic over M iff is a *filter* for the order $\leq$ [7] such that G intersects every dense set D

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[6] That is $q < p$ means conventionally that q forces a better approximation of f than p (smaller is better).

[7] This means (i) that $p \in G$ and $p \leq q \in P$ implies $q \in G$, and (ii) that for every $p, q \in G$, there exists a common smaller $r \in G$ satisfying $r \leq p$, $r \leq q$,

of conditions $D \in M$ [8] (be careful: $D \in M$, $D \subseteq P \in M$, $G \subseteq P$, but $G \notin M$).

If G is generic, the properties (2) and (3) above are automatically satisfied since the sets of conditions $D_n = \{p \in P : n \in \mathrm{dom}(p)\}$ for $n \in \omega$ and $E_\alpha = \{p \in P : \alpha \in \mathrm{range}(p)\}$ for $\alpha < \omega_1^M$ are dense: (2) means that $G \cap D_n \neq \emptyset$ and (3) means that $G \cap E_\alpha \neq \emptyset$.

Cohen's main results say that M can be extended using G to an inner model $M[G]$ where properties "forced" by P become valid. The first key result concerns the existence of $M[G]$.

Cohen's theorem (1963). *There exists a ZFC-model* $\mathcal{A} = M[G]$ *such that* (1) M *is an inner model of* $\mathcal{A}$, (2) G *is not a set in* M *but is a set in* $\mathcal{A}$, (3) $\mathcal{A}$ *is essentially unique (minimal).* □

An essential feature of forcing extensions is that it is possible to describe $M[G]$ using the language $\mathcal{L}_G$ which is the language $\mathcal{L}$ of M extended by a new symbol constant for G. We can define in M "names" for the elements of $M[G]$ even if they live outside M in V. As was emphasized by Patrick Dehornoy (2003), forcing is

> "as a field extension whose elements are described by polynomials defined on the ground field".

In particular, the validity of a formula φ in $M[G]$ can be coded by a *forcing relation* $p \Vdash \varphi$ *defined in* M. This is the second key result. The definition of $p \Vdash \varphi$ is rather technical but an excellent intuition is given by the idea of "localizing" truth, p being interpreted as a local domain (as an open set of some topological space), and $p \Vdash \varphi$ meaning that φ is "locally true" on p.

Forcing theorem. *For every generic* $G \subseteq P$, $M[G] \models \varphi$ *iff there exists a* $p \in G$ *s.t.* $p \Vdash \varphi$. □

Using forcing, we can in particulat add to $\mathbb{R}$ (i.e. to $\mathcal{P}(\omega)$) new elements called *generic reals*. Let P be the partial order of binary finite sequences $p = (p(0), \ldots, p(n-1))$. If $G \subseteq P$ is generic, $f = \cup G$ is a map $f : \omega \to \{0,1\}$ which is the characteristic function $f = 1_A$ of a *new* subset $A \subseteq \omega$ and $A \notin M$. Indeed, if $g : \omega \to \{0,1\}$ defines a subset $B \subseteq \omega$ which belongs to M, then the set of conditions $D_g = \{p \in P : p \nsubseteq g\} \in M$ is dense (if p is any finite sequence it can be extended to a sequence long enough to be different from g) and therefore $G \cap D_g \neq \emptyset$. But this means that $f \neq g$.

To prove the *negation* $\neg CH$ of CH, one adds to M a great number of generic reals. More precisely, one embeds ω_2^M into $\{0,1\}^\omega$ (isomorphic to $\mathbb{R}$) using as forcing conditions the set P of finite binary sequences of

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[8] I.e. there exists $p \in D$ such that $p \in G$.

$\omega_2^M \times \omega$. If G is generic, then $f = \cup G$ is a map $f : \omega_2^M \times \omega \to \{0,1\}$, that is an ω_2^M-family $f = \{f_\alpha\}_{\alpha < \omega_2^M}$ of generic reals $f_\alpha : \omega \to \{0,1\}$. Using density arguments one shows that f yields an embedding $\omega_2^M \hookrightarrow \{0,1\}^\omega$ in $M[G]$ and that ω_2^M *doesn't collapse* in $M[G]$ (because P is ω-saturated, i.e. there doesn't exist in P any infinite countable subset of incompatible elements). This implies immediately $\neg CH$.

Easton theorem is proved by *iterating* such constructions and adding to every regular $\aleph_\alpha$ as many new subsets as it is necessary to have $2^{\aleph_\alpha} = \aleph_{F(\alpha)}$.

As said Kanamori (2011, p.9), the method of forcing is a :

> "remarkably general and flexible method for extending models of set theory".

An extraordinary harvest of new results followed very quickly. For instance Azriel Levy defined the collapse of inaccessible cardinals λ (see below : if $\alpha < \lambda$, $2^\alpha < \lambda$). If κ is a regular cardinal and $\lambda > \kappa$ is inaccessible we can collapse all the $\alpha < \lambda$ in such a way that, in $M[G]$, λ would become the succesor $\lambda = \kappa^+$.

3.3 Absoluteness

Many philosophers and logicians who are "deflationist" regarding mathematical "ontology" consider that the only sentences having a well determined truth-value are those the truth-value of which is the same in all models of ZFC, and that sentences the truth-value of which can change depending on the chosen model are "inherently vague" (Feferman). Such an antiplatonist conception has drastic consequences.

Many elementary set operations (pairs, unions, intersections, inclusions, products, fonctions and their domains and codomains, are ZF-absolute. Ordinals are ZF-absolute. According to Schönfield theorem, first order arithmetic (PA) is ZF-absolute, that is invariant w.r.t. extensions of the universe. But, contrary to PA, structures and notions as fundamental as $\mathbb{R}$, Card(κ), $X \to \mathcal{P}(X)$, $X \to |X|$, and second order arithmetic (analysis, axiomatized by $\mathbf{Z}_2$), *are not ZF-absolute.*

$\mathcal{P}(X)$ is only "downward" absolute, that is if $Y \in \mathcal{P}(X)$ is valid in V it is valid in every sub-model of V. It is the same for Card(κ). So if κ is regular in $M[G]$ it is surely regular in M, but κ is regular in M,it can collapse in $M[G]$.

This "vagueness" is one of the main classical arguments of antiplatonists against non-constructive set theories. But, it has been emphasized by Hugh Woodin in his 2003 paper *Set theory after Russell. The journey back to Eden* that vagueness is not an admissible argument against platonism and shows only that it is necessary to *classify* the different models of ZF and ZFC. As he explained also in his talk at the Logic Colloquium held in Paris in 2000 (quotation from Dehornoy, 2003, p. 23):

> "There is a tendency to claim that the Continuum Hypothesis is inherently vague and that this is simply the end of the story. But any legitimate claim that CH is inherently vague must have a mathematical basis, at the very least a theorem or a collection of theorems. My own view is that the independence of CH from ZFC, and from ZFC together with large cardinal axioms, does not provide this basis. (...) Instead, for me, the independence results for CH simply show that CH is a difficult problem."

By the way, the CH was the first problem in Hilbert's list in the International Mathematical Congress in Paris in 1900.

> In fact, the strong variability of the possible models of ZFC is an argument in favor of the *irreducibility* of the continuum to a set of points which can be "*individuated*" by a symbolic description, and platonism can be interpreted as a search for a certain type of absoluteness of the continuum w.r.t. some theories stronger than ZF and ZFC.

As far as the CH is ZFC-provably valid for some classes of subsets of reals (see below), but not for the whole set $\mathcal{P}(\mathbb{R})$, CH is a question about the non absoluteness of $X \to \mathcal{P}(X)$.

It must be emphasized that well-known classical mathematical problems are undecidable in ZFC.

- Borel's conjecture. Let $A \subseteq \mathbb{R}$ be of *strong* measure 0 that is $A \subseteq \bigcup_{n \in \mathbb{N}} I_n$ of intervals I_n of length $\varepsilon_n > 0$ for every family of ε_n. It is Lebesgue measurable of measure 0. Must it necessarily be countable? A Cantor set is uncountable of Lebesgue measure 0 but not of strong measure 0. Wacław Sierpinski (1928) + Richard Laver (1976): undecidable.

- Whitehead's conjecture. If G is an abelian group and if the exact sequence $0 \to \mathbb{Z} \to E \to G \to 0$ is an abelian extension of G by $\mathbb{Z}$, does the sequence split ($\mathrm{Ext}^1(G, \mathbb{Z}) = 0$)? Shelah (1973): undecidable.

- Kaplanski's conjecture. Let $f : B \to C^0([0,1])$ be a morphism of Banach algebras. Must it necessarily be continuous? Garth Dales(1970) + Robert Solovay (1976) and Hugh Woodin (1987): undecidable.

To tackle this problem, we will have to look at two opposed strategies, both introduced by Gödel, one "minimalist" ("ontologically" deflationist) and the other "maximalist" ("ontologically" inflationist). To explain this point we first introduce some classes of sets of reals studied by what is called "descriptive set theory".

3.4 Borel and projective hierarchies

In descriptive set theory, one works in $\mathbb{R}$ or in $\mathcal{N} = \omega^\omega$ or in $\{0,1\}^\omega$, and, more generally, on metric, separable, complete, perfect (closed without isolated points) spaces $\mathcal{X}$ (Polish spaces). One considers in $\mathcal{X}$ different "nicely" definable classes of subsets Γ. The first is the *Borel hierarchy* constructed from the open sets by iterating the set theoretic operations of complementation and of *countable* projection $\mathcal{X} \times \omega \to \mathcal{X}$. If $P \subseteq \mathcal{X} \times \omega$ (that is, if P is a countable family of subsets $P_n \subseteq \mathcal{X}$), one considers the subset of $\mathcal{X}$ defined by $\exists^\omega P = \{x \in \mathcal{X} \mid \exists n\ P(x,n)\}$.[9] It is the union $\bigcup\limits_{n\in\omega} P_n$.

The Σ^0_1 are the open subsets of $\mathcal{X}$, the $\Pi^0_1 = \neg\Sigma^0_1$ are the closed subsets, $\Delta^0_1 = \Pi^0_1 \cap \Sigma^0_1$ the clopen subsets, and the Borel hierarchy B is defined by:

$$\Pi^0_n = \left\{\neg\varphi \mid \varphi \in \Sigma^0_n\right\} = \neg\Sigma^0_n,\ \Sigma^0_{n+1} = \exists^\omega\neg\Sigma^0_n = \exists^\omega\Pi^0_n,\ \Delta^0_n = \Pi^0_n \cap \Sigma^0_n.$$

It can be shown that this hierarchy is *strict*:

$$\begin{tikzcd}
 & \Sigma^0_n \arrow[dr] & \\
\Delta^0_n \arrow[ur] \arrow[dr] & & \Delta^0_{n+1} \\
 & \Pi^0_n \arrow[ur] &
\end{tikzcd}$$

One then defines the higher hierarchy of *projective* sets using a supplementary principle of construction, namely *continuous* projections $\mathcal{X} \times \mathcal{N} \to \mathcal{X}$, written $\exists^\mathcal{N}$. One gets a new hierarchy beginning with the class $\Sigma^1_1 = \exists^\mathcal{N}\Pi^0_1$ – the so called *analytic* subsets – and continuing with the classes:

$$\Pi^1_n = \left\{\neg\varphi \mid \varphi \in \Sigma^1_n\right\} = \neg\Sigma^1_n,\ \Sigma^1_{n+1} = \exists^\mathcal{N}\neg\Sigma^1_n = \exists^\mathcal{N}\Pi^1_n,\ \Delta^1_n = \Pi^1_n \cap \Sigma^1_n.$$

For instance, $P \subseteq \mathcal{X}$ is Σ^1_1 if there exists a *closed* subset $F \subseteq \mathcal{X} \times \mathcal{N}$ such that: $P(x) \Leftrightarrow \exists\alpha\ F(x,\alpha)$. In the same way, $P \subseteq \mathcal{X}$ is Σ^1_2 if

---

[9] $P(x,n) = P_n(x)$. We identify predicates $\varphi(x)$, $P(x,n)$, etc. with their extensions.

there exists an *open* subset $G \subseteq \mathcal{X} \times \mathcal{N} \times \mathcal{N}$ such that: $P(x) \Leftrightarrow \exists\alpha\ \forall\beta\ G(x, \alpha, \beta)$, etc.

More generally, one can define projective sets in V using the cumulative hierarchy V_α. Indeed P is projective if it is definable with parameters over $(V_{\omega+1}, \in)$. More precisely, $P \subset V_{\omega+1}$ is Σ^1_n if it is the set of sets x s.t. $(V_{\omega+1}, \in) \models \varphi(x)$ for a Σ_n formula $\varphi(x)$, that is a formula of the form $\varphi(x) = \exists x_1 \forall x_2 \ldots \psi$ with n quantifiers and a ψ having only *bounded* quantifiers.[10] In that sense, projective sets can be considered as the "reasonably" definable subsets of $\mathbb{R}$.

As the Borel hierarchy, the projective hierarchy is *strict* and it is a continuation of the Borel hierarchy according to:

Suslin theorem. $B = \Delta^1_1$. □

This theorem can be interpreted as a *construction principle*: it asserts that in the Δ^1_1 case, the complex operation of continuous projection can be reduced to an iteration of simpler operations of union and complementation.

There exist strict Π^1_n and Σ^1_n sets, which are very natural in classical analysis. For instance, in the functional space $C^0([0,1])$ of real continuous functions on $[0,1]$ endowed with the topology of uniform convergence, the subset

$$\left\{f \in C^0([0,1]) \mid f \text{ smooth}\right\}$$

is Π^1_1 (but not Δ^1_1).

> In the space $C[0,1]^\omega$ of countable sequences (f_i) of functions, the subset:
>
> $$\left\{(f_i) \in C[0,1]^\omega \;\middle|\; \begin{array}{c}(f_i) \text{ converges for the topology} \\ \text{of simple convergence}\end{array}\right\}$$
>
> is Π^1_1, and the subset:
>
> $$\left\{(f_i) \in C[0,1]^\omega \;\middle|\; \begin{array}{c}\text{a sub-sequence converges for the} \\ \text{topology of simple convergence}\end{array}\right\}$$
>
> is Σ^1_2 and every Σ^1_2-subset of $C[0,1]$ can be represented that way (Becker [1992]):
>
> **Becker representation theorem.** *For every Σ^1_2-set $S \subseteq C[0,1]$ there exists a sequence (f_i) such that*
>
> $$S = A_{(f_i)} = \left\{g \in C[0,1] \;\middle|\; \begin{array}{c}\text{a sub-sequence of } (f_i) \text{ converges} \\ \text{towards } g \text{ for the topology} \\ \text{of simple convergence}\end{array}\right\}.$$

---

[10] Bounded quantifiers are of the form $\exists y \in z$ and $\forall y \in z$.

Another examples are given by the compact subsets $K \in \mathcal{K}(\mathbb{R}^n)$ of $\mathbb{R}^n$: for $n \geq 3$,

$$\{K \in \mathcal{K}(\mathbb{R}^n) \mid K \text{ arc connected}\}$$

is Π_2^1,

and for $n \geq 4$,

$$\{K \in \mathcal{K}(\mathbb{R}^n) \mid K \text{ simply connected}\}$$

is also Π_2^1.

4 The "minimalist" strategy of the constructible universe

4.1 Absolute constructivity$L$

The first Gödelian strategy for constraining the structure of ZF-universes consisted in *restricting* the universe V. It is the strategy – refered to as $V = L$ – of *constructible* sets (Gödel 1938). It is "ontologically deflationary".

To define L, one substitutes, in the construction of the cumulative hierarchy V_α of V by means of a transfinite recursion on the $X \to \mathcal{P}(X)$ operation, the power sets $\mathcal{P}(X)$ – which are not ZF-absolute – with *smaller* sets $\mathcal{D}(X) = \{Y \subseteq X \mid Y \text{ elementary}\}$ (where "elementary" means definable by a *first order* formula over the structure $\langle X, \in \rangle$) – which are ZF-absolute. This means that Y is definable by a first-order formula $\varphi(z, a_i)$ with parameters $a_i \in X$ i.e. $z \in Y \Longleftrightarrow \varphi(z, a_i)$. If we accept second order formulas then we get HOD and not L.

L is then defined as V using a transfinite recursion on ordinals: $L_0 = \emptyset$, $L_{\alpha+1} = \mathcal{D}(L_\alpha)$, $L_\lambda = \bigcup_{\alpha<\lambda} L_\alpha$ if λ is a limit ordinal, and $L = \bigcup_{\alpha \in On} L_\alpha$. L is absolute and its absoluteness comes from the fact that each level L_α is constructed using only unambiguous formulae and parameters belonging to the previous stages L_β, $\beta < \alpha$.

Gödel (1938, 1940) has shown that if $V = L$ it is possible to define a *global wellordering* on L, which is a very strong form of *global AC*. The well-order relation is defined by a transfinite induction on the levels α. If X and Y are of different levels their order is the order of their respective levels. If they are of the same level, their order is first that of the Gödel numbers of their minimal defining formulae, and then the order of their parameters (which are of lower order and therefore wellordered by the induction hypothesis). Gödel also proved the fundamental result that in ZF we have the *generalized continuum hypothesis*: $(V = L) \vdash GCH$. It is due to the fact that, if $\alpha \geq \omega$, $|L_\alpha| = |\alpha|$. We have $\mathcal{P}^L(\omega_\alpha) \subset L_{\omega_{\alpha+1}}$

but $\left|L_{\omega_{\alpha+1}}\right| = \aleph_{\alpha+1}$ and therefore $\left|\mathcal{P}^L(\omega_\alpha)\right| \leq \aleph_{\alpha+1}$. So, in contrast with the "width" of V which increases as a "cone", the "width" of L increases by steps. $|L_{\omega+n}| = |\omega+n| = \aleph_0$ (while $|V_{\omega+1}| = |\mathcal{P}(V_\omega)| > \aleph_0$ and we have to wait until ω_1 to get $|L_{\omega_1}| = |\omega_1| > \aleph_0$.

L is in fact the *smallest* inner model of V:

(i) $On \subset L$,

(ii) L is *transitive*: if $Y \in_V X$ and $X \in_L L$, then $Y \in_L L$,

(iii) $(L, \in_L)$ is a model of ZF.

It can be defined in V by a sentence $L(x) =$ "x is constructible" which is *independent* of V (ZF-absolute), and in that sense, it is a *canonical* model of ZFC.

Remark. It must be emphasized that the constructible universe L is not strictly constructive since it contains the class On of ordinals which is non-constructive. But the characteristic property of L is that it reduces non-constructivity exactly to On.

In the constructible universe L there exists a Δ^1_2-wellorder relation $<$ on $\mathbb{R}$. But, according to a theorem due to Fubini, such a wellordering *cannot be Lebesgue measurable* and there exist therefore in L Δ^1_2 sets which, despite the fact they belong to the low levels of the projective hierarchy and are "simple" and "nice" to define, are nevertheless not Lebesgue measurable and therefore not well-behaved. This astonishing property is very troublesome.

With regards to CH, one uses the fact that the Δ^1_2-wellorder relation $<$ on $\mathbb{R}$ is a fortiori Σ^1_2, and that the Σ^1_2 are the $\aleph_1$-Suslin sets. If χ is an infinite cardinal, $P \subseteq \mathbb{R}$ is called a χ-Suslin set if it exists a closed subset $F \subseteq \mathbb{R} \times \chi^{\mathbb{N}}$ s.t. $P = \exists^{\chi^{\mathbb{N}}} F$ (i.e. P is the projection of F). The Σ^1_1 are, by definition, the $\aleph_0$-Suslin sets. Indeed, if $\chi = \aleph_0$ then $P = \exists^{\mathbb{R}} F$ and therefore $P \in \Sigma^1_1$. A theorem of Martin says that $P \subseteq \mathcal{X}$ is an $\aleph_n$-Suslin set iff $P = \bigcup_{\xi < \aleph_n} P_\xi$ with P_ξ Borelians.[11] As the wellordering $<$ on $\mathbb{R}$ is Σ^1_2, according to a theorem of Schönfield, its ordinal is $< \aleph_2$ and CH is therefore valid.

In spite of its intrinsic limitations, L is a very interesting model of ZFC, which possesses a "fine structure" interpolating between the different L_α and very rich *combinatorial* properties investigated by Ronald Jensen, the "consummate master of constructibility" according to Akihiro Kanamori. One of its main properties is the following. Let us first define what is a *club* ("closed unbounded" subset) $C \subseteq \alpha$ of a limit

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[11] See Moschovakis [1980], p. 97.

ordinal α: C is closed for the order topology (i.e. limits in C belong to C: if $\beta < \alpha$ and $\mathrm{Sup}\,(C \cap \beta) = \beta$, then $\beta \in C$) and unbounded in α (for every $\beta < \alpha$ there exists an element $\gamma \in C$ s.t. $\beta < \gamma$). For a cardinal κ, let $\Box_\kappa$ be the property that there exists a sequence of clubs $C_\alpha \subseteq \alpha$ with limit ordinals $\alpha < \kappa^+$ s.t. C_α is of order type $\leq \kappa$ (and $< \kappa$ if $\mathrm{cf}\,(\alpha) < \kappa$) and if λ is a limit point of C_α then $C_\alpha \cap \lambda = C_\lambda$. $\Box_\kappa$ is used to construct systematically and coherently bijections between κ and ordinals $\kappa \leq \alpha < \kappa^+$ by cofinalizing the α by clubs. As explain Matthew Foreman and Menachem Magidor (1987),

> "[It] is useful for proving many combinatorial results, yielding a general method for passing singular limit cardinals in inductive constructions."

In L, these constructions are always possible according to Jensen:

Theorem (Jensen, 1970). $V = L \models \forall \kappa \, \Box_\kappa$. $\Box$

$\Box_\kappa$ constrains the structure of the *stationary* subsets S of κ^+ ($S \subset \kappa^+$ is stationary if $S \cap C \neq \emptyset$ for every club C in κ^+). They cannot reflect at some ordinal $\alpha < \kappa$ of cofinality $\mathrm{cf}\,(\alpha) > \omega$, where "reflect" means "remaining stationary in α".

In spite of its interest, the structure of L is rather pathological with regards to the continuum and many of the above results are in some sense counterintuitive. They result from the fact that the AC, which implies the existence of very complicated and irregular sets, remains valid in L and that the axiom of constructibility $V = L$ forces some of them to exist *inside* the projective hierarchy which should be composed only of relatively simple and regular sets: nicely definable sets are not necessarily well-behaved.

It is the reason why many specialists consider that the strategy $V = L$ is dramatically too restrictive and, moreover, not philosophically justifiable. For instance, John Steel (2000) considers L as "pathological" and claims:

> "The central idea of descriptive set theory is that definable sets of reals are free from the pathologies one gets from a wellorder of the reals. Since $V = L$ implies there is a Δ^1_2 wellorder of the reals, under $V = L$ this central idea collapses low in the projective hierarchy, and after that there is, in an important sense, *no* descriptive set theory. One has instead infinitary combinatorics on $\aleph_1$. This is certainly not the sort of theory that looks useful to Analysts."

4.2 Relative constructivity

One can generalize the concept of constructibility in two ways and define L-like universes.[12] First (Azriel Levy in the 1950s), if A is any set, one can relativize definability to A taking $\mathcal{D}_A(X) = \{Y \subseteq X \mid Y$ definable by a first order formula of the structure $\langle X, \in, A \cap X \rangle)\}$. One gets that way the universe, called $L[A]$, of constructible sets relative to A, which is the smallest inner model s.t. for every $X \in L[A]$ we have $A \cap X \in L[A]$. In $L[A]$ the only remaining part of A is $A \cap L[A] \in L[A]$. As L, $L[A]$ satisfies AC.

(We can note that the projective subsets are the $A \subseteq L_1[\mathbb{R}] \cap \mathbb{R}$.

On the other hand (Andràs Hajnal also in the 1950s), one can start the recursive construction of L not with $L_0 = \emptyset$ but with the transitive closure of $\{A\}$, $L_0(A)$. One gets that way $L(A)$ which is the smallest inner model containing On and A. If there is a wellordering on A (it the case if AC is valid), then $L(A)$ is globally wellordered for the same reasons as L. In particular, $L(\mathbb{R})$ is a good compromise between the non-constructibility of $\mathbb{R}$ and the constructibility from $\mathbb{R}$ of the rest of the universe.

One could think that generalizations of constructibility such as $L(A)$ or $L[A]$ would overcome the problems raised by L. But it is not the case. Hence a "paradigm shift" and a critical change of strategy.

5 The "maximalist" strategy of large cardinals

It is therefore justified to reverse the strategy and to look for *additional* axioms, which could be thought of as "natural", for ZF and ZFC. Gödel himself from 1946 began to plan the introduction of "stronger and stronger axioms of infinity" and imagined "extremely strong axioms of infinity of an entirely new kind". His strategy results from a deep hypothesis on *reflection* properties:

> "Any proof of a set-theoretic theorem in the next higher system above set theory (i.e. any proof involving the concept of truth ...) is replaceable by a proof from such an axiom of infinity." (quoted by Kanamori (2011), p4)

After him many set-theoretis tried to *generalize* to such augmented axiomatics the search of canonical models and fine combinatorial structures. As was emphasized by John Steel (2004):

> "In extending ZFC, we are attempting to *maximize interpretative power*".

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[12] See Kanamori [1994], p. 34.

And there is place for philosophy in such a maximizing strategy since the problem is not only to find a solution to the continuum problem but also to understand what "to be a solution" means. By the way, to study such "maximizing" large cardinals models is perfectly compatible with a minimalist perspective: to retrieve L, one has only to relativize the theory to the constructible subuniverse L since $ZFC +$ "$V = L$" $\vdash \varphi$ is equivalent to $ZFC \vdash \varphi^L$. As explained by Steel (2004), suppose that the philosopher A believes in L and the philosopher B in $L[G]$ with G forcing the adjunction of ω_2 reals to the model of $\mathbb{R}$ in L. A believes in CH and B in $\neg CH$, but B can interpret the formulae φ of A as its own φ^L and A can interpret the formulae φ of B as forced φ (the truth of $\Vdash \varphi$ being definable in the ground model L, see above). There is therefore no real conflict.

Different "maximizing" strategies have been considered in order to enhance the interpretative power and the consistency strength of ZFC:

1. Iterate transfinitely theories $T_{\alpha+1} = T_\alpha +$ "consistency of T_α" starting from ZF or ZFC.

2. Postulate "good" regularity properties of projective sets, and therefore of the continuum.

3. Make the theory of the continuum "*rigid*", that is define *under which conditions the properties of* $\mathbb{R}$ *cannot be further modified by forcing.*

Strategy (3) tries to *reduce* – and even to neutralize – the variability induced by forcing. The ideal aim would be forcing invariance to make the theories of $\mathbb{R}$ and $\mathcal{P}(\mathbb{R})$ in some sense as rigid as first order arithmetic. It is an extremely difficult program and we will first evoke some classical results concerning $\mathbb{R}$. CH concerns $\mathcal{P}(\mathbb{R})$ the forcing invariance of which is the object of more recent works of Woodin.

But we first emphasize the fact that strategies (1), (2) converge towards the introduction of *large cardinal axioms* (*LCAs*) which express the existence of higher infinities. Indeed, it seems that every "maximizing" strategy is in some sense equivalent to a LCA.

> Look for instance at the *Proper Forcing Axiom* PFA. A forcing P is called *proper* if, for every regular uncountable cardinal λ, it preserves the stationary subsets of $[\lambda]^\omega$ (the set of countable subsets of λ).
>
> **Proper Forcing Axiom.** If the forcing P is proper and if the D_α's are dense subsets indexed by the countable ordinals

> $\alpha < \omega_1$, then there exists a filter $G \subseteq P$ intersecting all the D_α's. (Compare with the definition of G being generic). $\square$
>
> Many results are known for PFA. It implies $2^{\aleph_0} = \aleph_2$ (and therefore $\neg CH$), it implies projective determinacy PD (Woodin) and AD (Steel, 2007) for the inner model $L(\mathbb{R})$.[13] As far as its consistency strength is concerned, it is known that [14]
>
> $$\mathrm{Con}\,(\exists\kappa \text{ supercompact}) \Rightarrow \mathrm{Con}\,(PFA) \Rightarrow \mathrm{Con}\,(\exists\kappa \text{ measurable})\,.$$
>
> It is conjectured that in fact PFA is equiconsistant with "$\exists\kappa$ supercompact" (see below for a definition of supercompactness).

It also seems that there exists a wellordering of *LCAs* which can be defined by the inclusion of their sets of Σ^1_2 (and even Π^0_1) consequences. As emphasized by John Steel (2000):

> "It seems that the consistency strengths of all natural extensions of ZFC are wellordered, and the large cardinal hierarchy provides a sort of yardstick which enables us to compare these consistency strengths."

Philosophically speaking, the nominalist confusion between a strong "quasi-ontology" for sets and a realist true ontology of abstract idealities has disqualified such axioms. But I think that such a dogmatic prejudice has been a great philosophical mistake. Indeed, one of the best philosophical formulation of incompleteness is precisely to say that a "good" theory of the continuum requires a very strong "quasi-ontology" for sets, a *maximal one*, not a minimal one.

A "good" regularity of the continuum entails *for objective reasons* a strong platonist commitment concerning higher infinities. Why?

Because the regularity of classes of $A \subset R$ depends on the "height" of V.

This key point has been perfectly emphasized by Patrick Dehornoy:

> "properties which put into play objects as 'small' as sets of reals (as low in the cumulative hierarchy) are related to other properties which put into play very 'huge' objects which seem very far from them."[15]

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[13] See below §§ 7.2 and 8 for a definition of PD and AD.

[14] If A is an axiom, Con (A) means "consistency of the theory $ZFC + A$".

[15] Dehornoy [1989].

Some specialists call "reverse descriptive set theory" this remarkable equivalence between properties of regularity of projective sets and $LCAs$.

There are many theorems showing that the platonist cost of a good theory is very high. Let us for instance mention one of the first striking theorems proved by Robert Solovay using forcing. Let CM be the axiom of existence of a measurable cardinal (see below for a definition).

Solovay theorem (1969). *$ZFC + CM \vdash$ every Σ_2^1 is "regular" (where "regular" means properties such as Baire property, Lebesgue measurability, and perfect set property).*[16] $\square$

6 Regularity of projective sets

6.1 The regularity of analytic sets

The French school (Borel, Baire, Lebesgue) and the Russian and Polish schools (Suslin, Luzin, Sierpinski) initiated the study of the Borel and projective classes and achieved deep results concerning their *regularity* and their *representation* where "regularity" means Lebesgue measurability, or the perfect set property (to be countable or to contain a perfect subset, i.e. a closed subset without isolated point), or the Baire property (to be approximated by an open subset up to a meager set, i.e. a countable union of nowhere dense sets).

Regularity is non trivial in ZFC since the axiom of choice enables to construct non regular sets (Bernstein, Vitali).

The first regularity theorem is the celebrated:

Cantor-Bendixson theorem. *If $A \subseteq \mathbb{R}$ is* closed, *then A can be decomposed in a unique way as a disjoint union $A = P + S$ where P is* perfect *and S* countable. $\square$

As a perfect set P is of cardinality $|P| = 2^{\aleph_0}$, the *continuum hypothesis CH holds for the closed sets Π_1^0*.

Another early great classical theorem of regularity is the:

Theorem. Borelians (Δ_1^1) are "regular".

Luzin-Suslin theorem. *The analytic subsets Σ_1^1 are "regular", shares the perfect subset property and CH is therefore true for the Σ_1^1 sets.* $\square$

In the same way, one can show that the Π_1^1 are Lebesgue measurable.

In 1925 Luzin asked if all projective subsets are "regular". It was a good question since it is in fact *impossible* to show in ZF (no AC) that the Δ_1^1 and Σ_2^1 share the perfect set property and to show in ZFC that the Δ_2^1 share the Baire property. In fact – and it has been a revolutionary result – many of the "natural" properties of the projective sets

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[16] See Moschovakis [1980], p. 284.

go *far beyond* the demonstrative strength of ZF and ZFC. It is therefore methodologically and philosophically justified to look for additional axioms.

6.2 Projective determinacy and the "regularity" of the continuum

A very interesting regularity hypothesis is the so called *determinacy* property. One considers infinite two-players games with perfect information on sets X. Each player (I and II) plays in turn an element a of X:

$$\begin{tikzcd}
I & a_0 \arrow[dr] & & a_2 \arrow[dr] & & ... \\
II & & a_1 \arrow[ur] & & a_3 \arrow[ur] &
\end{tikzcd}$$

At the end of the game we get a sequence $f \in X^{\mathbb{N}}$. Let $A \subset X^{\mathbb{N}}$. The player I (resp. II) wins the play f of the game $G = G_X(A)$ associated to A if $f \in A$ (resp. if $f \notin A$).

Definition. *$A \subset X^{\mathbb{N}}$ is called* determined *(written* $\mathrm{Det}(A)$ *or* $\mathrm{Det}\, G_X(A)$*) if one player has a winning strategy. Therefore A is determined iff*

$$\exists a_0 \forall a_1 \exists a_2 ...(a_0, a_1, a_2, ...) \in A.$$

Determinacy is a strong property of "regularity". Indeed, for every $A \subset \mathbb{R}$ ($\mathbb{R}$ being identified with $\mathcal{N} = \omega^\omega$, case where $X = \mathbb{N}$), $\mathrm{Det}(A) \Rightarrow$"$A$ satisfies the Baire and the perfect subset properties, and is Lebesgue measurable".

The first theorem linking determinacy with the projective hierarchy has been the key result:[17]

Gale-Stewart Theorem(1953). *$ZFC \vdash$ the open and closed subsets A of $X^{\mathbb{N}}$ (the Σ^0_1 and Π^0_1) are determined.* (It implies Cantor-Bendixon) ☐

After many efforts, Donald Martin proved a fundamental theorem which concluded a first stage of the story:

Martin theorem (1975). *$ZFC \vdash$ Borel sets (the $\Delta^1_1 \subset X^{\mathbb{N}}$) are determined.* ☐

This celebrated result – that Giorgio Venturi and Matteo Viale (2018) called "a major surprise" – shows that ZFC is the "good" axiomatic for the Borel subsets of $\mathbb{R}$. But, it is the *limit* of what is provable in ZFC. Indeed, ZFC *cannot* imply the determinacy of Σ^1_1-sets since in the constructible model L of ZFC there exist Σ^1_1-sets that don't share the perfect set property. As for Π^1_1-sets, their determinacy implies the

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[17] See Grigorieff [1976] and Moschovakis [1980], p. 288.

measurability of the Σ_2^1-sets, but in L there exists a Δ_2^1-wellorder of $\mathbb{R}$, which, according to Fubini theorem, cannot be Lebesgue measurable.

So, in what concerns descriptive set theory, the conclusion is that ZFC correctly axiomatizes not general sets but only specific classes of sets such as Borelians. With ZFC we are therefore *very far* from a correct comprehension of the structure of the continuum. To go further we need LCA.

7 The necessity of large cardinals and "reverse" descriptive set theory

We are now going to tackle a whole set of highly technical problems. A good reference are the excellent articles by Peter Koellner in the *Stanford Encyclopedia of Philosophy* (Independence of large cardinals, The continuum hypothesis, Large cardinals and determinacy).

7.1 Elementary embeddinds and ultrapowers

To go further we have to introduce the very important concepts of an *elementary embedding* and of an *ultrapower* $V^{\mathcal{U}}$ where $\mathcal{U}$ is an ultrafilter on a set S.

If $j : \mathcal{A} \prec \mathcal{A}'$ is an embedding of a model $\mathcal{A}$ in a model $\mathcal{A}'$, *elementarity* means that $\mathcal{A}'$ has exactly the same first-order theory as $\mathcal{A}$ in the very strong language $\mathcal{L}_{\mathcal{A}}$ where there exist names for *every* element of $\mathcal{A}$ (that is, for every set x and first-order formula φ, we have $\mathcal{A} \models \varphi(x)$ iff $\mathcal{A}' \models \varphi(j(x))$. So, for first-order logic, $\mathcal{A}'$ adds only *indiscernible* elements. A less constraining relation is *elementary equivalence*: $\mathcal{A}$ and $\mathcal{A}'$ are elementary equivalent, $\mathcal{A} \equiv \mathcal{A}'$, if they have the same first-order theory in the standard language $\mathcal{L}$ used for talking about models such $\mathcal{A}$. In that case, elements of $\mathcal{A}$ characterized by first-order sentences can be substituted for other elements of $\mathcal{A}'$. It is no longer the case for an elementary embedding.

If j is an elementary embedding $j : M \prec M^*$ of models of ZFC where M^* is an *inner* model of M (so M is embedded into one of its sub-model) and if $\alpha \in On(M)$ is an ordinal in M, one has $j(\alpha) \in On(M^*) \subset On(M)$ and, because of the elementarity of j , $\alpha < \beta \Leftrightarrow j(\alpha) < j(\beta)$. This implies $j(\alpha) \geq \alpha$. One shows that if j is not trivial there exists necessarily an ordinal α s.t. $j(\alpha) > \alpha$. Let χ be the smallest of these α. It is called the *critical ordinal* $\mathrm{crit}(j)$ of j.

If S is a set, a *filter* $\mathcal{U}$ over X is a set of subsets of X, $\mathcal{U} \subseteq \mathcal{P}(X)$, s.t. the complementary set of $\mathcal{U}$ in the Boolean algebra $\mathcal{P}(X)$ is an ideal.[18] $\mathcal{U}$

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[18] This means (i) $\emptyset \notin \mathcal{U}$, (ii) if $U \in \mathcal{U}$ and $U \subseteq V$ then $V \in \mathcal{U}$, (iii) if $U, V \in \mathcal{U}$ then $U \cap V \in \mathcal{U}$.

is an *ultrafilter* if it is maximal, namely if for every $U \subseteq X$, either $U \in \mathcal{U}$ or $X - U \in \mathcal{U}$. For every $x \in X$, $\mathcal{U}_x = \{U \subseteq X : x \in U\}$ is an ultrafilter called "principal". A non-principal ultrafilter is called "free". Pricipal ultrafilters are in some sense trivial while free ultrafilters can explore infinite horizons. A typical exemple is the ultrafilter of the $U \subseteq \mathbb{N}$ s.t. $\mathbb{N} - U$ is finite.

Let us now define the ultrapower $V^{\mathcal{U}}$ for such an ultrafilter. The elements of $V^{\mathcal{U}}$ are the maps $f : S \rightarrow V$, f and g being equivalent if they are equal almost everywhere (a.e.), that is if $\{s \in S : f(s) = g(s)\} \in \mathcal{U}$. Any element X of V is represented by the constant map $f_X(s) = X$ and this defines a canonical embedding $j : V \hookrightarrow V^{\mathcal{U}}$. If $\varphi(x_1, \ldots, x_n)$ is a formula of the language of V, $\varphi(f_1, \ldots, f_n)$ is valid in $V^{\mathcal{U}}$ $\left(V^{\mathcal{U}} \models \varphi(f_1, \ldots, f_n)\right)$ iff $\varphi(f_1, \ldots, f_n)$ is valid a.e., that is if

$$\{s \in S : V \models \varphi(f_1(s), \ldots, f_n(s))\} \in \mathcal{U}.$$

One shows that $V^{\mathcal{U}}$ is *well-founded* if the ultrafilter $\mathcal{U}$ is ω_1-complete and that there exists in that case an isomorphism between $\left\langle V^{\mathcal{U}}, \in_{\mathcal{U}} \right\rangle$ and $\langle M_{\mathcal{U}}, \in \rangle$ where $M_{\mathcal{U}} = \mathrm{Ult}(V,\mathcal{U})$ is an inner model (Mostowski collapsing lemma). A fundamental theorem of Jerzy Łoš (1955, he was a student of Tarski) says that $j : V \prec M_{\mathcal{U}}$ is an elementary embedding.

7.2 Determinacy of projective sets

To prove determinacy results for projective sets beyond Δ^1_1, one must introduce additional axioms and many converging results show that the most natural are large cardinal axioms. The first example was introduced by Stan Ulam.

Definition. *A cardinal $\kappa > \omega$ is* measurable *if it bears a free ultrafilter $\mathcal{U}$ which is κ*-complete *(that is stable w.r.t. κ-infinite intersections $\bigcap\limits_{\lambda<\kappa} X_\lambda$ with $\lambda < \kappa$). It is equivalent to say that κ bears a* measure *μ with range $\{0,1\}$ (with $\mu(\kappa) = 1$), diffuse (without atoms: $\forall \xi \in \kappa$ we have $\mu(\{\xi\}) = 0$) and κ-additive. The equivalence is given by $\mu(A) = 1 \Leftrightarrow A \in \mathcal{U}$ and $\mu(A) = 0 \Leftrightarrow \kappa - A \in \mathcal{U}$.* ☐

A measurable cardinal has "small" and "large" infinite subsets where "small" and "large" are as uncommensurable as finite and infinite in $\mathbb{N}$. A first typical result was another theorem due to Donald Martin:

Martin theorem (1970). $ZFC + MC \vdash \mathrm{Det}\left(\Sigma^1_1\right)$. ☐

This remarkable result – whose proof was "ground breaking" (Kanamory (2011) p.24) – shows that the regularity of projective sets beyond the Borelians Δ^1_1 effectively depends upon *LCAs*.

Corollary: Solovay theorem (1969). $ZFC + MC \vdash$ *"the Σ^1_2-sets are 'regular' ".* ☐

But Solovay also showed that $ZFC + MC \nvdash PD$ (where PD is the axiom of *Projective Determinacy*: every projective $A \subseteq \mathbb{R}$ is determined, see below) since $ZFC+PD \vdash \mathrm{Cons}(ZFC+MC)$ and therefore if $ZFC+MC \vdash PD$ we would have $ZFC + MC \vdash \mathrm{Cons}(ZFC + MC)$, which would contradict Gödel theorem.

A fundamental result on the *intrinsic limitations* of the constructive universe L is the following:

Scott theorem (1961). *MC is false in $V = L$ and therefore $ZFC \nvdash CM$.* $\square$

It is a corollary of the following

Theorem. *If the free ultrafilter $\mathcal{U}$ on the measurable cardinal κ is κ-complete, then* $\mathrm{crit}(j) = \kappa$ *and therefore $j(\kappa) > \kappa$.* $\square$

Indeed, suppose there exists a MC and let κ be the least MC. Now suppose that $V = L$. Elementarity implies $M_{\mathcal{U}} = L$ since $M_{\mathcal{U}}$ is an inner model satisfying the axiom of constructibility and therefore $L \subseteq M_{\mathcal{U}} \subseteq V = L$. Then in $M_{\mathcal{U}} = L = V$, $j(\kappa)$ is the least MC, which contradicts $j(\kappa) > \kappa$. $\square$

As Woodin emphasized (p.520) in his contribution to the 2010 International Congress of Mathematians,

> "(If $V = L$) projective determinacy must fail and moreover there are pathological projective sets."

Measurable cardinals κ are particularly important because (Keisler) they are exactly the *critical* cardinals κ of elementary embeddings $j : V \to M$. We consider the $\mathcal{U} \subset \mathcal{P}(\kappa = \mathrm{crit}(j))$ defined by $X \in \mathcal{U}$ if $\kappa \in j(X)$. Then $\mathcal{U}$ is a free ultrafilter κ-complete on κ and κ is therefore measurable.

They are very large; such a κ is *regular* (there exists no unbounded $f : \lambda \to \kappa$ with $\lambda < \kappa$), *strongly inaccessible* ($\forall\lambda < \kappa$, $2^{\lambda} < \kappa$), and preceded by κ strongly inaccessible smaller cardinals. But as large as they may be, MCs guarantee only the determinacy of the lowest post-Borelian level of definable subsets of $\mathbb{R}$. To guarantee the determinacy of all projective subsets, one needs much stronger axioms, such as PD, which are not entailed by MC (see Solovay's remark above)

As we will see below, many specialists consider that Projective Determinacy is a "good" axiomatic for $\mathbb{R}$. Indeed, PD is "empirically complete" for the projective sets and $ZFC + PD$ "rigidifies" the properties of projective sets w.r.t. forcing: it makes them "forcing-absolute" or "generically absolute".

One can also consider the even stronger axiom proposed by Woodin : "$L(\mathbb{R})$ satisfies AD" where $L(\mathbb{R})$ is the constructible closure of $\mathbb{R}$ (i.e.

the smallest inner model containing *On* and $\mathbb{R}$, see above) and the *Axiom of Determinacy AD* (introduced in 1962 by Jan Mycielski and Hugo Steinhaus means that *every* subset (not necessarily projective) of $\mathbb{R}$ is determined. *AD* is incompatible with *AC* since, according to Fubini theorem, *AC* enables to construct non-Lebesgue measurable, and therefore non-determined, subsets of $\mathbb{R}$.

AD is necertheless compatible with the weak choice axiom that every *countable* set of $A \subset \mathbb{R}$ havs a choice fonction. Solovay and Martin proved that *AD* implies that ω_1 and ω_2 are already measurable and that $\omega_{n\geq 3}$ are singular with cofinality ω_2.

8 The transcendance of $\mathbb{R}$ over L and the set $0^{\#}$ (0 sharp)

8.1 Indiscernible ordinals

Once we accept the relevance and the legitimacy of *LCAs*, we need some tools for measuring the transcendence of V over L. A first possibility is given by what are called *indiscernible* ordinals (Silver, 1966 in his Berkeley thesis) which enable to construct the simplest canonical non-constructible real. We consider the levels of L of the form $\langle L_\lambda, \in \rangle$ with λ a limit ordinal. A set I of ordinals in this cumulative hierarchy L_λ of constructive sets up to level λ is called a set of indiscernibles if, for every n-ary first-order formula $\varphi(x_1, \ldots, x_n)$, the validity of φ on I is independent of the choice of the x_i's: that is for every sequences $c_1 < \ldots < c_n$ and $d_1 < \ldots < d_n$ in I

$$L_\lambda \models \varphi(c_1, \ldots, c_n) \text{ iff } L_\lambda \models \varphi(d_1, \ldots, d_n).$$

This means that the ordinals of I cannot be separated using first-order formulae.

When it exists, the set S of *Silver indiscernibles* is characterized by the following properties, which express that, for all *uncountable* cardinals κ, all the L_κ's share essentially the same first-order structure as $L_{\aleph_1}$:

1. $\kappa \in S$ (all uncountable cardinals of V are indiscernible in L).
2. $S_\kappa = S \cap \kappa$ is of order-type κ and S_κ is closed and unbounded (club, see above) in κ if κ is regular (S_κ is nicely distributed in κ up to its horizon).
3. $S_\kappa = S \cap \kappa$ is a set of indiscernibles for $\langle L_\kappa, \in \rangle$ (the relation between S_κ and L_κ increases nicely up to the limit relation between S and L.

4. S_κ *generates* L_κ in the sense that the *Skolem hull* of $S_\kappa = S \cap \kappa$ in L_κ is equal to L_κ: $\mathrm{Hull}^{L_\kappa}(S_\kappa) = L_\kappa$, where the Skolem hull of $I \subset L_\kappa$ is constructed by adding for every $(n+1)$-ary formula $\varphi(y, x_1, \ldots, x_n)$ with $x_i \in I$ a Skolem term $t_\varphi(x_1, \ldots, x_n)$ which is the smallest y (for the wellorder of L) s.t. $\varphi(y, x_1, \ldots, x_n)$ if such an y exists and 0 otherwise. In other words every constructible element $a \in L_\kappa$ is *definable* by a definite description with indiscernibles parameters in S_κ.

This can be generalized to structures $\mathcal{M} = \langle M, E \rangle$ with a binary relation E looking like $\in$ (that is, which are elementary equivalent to some $\langle L_\lambda, \in \rangle$ for λ a limit ordinal) and with $I \subset M$. In that case, we have $\mathrm{Hull}^{\mathcal{M}}(I) \prec \mathcal{M}$ and in fact $\mathrm{Hull}^{\mathcal{M}}(I)$ is the smallest elementary substructure of $\mathcal{M}$ containing I. Let $\Sigma = \Sigma(\mathcal{M}, I)$ be the set of formulae φ which can be satisfied by $\mathcal{M}$ on I. This defines particular sets of formulae called *EM*-sets (from Ehrenfeucht-Mostowski, 1956). The EM theorem says that if Σ is a theory having infinite models and if $\langle I, < \rangle$ is any total well-ordering of infinite order-type $\alpha \geq \omega$, then there exists a model $\mathcal{M}$ of Σ containing I for which I is a set of indiscernibles, and moreover, $\mathcal{M}$ can be chosen in such a way as to be the *Skolem hull* of I: $\mathcal{M} = \mathrm{Hull}^{\mathcal{M}}(I)$. Such an $(\mathcal{M}, I)$ is essentially unique and its transitive collapse (isomorphism with a structure where E becomes $\in$) is written $(\mathcal{M}(\Sigma, \alpha), I(\Sigma, \alpha))$ where α is the order type of I ($I(\Sigma, \alpha)$ is therefore a set of true $\in$-ordinals).

One can develop a theory of *EM*-sets and of their well-foundedness. If Σ is well-founded (i.e. if $\mathcal{M}(\Sigma, \alpha)$ is well-founded for every ordinal α) and if α is a limit ordinal, then $\mathcal{M}(\Sigma, \alpha)$ is isomorphic to a $\langle L_\lambda, \in \rangle$. Moreover, if $I(\Sigma, \kappa)$, with $\kappa > \omega$ an uncountable cardinal, is unbounded in the class of ordinals of $\mathcal{M}(\Sigma, \alpha)$ (and it is then the case for every ordinal $\alpha > \omega$), and if for every ordinal $\gamma < i_\omega$ (the ω-th element of $I(\Sigma, \kappa)$) we have $\gamma \in \mathrm{Hull}^{\mathcal{M}}(\{i_n\})$ (and it is then the case for every ordinal $\alpha > \omega$), then $\mathcal{M}(\Sigma, \kappa) = \langle L_\kappa, \in \rangle$, $I(\Sigma, \kappa)$ is closed unbounded in κ and if $\tau > \kappa > \omega$ then $I(\Sigma, \tau) \cap \kappa = I(\Sigma, \kappa)$.

Let us return to Silver indiscernibles. If such a Σ exists, S is defined by

$$S = \bigcup \{ I(\Sigma, \kappa) : \kappa \text{ uncountable cardinal} \}.$$

The uniqueness of S is a consequence of the unicity of such a Σ:

Theorem. *Such a Σ is unique and is the set of n-ary sentences φ s.t. $L_{\aleph_\omega} \models \varphi(\aleph_1, \ldots, \aleph_n)$. It is called "zero sharp" and written $0^\#$ (see below).* □

If there exists an uncountable limit cardinal κ s.t. $\langle L_\kappa, \in \rangle$ possesses an uncountable set I of indiscernibles, then S exists. The existence of S

is also implied by large cardinal hypothesis as for instance:

Theorem. *If there exists a MC then S exists and moreover $L_\kappa \prec L_\lambda$ for every uncountable $\kappa < \lambda$.* $\square$

The existence of S under MC means that after the first uncountable level $L_{\aleph_1}$ all the L_κ share essentially the same first-order theory. V transcends L "in a drastic way" (Kanamori, 2011, p.16), but in such a way that it makes L as simple as possible, the first uncountable level $L_{\aleph_1} = \mathrm{Hull}^{L_{\aleph_1}}(S \cap \aleph_1)$ determining the *whole* theory of L.

A deep consequence is that the *truth* in L becomes definable in V. Indeed, let $\varphi(x_1, \ldots, x_n)$ be a formula. There exists an uncountable cardinal κ s.t.

$$\text{for all } (x_i) \in L_\kappa,\ L \models \varphi(x_i) \text{ iff } L_\kappa \models \varphi(x_i) \ .$$

As $L_\kappa \prec L_\lambda$ if $\kappa < \lambda$, we have

$$L \models \varphi(x_i) \text{ iff } L_\lambda \models \varphi(x_i) \text{ for all } \lambda \geq \kappa \ .$$

Now, we arithmetize the situation. Let $T = \{\ulcorner \varphi \urcorner : L_{\aleph_1} \models \varphi\}$ be the set of Gödel numbers of the φ valid in $L_{\aleph_1}$ and therefore in all the L_κ (κ uncountable) by elementarity. Then

$$L \models \varphi \text{ iff } \ulcorner \varphi \urcorner \in T$$

defines the truth in L. This is not in contradiction with Gödel-Tarski uncompleteness theorems since $\aleph_1$ and T are *not definable* in L and therefore the truth of L is not definable in L.

8.2 The set $0^{\#}$

As $L_{\aleph_\omega} \prec L$ and $\aleph_i \in S$ for $i > 0$ in ω, we can represent the indiscernibles in formulae by some of the $\aleph_i$'s and restrict to $L_{\aleph_\omega}$, which contains all the $\aleph_i$'s. Then, $L \models \varphi(x_i)$ for $x_i \in S$ iff $L_{\aleph_\omega} \models \varphi(\aleph_i)$. Solovay called $0^{\#}$ (zero sharp) the set (if it exists) defined by

$$0^{\#} = \{\varphi : L_{\aleph_\omega} \models \varphi(\aleph_i)\}$$

which is the set of formulae true on the indiscernibles of L. Via Gödelization $0^{\#}$ becomes a set of integers (also written $0^{\#}$) and can therefore be coded by a real (also written $0^{\#}$).

We must emphasize the fact that, as $L_{\aleph_1} \prec L$, every constructible set $x \in L$ which is *definable* in L is *countable* since its definite description is valid in $L_{\aleph_1}$ by elementarity and therefore $x \in L_{\aleph_1}$. More generally for every infinite constructible set $x \in L$ we have $|\mathcal{P}(x)|^L = |x|$. Since the existence of a measurable cardinal implies that $0^{\#}$ exists, we have:

Corollary. *If there exists a MC, the constructible continuum $\mathbb{R}^L$ is countable.* □

Via arithmetization through Gödel numbers, the non-constructible set $0^{\#}$ can be considered as a very special subset of $\omega = \mathbb{N}$ which does not belong to L, or as a very special real number coding the truth in L. Its existence implies that every uncountable cardinal κ of V is an indiscernible of L and shares *all* large cardinal axioms verified by L.

A property equivalent to the existence of $0^{\#}$ is the *non-rigidity* of L:

Theorem (Kunen, Stanford, thesis, 1968**).** $0^{\#}$ exists iff there exists a non-trivial elementary embedding $j : L \prec L$ (this presuppose $V \neq L$ and j non-trivial, see below). □

Indeed, as $\mathrm{Hull}^L(S) = L$, for every $x \in L$ there exists a Skolem term t s.t. $x = t\left(i_{\alpha_1}, \ldots, i_{\alpha_n}\right)$, i_α being the α-th element of S. j is then simply defined by *the shift* on indiscernibles

$$j(x) = j\left(t\left(i_{\alpha_1}, \ldots, i_{\alpha_n}\right)\right) = t\left(i_{\alpha_1+1}, \ldots, i_{\alpha_n+1}\right).$$

One shows that it is an elementary embedding and, as $j\left(i_0\right) = i_1 \neq i_0$, j is non-trivial. □

By the way, this proves again that $V \neq L$ since another celebrated theorem of Kunen proves that

Theorem (Kunen, 1971). *$ZFC \vdash$ there exists no $j : V \prec V$.*

The *AC* is essential since without it one can prove the existence of $j : V \prec V$. $\kappa = \mathrm{crit}\,(j)$ is then an extremely large cardinal "beyond choice" called a (William) *Reinhardt* cardinal.

The existence of $0^{\#}$ is a principle of *transcendence* of V over L expressing that V is very different from L. If $0^{\#}$ *doesn't exist*, then V looks like L (L is a good approximation of V) according to the result:

Covering lemma (Jensen). *If $0^{\#}$ doesn't exist, then if x is an uncountable set of ordinals there exists a constructible set $y \supseteq x$ of the same cardinality as x. So, every set x of ordinals can be covered by a constructible set $y \supseteq x$ of cardinality $|y| = |x|.\aleph_1$.* □

Corollary. *If $0^{\#}$ doesn't exist, the covering lemma implies that, for every limit singular cardinal κ of V, we have $\left(\kappa^+\right)^L = \kappa^+$, which shows that V and L are quite similar.* □

Indeed (see Jech [1978], p. 358), if $\lambda = \left(\kappa^+\right)^L$ and if $\lambda < \kappa^+$, then $|\lambda| = \kappa$. But as κ is singular, we would have $\mathrm{cf}\,(\lambda) < |\lambda|$ and this is impossible since λ is regular in L and $\lambda \geq \omega_2$. For, if x is an unbounded subset of λ of cardinal $|x| = \mathrm{cf}\,(\lambda)$, it can be covered by a *constructible* subset $y \in L$ of λ of cardinal $|y| = |x|.\aleph_1$ and, as λ is regular in L, $|y| = |\lambda|$. So, $|\lambda| = \aleph_1 .\,\mathrm{cf}\,(\lambda)$ and, as $\lambda \geq \omega_2$, $|\lambda| = \mathrm{cf}\,(\lambda)$. □

Corollary. *If GCH fails at a strong limit singular cardinal, then $0^{\#}$ exists.* □

Indeed, if $0^{\#}$ doesn't exist, for such a singular cardinal κ we have $(\kappa^{+})^{L} = \kappa^{+}$. As L satisfies GCH , $(2^{\kappa})^{L} = (\kappa^{+})^{L} = \kappa^{+}$.[19] Now, κ is a strong limit by hypothesis (i.e. $\lambda < \kappa \Rightarrow 2^{\lambda} < \kappa$) and this implies $\kappa^{\mathrm{cf}(\kappa)} = 2^{\kappa}$ and moreover, since κ is singular and therefore $\mathrm{cf}\,(\kappa) < |\kappa|$, $2^{\mathrm{cf}(\kappa)} < \kappa$. Let $x \in A = [\kappa]^{\mathrm{cf}(\kappa)}$ be a subset of κ of cardinal $\mathrm{cf}\,(\kappa)$. It is covered by a constructible subset $y \in L$ of κ of cardinal $|y| = \lambda = \aleph_1.\,\mathrm{cf}\,(\kappa)$. Then A can be covered by the union of the $[y]^{\mathrm{cf}(\kappa)}$ for such Y. But $\left|[y]^{\mathrm{cf}(\kappa)}\right| = \lambda^{\mathrm{cf}(\kappa)} = (\aleph_1.\,\mathrm{cf}\,(\kappa))^{\mathrm{cf}(\kappa)} = 2^{\mathrm{cf}(\kappa)}$, and by hypothesis $2^{\mathrm{cf}(\kappa)} < \kappa$. Now, there exist at most $|\mathcal{D}\,(\kappa)|$ such y [20] and $|\mathcal{D}\,(\kappa)| = (\kappa^{+})^{L} = \kappa^{+}$. All this implies $|A| = \kappa^{\mathrm{cf}(\kappa)} = \kappa^{+}$. As $\kappa^{\mathrm{cf}(\kappa)} = 2^{\kappa}$, we have $2^{\kappa} = \kappa^{+}$, that is GCH. $\square$

Jensen's covering Lemma is called a "prodigious progress" by Kanamori (2011, p.30) because it expresses "a global affinity between V and K" and solves the Singular Cardinal Hypothesis (SHC), namely (see above) that if κ is a singular strong limit cardinal (regular strong limit = inaccessible) then $2^{\kappa} = \kappa^{+}$. It implies a *dichotomy* :

1. either if κ is singular then $\kappa^{+} = (\kappa^{+})^{L}$ (L is close to V),
2. or if κ is uncountable it is inaccesible in L (L is far from V).

There are many ways for insuring that $0^{\#}$ exists, for instance PFA since PFA implies the failure of $\square_{\kappa}$ for every κ (Todorcevic, 1980). More generally, many results show that a failure of $\square_{\kappa}$ is linked with $LCAs$. For instance:

- Jensen: if $\square_{\kappa}$ fails for some singular κ, there exists an inner model M with a strong cardinal.
- Solovay: if κ is supercompact, then $\square_{\lambda}$ fails for every $\lambda > \kappa$.

This is related to the fact that properties such as the covering between V and M allow to reflect $\square_{\kappa}$ from M to V and therefore, if $\square_{\kappa}$ fails in V for κ singular it is because there exist some LCs violating the covering lemma.

When $0^{\#}$ exists, a very interesting structure to look at is $L\left[0^{\#}\right]$. It can be shown [21] that $L\left[0^{\#}\right]$ has a fine structure, satisfies the Global Square property $\forall\kappa\ \square_{\kappa}$ and, for every singular cardinal κ of V, $(\kappa^{+})^{L[0^{\#}]} =$

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[19] As, if $0^{\#}$ exists, $(2^{\kappa})^{L} = \kappa$ for every infinite cardinal since κ^{+} is inaccessible in L, $(2^{\kappa})^{L} = \kappa^{+} > \kappa$ is a counter exemple when $0^{\#}$ doesn't exist.

[20] Recall that $\mathcal{D}(x)$ is the set of constructible parts of x (see above).

[21] See e.g. Steel [2001].

κ^+. Iterating the sharp operation, one can get an increasing sequence of models M mildly transcendent over L which have also a fine structure, satisfy the Global Square property and, for every singular cardinal κ of V, $(\kappa^+)^M = \kappa^+$, and are smaller than the first inner model possessing a measurable cardinal. Under the hypothesis that $a^\#$ exists for every set of ordinals a ($a^\#$ is defined as $0^\#$ but in $L[a]$), one can even go beyond this limit, up to the existence of a supercompact cardinal.

8.3 The hierarchical structure of V beyond L.

The equivalence between the existence of $0^\#$ and the existence of a non-trivial elementary embedding $j : L \prec L$ enables to clarify the structure of V beyond L [22]. Let $\kappa = \text{crit}\,(j)$ and let $\mathcal{U}$ be the set of subsets $u \subseteq \kappa$ s.t. $u \in L$ (i.e. $u \in L \cap \mathcal{P}(\kappa)$) and $\kappa \in j(u)$. $\mathcal{U}$ is trivially a filter. It is an ultrafilter since for every $u \in L \cap \mathcal{P}(\kappa)$ either $\kappa \in j(u)$ or $\kappa \notin j(u)$. It is a free ultrafilter since if $u \subset \kappa$ is bounded, then $\kappa \notin u$, $j(u) = u$, $\kappa \notin j(u)$ and therefore $u \notin \mathcal{U}$. Moreover, it is κ-complete w.r.t. L since if $u_\alpha \in \mathcal{U} \cap L$ is a family with $\alpha < \beta < \kappa$ then $\bigcap_{\alpha<\beta} u_\alpha \in \mathcal{U}$. One can show that the ultrapower $L^{\mathcal{U}}$ is well-founded. Due to Łoš theorem, the embedding $i : L \prec L^{\mathcal{U}}$ is elementary. But the Mostowski collapsing lemma implies that $\langle L^{\mathcal{U}}, \in_{\mathcal{U}} \rangle \simeq \langle M_{\mathcal{U}}, \in \rangle$ for some transitive inner model $M_{\mathcal{U}}$. But necessarily $M_{\mathcal{U}} = L$ by minimality of L and via this isomorphism $i : L \prec L^{\mathcal{U}}$ becomes an elementary embedding $j : L \prec L$. It can be shown that, if $\lambda = (\kappa^+)^L$ then $\mathcal{M} = \langle L_\lambda, \in, \mathcal{U} \rangle$ is a model of $ZF - \{\text{Power Set axiom}\}$ where κ becomes the largest cardinal, $\mathcal{U}$ remains a free ultrafilter κ-complete with "good" technical properties ("normality" and "amenability"). Such a procedure can be *iterated* on the ordinals. Starting from a $\mathcal{M}_0 = \langle L_{\lambda_0}, \in, \mathcal{U}_0 \rangle$, one gets a $\mathcal{M}_1 = \langle L_{\lambda_1}, \in, \mathcal{U}_1 \rangle$, etc. The successive $\mathcal{M}_\alpha$ yield a sequence of critical cardinals κ_α which are indiscernibles for L.

9 Martin-Steel-Woodin theorem

9.1 LC and reflection

To measure the size of large cardinals, the best way is to use associated *reflection* phenomena which are of a very deep philosophcal value.[23] Intuitively, reflection means that the properties of the whole universe V are reflected in sub-universes. As was emphasized by Matthew Foreman (1998, p. 6):

> "Any property that holds in the mathematical universe should

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[22] See e.g. Schimmerling [2001].

[23] See Martin-Steel [1989], Patrick Dehornoy [1989].

> hold of many set-approximations of the mathematical universe."

Definition. *A cardinal* κ reflects *a relation* $\Phi(x,y)$ *defined on ordinals if every solution* $y \geq \kappa$ *parametrized by* $x < \kappa$ *can be substituted for by a solution* $y < \kappa$:

$$\forall \alpha (\in On) < \kappa \ \left[\exists \beta \geq \kappa \ \Phi(\alpha,\beta) \Rightarrow \exists \beta^* < \kappa \ \Phi(\alpha,\beta^*) \right].$$

Let j be an elementary embedding $j : M \prec M^*$. $\kappa = \mathrm{crit}\,(j)$ is a large – in fact at least measurable – cardinal, which increases indefinitely when M^* moves near to M, the limit $M^* = M$ being inconsistent according to Kunen theorem.

To see that it is a reflection phenomenon, let $\Phi(\alpha,\kappa)$ be a relation that holds in M for $\alpha < \kappa$. If M^* is sufficiently close to M for $\Phi(\alpha,\kappa)$ to remain true in M^*, then $M^* \models \exists (x < j(\kappa)) \Phi(\alpha, x)$ (it is sufficient to take $x = \kappa$). But, according to the elementarity of the embedding j, this is equivalent to $M \models \exists (x < \kappa) \Phi(\alpha, x)$.

To go beyond measurable cardinals, specialists use the following technique. Let V_α be the cumulative hierarchy of sets up to level α. For κ critical (and therefore measurable), one has $V_\kappa^{M^*} = V_\kappa^M$ (that is the equality of M and M^* up to level κ). It is natural to stengthen this condition.

Definition. *A cardinal* κ *is called* superstrong *in* M *if there exists an elementary embedding* j *s.t.* $V_{j(\kappa)}^{M^*} = V_{j(\kappa)}^M$ *(that is* $V_{j(\kappa)}^{M^*} \subset M$ *and* $M = M^*$ *up to* $j(\kappa)$ *and not only up to* κ *).* □

We will see below that much larger cardinals, called *supercompact*, are the way to Woodin's Ultimate-L models.

But for now let's insist on the fact that between measurable and superstrong cardinals, Hugh Woodin introduced another class of large cardinals.

> "A Woodin cardinal (...) is an important example of concept formation through method" (Kanamori, 2011, p.44)

Definition. *A cardinal* δ *is called a Woodin cardinal if for every map* $F : \delta \to \delta$*, there exists* $\kappa < \delta$ *and an elementary embedding* j *of critical ordinal* κ *s.t.* $F|_\kappa : \kappa \to \kappa$ *and* $V_{j(F(\kappa))}^{M^*} = V_{j(F(\kappa))}^M$ *(that is* $M = M^*$ *up to* $j(F(\kappa))$ *).* □

Woodin has shown that:

1. if δ is a Woodin cardinal, there exist infinitely many smaller measurable cardinals $\chi < \delta$,

2. if λ is a superstrong cardinal, there exist infinetely many smaller Woodin cardinals $\delta < \lambda$.

A key result is the Martin-Steel theorem which evaluates exactly the "cost" of determinacy:

Martin-Steel theorem (1985). *If there exist n Woodin cardinals δ_i, $i = 1, \ldots, n$, dominated by a measurable cardinal κ ($\kappa > \delta_i$ for all i), then $ZFC \vdash \text{Det}\left(\Pi^1_{n+1}\right)$.* $\square$

The converse is due to Woodin.

Corollary (Woodin, 1984). *If there exists a countable infinity of Woodin cardinals dominated by a measurable cardinal, in particular if there exists a superstrong cardinal λ, then* Projective Determinacy *is valid (all the projective subsets of $\mathbb{R}$ are determined). So the* projective CH *is also valid.*[24] $\square$

As explained by Woodin, this fundamental result was "unexpected" and has been the turning point of LCA legitimacy. It shows in a spectacular way that LC existence has a "tremendous influence" on the structure on $\mathbb{R}$. Under good LCA, second order arithmetics becomes "sealed", "rigidified", *generically absolute* and complete in the sens that its properties can no longer be changed by forcing (ut this does not settle CH since CH belongs to *third-order* arithmetics).

It is for this reason that specialists consider that $ZFC+$ Projective Determinacy is a "good" axiomatic for $\mathbb{R}$ in the sense of descriptive set theory.

Some years later, the theorem was deepened by a "remarkable achievement" and "crowning equi-consistancy result" (Kanamory, 2011, p.45) which shows that under this LCA $L(\mathbb{R})$is "sealed", rigidified :

Martin-Steel-Woodin theorem (1987). *If there exists a countable infinity of Woodin cardinals dominated by a measurable cardinal, in particular if there exists a superstrong cardinal λ, then $L(\mathbb{R})$ (the smallest inner model of V containing the ordinals On and $\mathbb{R}$, see above) satisfies the axiom of complete determinacy AD: every $A \subseteq \mathbb{R}$ in $L(\mathbb{R})$ is determined. (This result is stronger than the previous one since $\mathcal{P}(\mathbb{R}) \cap L(\mathbb{R})$ is a larger class than the projective class.)* $\square$

AD is incompatible with AC since AC enables the construction of a non-determined well ordering on $\mathbb{R}$ (see above).[25] So $ZFC + AD$ is inconsistant but there exist ZFC-models of $ZT + AD$.

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[24] Projective Determinacy is also valid under PFA (Woodin, see above).

[25] So, the inner model $L(\mathbb{R})$ of a ZFC–model V can violate AC.

9.2 Woodin's dream

But the most significative results would concern the situation where, beyond $L(\mathbb{R})$, no property of $\mathbb{R}$ could be further modified in a *forcing* extension. In that case, the theory of the continuum would become *rigid* w.r.t. forcing. It is Woodin's dream.

Woodin and Shelah have shown that it is possible to approximate this ideal goal if there exists a *supercompact* cardinal κ. A cardinal κ is γ-supercompact if there exists an elementary embedding $j : V \prec M$ with. $\mathrm{crit}(j) = \kappa$ (so that $\kappa < j(\kappa)$) and $\gamma < j(\kappa)$ s.t. $M^{\gamma} \subseteq M$. "κ is measurable" is equivalent to "κ is κ-supercompact". If κ is 2^{κ}-supercompact, then κ is measurable *in* M since every ultrafilter on κ is in M. κ is supercompact if it is γ-supercompact for every $\gamma \geq \kappa$.[26]

We will now venture into this dream.

10 Woodin's Ω-logic

Up to now, we looked for extensions of ZFC by large cardinal axioms ($LCAs$) which were "good" in the perspective of the descriptive set theory of the continuum. But if $LCAs$ can decide properties of regularity of $\mathbb{R}$, they *cannot* settle CH since a "small" forcing (adding $\aleph_2$ new subsets to ω) is sufficient to force $\neg CH$ from a CH-model and such a small forcing remains possible irrespective of what $LCAs$ are introduced (Levy-Solovay theorem, 1967). We need therefore a new strategy. As we have just seen, the most natural one is to try to make the properties of the continuum *immune* w.r.t. to forcing, that is to make the continuum in some sense *rigid*. The deepest contemporary results in this perspective are provided by Woodin's recent works on Ω-logic and Ultimate-L.

We look for theories sharing *some absoluteness properties relatively to forcing*. This special kind of relative absoluteness is called " *conditional generic absoluteness*".[27]

The fragment of V where CH "lives" naturally is $(H_2, \in)$ where $(H_k, \in)$ is the set of sets x which are hereditary of cardinal $|x| < \aleph_k$. The fragment $(H_0, \in) = V_{\omega}$ is the set of hereditary finite sets and, with the axioms ZF minus the axiom of infinity, is equivalent to first order arithmetic $\langle \omega = \mathbb{N}, +, ., \in \rangle$ with Peano axioms. In one direction, $\mathbb{N}$ can be retrieved from H_0 using von Neumann's construction of ordinals and, conversely, H_0 can be retrieved from $\mathbb{N}$ via Ackermann's trick: if p, q are integers, $p \in q$ iff the p-th digit in the binary extension of q is 1. For first order arithmetic, Peano axioms are "empirically" and practically complete in spite of Gödel incompleteness theorem. The following classical

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[26] See Dehornoy [2003].

[27] See Steel [2004]: "Generic absoluteness and the continuum problem".

result expresses their rigidity:

Schönfield theorem. *H_0 is absolute and a fortiori forcing-invariant. Incompleteness cannot be manifested in it using forcing.* □

We can therefore consider ZFC as a "good" *canonical* theory for first order arithmetic. But it is no longer the case for larger fragments of V.

The fragment $(H_1, \in)$ of V composed of countable sets of finite ordinals is isomorphic to $\langle \mathcal{P}(\omega) = \mathbb{R}, \omega, +, ., \in \rangle$ and corresponds to second order arithmetic (i.e. analysis). The definable subsets $A \subseteq \mathcal{P}(\omega)$ are the projective subsets[28] and therefore H_1 can be considered as the fragment of V where the projective sets live. We have seen that to settle and "freeze" or "seal" most of its higher order properties (regularity of projective sets) w.r.t. forcing, we *need LCAs* and in particular projective determinacy PD. Without such appropriate LCA second order arithmetics (analysis) *would not be* forcing-stable and generically absolute.

As is emphasized by Woodin (2003, quoted in Dehornoy [2003]):

> "*Projective Determinacy* settles (in the context of ZFC) the classical questions concerning the projective sets and moreover Cohen's method of forcing *cannot* be used to establish that questions of second order number theory are formally unsolvable from this axiom. (...) I believe the axiom of *Projective Determinacy* is as true as the axioms of Number Theory. So I suppose that I advocate a position that might best be described as *Conditional Platonism*."

The "good" LCA is "there exists a proper class of Woodin cardinals" (PCW) : for every cardinal κ there exists a Woodin cardinal $> \kappa$). Then, we have :

Theorem (Woodin, 1984). *$ZFC + PCW \vdash H_1$ is immune relatively to forcing in the sense its properties are forcing-invariant.* □

As PCW implies at the same time PD and forcing-invariance for H_1, it can be considered as a "good" *canonical* theory, "empirically" and "practically" complete (marginalizing incompleteness) for $(H_1, \in)$, that is for analysis (second order arithmetic). PCW implies the *generic completeness* result that all the $L[\mathbb{R}]$ of generic extensions $V[G]$ are elementary equivalent.

The idea is then to try to generalize properties of absoluteness relative to forcing. The general strategy for deciding that way ZFC-undecidable

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[28] A set $A \subseteq \mathcal{P}(\omega)$ is definable in H_1 (with parameters in H_1) iff there exists a first-order formula $\varphi(x, y)$ and a parameter $b \in \mathcal{P}(\omega)$ s.t. $A = \{a \in \mathcal{P}(\omega) \,|H_1 \models \varphi(a, b)\}$. If $\pi : \mathcal{P}(\omega) \to [0, 1] \simeq \mathbb{R}$ is given by $\pi(a) = \sum_{i \in a} \frac{1}{2^i}$, then $X \subseteq [0, 1]$ is projective iff $A = \pi^{-1}(X)$ is definable in H_1.

properties φ in a fragment H of V is described in the following way by Patrick Dehornoy (2003):

> "every axiomatization freezing the properties of H relatively to forcing (i.e. neutralizing forcing at the level H) implies φ".

The main problem tackled by Woodin was to apply this strategy to the fragment $(H_2, \in)$ of V which is associated to the set $\mathcal{P}(\omega_1)$. $\mathcal{P}(\omega_1)$ is not $\mathcal{P}(\mathbb{R})$ if $\neg CH$ is satisfied, but nevertheless it is possible to code CH by an H_2-formula φ_{CH} s.t. $H_2 \models \varphi_{CH}$ is equivalent to CH.[29]

The problem with H_2 is that, as we have already seen (but let's say it again), "small" forcings preserve LCA [30] and in particular (Levy-Solovay theorem) a small forcing of cardinal $\aleph_2$ that enables to violate CH by adding $\aleph_2$ subsets to $\mathbb{N}$ preserves $LCAs$. Therefore H_2 *cannot* be rigidified by $LCAs$.[31] Whatever the large cardinal hypothesis A may be, there will be always generic extensions M and N of V both satisfying A such that $M \models CH$ and $N \models \neg CH$. As CH is equivalent to a Σ_2^1 formula, M and N cannot be elementary equivalent after the Σ_2^1 level.

Woodin's first fundamental idea to overcome the dramatic difficulties of the problem at the H_2 level was to *strengthen logic* by restricting the admissible models and constructing a new logic adapted to forcing-invariance or "*generic invariance*". As he explains in his key paper on "The continuum hypothesis" (2001, p. 682):

> "As a consequence (of generic invariance), any axioms we find will yield theories for $\langle H[\omega_2], \in \rangle$, whose 'completeness' is immune to attack by applications of Cohen's method of forcing, just as it is the case for number theory."

In a first step, he introduced the notion of Ω-validity $\models_\Omega$ also called at the beginning Ω^*-derivability $\vdash_{\Omega^*}$.

Definition. *T being a theory in ZFC, we have $T \models_\Omega \varphi$ iff φ is valid in every generic extension where T is valid, that is iff for every generic extension $V[G]$ and every level α, $(V_\alpha)^{V[G]} \models T$ implies $(V_\alpha)^{V[G]} \models \varphi$.*$\square$

Of course $\models$ implies $\models_\Omega$. But the converse is trivially false: there exists Ω-valid formulae which are undecidable in ZFC, for instance

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[29] The point is rather technical. Woodin has shown that if $\neg CH$ is valid (i.e. $\mathbb{R} > \omega_1$), then $\mathcal{P}(\mathbb{R})$ doesn't belong to H_2 and is already too big for freezing (neutralizing the effects of forcing) the fragments of V containing it.

[30] Large cardinal axioms are axioms of the form $A = \exists\kappa\psi(\kappa)$ which share the property that if $V \models A$ then the cardinal κ is inaccessible and $\psi(\kappa)$ is forcing-invariant for every forcing extension $V[G]$ of forcing cardinal $< \kappa$ ("small" forcings).

[31] See Dehornoy [2003].

Con (ZFC). Indeed, if $(V_\alpha)^{V[G]} \models ZFC$ then $(V_\alpha)^{V[G]}$ is a model of ZFC and $(V_\alpha)^{V[G]} \models \mathrm{Con}\,(ZFC)$ So, $ZFC \models_\Omega \mathrm{Con}\,(ZFC)$, but of course (Gödel) $ZFC \nvDash \mathrm{Con}\,(ZFC)$.

It must be emphasized that Ω-validity *doesn't* satisfy the *compactness* property: there exist theories T and formulae φ s.t. we have $T \models_\Omega \varphi$ even if for every finite subset $S \subset T$ we have $S \nvDash_\Omega \varphi$.[32]

By construction, Ω-validity $\models_\Omega$ is itself forcing-invariant:[33]

Theorem ($ZFC + PCW$). *If* $V \models$ "$T \models_\Omega \varphi$" *then* $V[G] \models$ "$T \models_\Omega \varphi$" *for every generic extension of* V. ☐

Woodin investigated deeply this new "strong logic". In particular he was able to show that, under suitable *LCAs*, *CH* rigidifies V at the Σ_1^2-level (Σ_1 formulae for $V_{\omega+2}$):

Theorem (Woodin, 1984). *Under* PCW_{meas} *(there exists a proper class of measurable Woodin cardinals) and* CH, *Ω-logic is generically complete at the* Σ_1^2*-level: for every* φ *of complexity* Σ_1^2 *either* $ZFC + CH \models_\Omega \varphi$ *or* $ZFC + CH \models_\Omega \neg\varphi$. *All generic extensions* M *and* N *of* V *satisfying both* CH *are* Σ_1^2 *elementary equivalent.* ☐

The metamathematical meaning of this result of *conditional generic absoluteness* is that if a problem is expressed by a Σ_1^2-formula φ then it is "settled by *CH*" and immunized against forcing under appropriate *LCAs*. But:

Theorem (Abraham, Shelah). *This is false at the* Σ_2^2 *level. For every large cardinal hypothesis* A *there exist generic extensions* M *and* N *satisfying both* CH *s.t. in* M *there exist a* Σ_2^2*-wellorder of* $\mathbb{R}$ *while in* N *all the* Σ_2^2*-subsets of* $\mathbb{R}$ *are Lebesgue measurable.* ☐

In a second step, Woodin interpreted the Ω-validity $T \models_\Omega \varphi$ as the semantic validity for an Ω-logic whose *syntactic* derivation $T \vdash_\Omega \varphi$ had to be defined. His idea was to witness the Ω-proofs by particular sets that, under *PCW*, generalize the projective sets and can be interpreted without ambiguity in every generic extension.

It is the most difficult part of his work, not only at the technical level but also at the philosophical level since proofs are defined without proof rules and a proof calculus. The idea is to generalize the fact, that *via Gödelisation*, a classical proof is *coded* by a finite subset of $\mathbb{N}$. Ω-syntax allows as "proofs" infinite subsets of $\mathbb{N}$, that is real numbers $x \in \mathbb{R}$ and accept special subsets $A \subseteq \mathbb{R}$ of such infinitary proofs. The definition (under *PCW*) is the following:

Definition (PCW). $T \vdash_\Omega \varphi$ *iff there exists* a universally Baire *(UB) set* $A \subseteq \mathbb{R}$ *s.t. for every* A*-closed countable transitive model (ctm)* M *of* T *we have* $M \models \varphi$ *(in other words* $M \models$ "$T \models_\Omega \varphi$"*).* ☐

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[32] See Bagaria et al. [2005].

[33] See Woodin [2004].

A subset $A \subseteq \mathbb{R}$ of $\mathbb{R}$ is called UB if for every continuous map $f : K \rightarrow \mathbb{R}$ with source K compact Hausdorff, $f^{-1}(A)$ has the Baire property (there exists an open set U s.t. the symetric difference $f^{-1}(A) \Delta U$ is meager). If $A \subseteq \mathbb{R}$ is UB, it is interpreted canonically in every generic extension $V[G]$ as $A_G \subseteq \mathbb{R}^{V[G]}$. This is due to the fact that there exists a *tree presentation* of A. One identifies $\mathbb{R}$ with ω^ω and one considers trees $T \subset (\omega \times \gamma)^\omega$ and the projections $p[T]$ on ω^ω of their infinite branches:

$$p[T] = \{x \in \omega^\omega \ \mid \exists z \in \gamma^\omega \text{ with } (x \mid_n, z \mid_n) \in T, \forall n \in \omega\}.$$

A subset $A \subseteq \mathbb{R}$ is UB iff there exist trees T and S s.t. $p[T] = A$ and $p[S] = \omega^\omega - A$ in every generic extension $V[G]$. $p[T]$ yields a canonical interpretation of A in every generic extension $V[G]$. A ctm M is called A-closed if, for every ctm $N \supseteq M$, $A \cap N \in N$, in particular for every generic extension $V[G]$ and $N = M[G]$ we have $A \cap M[G] \in M[G]$. If A is Borelian, every ctm is always A-closed.[34] But it is no longer the case for general UB sets.

As far as, in the definition of $T \vdash_\Omega \varphi$, the class of admissible models is *restricted* to A-closed ctms, logic becomes strengthened. Of course, $T \vdash \varphi$ implies $T \vdash_\Omega \varphi$, but the converse is false for the same reasons as for $\vDash_\Omega$. Indeed, $ZFC \vdash_\Omega \text{Con}(ZFC)$ because every suitable ctm provides a model of ZFC and validates therefore $\text{Con}(ZFC)$.

More technically, what is really needed for the definition $T \vdash_\Omega \varphi$ are UB sets $A \subseteq \mathbb{R}$ sharing the following two properties:

1. $L(A, \mathbb{R}) \vDash AD^+$, where AD^+ is a strengthening of the axiom of determinacy saying that not only all $A \subseteq \mathbb{R} \simeq \omega^\omega$ are determined, but also all the $\pi^{-1}(A)$ for all maps $\pi : \lambda^\omega \rightarrow \omega^\omega$ with an ordinal $\lambda < \mathfrak{c}^+$;

2. every $A \subset \mathcal{P}(\mathbb{R}) \cap L(A, \mathbb{R})$ is UB.[35]

PCW implies these two properties and is therefore a good hypothesis.

It must be emphasized that this definition of Ω-provability is very original. As explain Joan Bagaria, Neus Castells and Paul Larson in their "Ω-logic primer":

> "The notion of Ω-provability differs from the usual notions of provability, e.g., in first-order logic, in that there is no deductive calculus involved. In Ω-logic, the same UB set may witness the Ω-provability of different sentences. For instance,

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[34] If M is A-closed for every A of Π^1_1-complexity, then M is well-founded.

[35] See Woodin [2000].

> all tautologies have the same *proof* in Ω-logic, namely $\emptyset$. In spite of this, it is possible to define a notion of height of proof in Ω-logic."

As Patrick Dehornoy explained to me (private communication), in Ω-logic a proof $\vdash_{\Omega} \varphi$ is a certificate of some property of the formula φ. This witnessing is no longer a derivation iterating syntactic rules but a UB subset of $\mathbb{R}$. What is common to classical and Ω-logics is that a very "small" object endowed with a precise internal structure warrants the validity of φ in a lot of immensely large models.

Woodin proved that Ω-logic is *sound*: if $T \vdash_{\Omega} \varphi$ then $T \models_{\Omega} \varphi$, i.e. (under PCW) if $\vdash_{\Omega} \varphi$ then $\models \varphi$ in all ZFC-models $(V_{\alpha})^{V[G]}$. He then formulated the main conjecture:

Ω-conjecture (1999). *Ω-logic is complete: if $\models_{\Omega} \varphi$ then $\vdash_{\Omega} \varphi$.* □

As he emphasized in Woodin (2002, p. 517):

> "If the Ω-conjecture is true, then generic absoluteness is equivalent to absoluteness in Ω-logic and this in turn has significant metamathematical implications".

Indeed (Dehornoy, 2007), the Ω-conjecture means that any formula φ valid in a lot of immensely large models satisfying $LCAs$ are certified by UB subsets of $\mathbb{R}$. The key fact proved by Woodin is the link of the concept of Ω-provability with the existence of *canonical models* for $LCAs$ (that is models which are in a certain way minimal and universal, as L for $ZFC + CH$). The Ω-conjecture expresses essentially the hypothesis that every LCA admits a canonical model.

Theorem. *A sentence φ is Ω-provable ($\vdash_{\Omega} \varphi$) iff $ZFC + A \vdash \varphi$ for some large cardinal axioms A admitting a canonical model.* □

Now, the key point is that when H_2 is rigidified, CH becomes automatically *false*.

> "If the theory of the structure $\langle \mathcal{P}(\omega_1), \omega_1, +, ., \in \rangle$ is to be resolved on the basis of a good axiom then necessarily CH is false."

The idea is that if the theory T of $\mathcal{P}(\omega_1)$ is completely unambiguous in the sense that there exists an axiom A s.t. $T \models \varphi$ iff $A \models_{\Omega}$ "$T \models \varphi$", then CH is necessarily false since the theory of $\mathcal{P}(\mathbb{R})$ cannot share this property.

Woodin theorem (2000, **under** PCW)**.** *(i) For every "solution" for H_2 (that is axioms freezing the properties of H_2 w.r.t. forcing) based on an Ω-complete axiom A (i.e. for every $\varphi \in H_2$, either $ZFC + A \vdash_{\Omega}$*

“$(H_2, \in) \models \varphi$” or $ZFC + A \vdash_\Omega$ “$(H_2, \in) \models \neg\varphi$”), CH is false. (ii) If the Ω-conjecture is valid, every “solution” for H_2 is based on an Ω-complete axiom and therefore CH is false. ☐

The proof uses Tarski results on the impossibility of defining truth and is quite interesting (Woodin 2001, p. 688). Let

$$\Gamma = \{\ulcorner\varphi\urcorner : ZFC + A \vdash_\Omega \text{“}(H_2, \in) \models \varphi\text{”}\}$$

be the (extremely complicated) set of Gödel numbers of the sentences Ω-valid in H_2. By hypothesis, Γ is Ω-recursive in the sense there exists a UB set B s.t. Γ is definable and recursive in $L(B, \mathbb{R})$. Now, PCW implies that Γ being Ω-recursive, it is definable in $(H(\mathfrak{c}^+), \in)$. If CH would be valid, then $\mathfrak{c} = \omega_1$, $H(\mathfrak{c}^+) = H_2$ and Γ would be definable in H_2, which would violate Tarski theorem.

It is in that sense Woodin (2001, p. 690) can claim:

> “Thus, I now believe the Continuum Hypothesis is solvable, which is a fundamental change in my view of set theory”.

11 Ultimate-L and the inner model programme

11.1 The ideal idea

An interesting strategy – called the *inner model program* – is to ask to what extent one can enlarge the universe of constructible sets L into L-like models $L[E]$ which accomodate a LC κ of a certain type. The idea is that the constructible strategy is “good” but not compatible with the LCs needed to have good regularity properties of the continuum. The ideal would therefore be to preserve the “good” structure of L while eliminating its drawbacks. As emphasized by Venturi and Viale (p.34)

> “(The) inner model program (...) tries to build canonical models of set theory that display similarities with L, but that nonetheless are compatible with all known large cardinals.”

Since the 1970s, Powell, Mitchell, Dodd, Jensen, Steel proved a lot of results using the technique of *extenders*. Intuitively, extenders are systems of compatible ultrafilters enabling to “extend” “small” ultrafilters to “large” ultrafilters in such a way that the corresponding ultrapowers fit. For instance if $j : V \to M$ is an elementary embedding with $\text{crit}(j) = \kappa$, and if we have an ultrafilter whose ultrapower gives an elementary embedding $i : V \to N$, we can re-embed N into M using j, which gives an elementary embedding $h : N \to M$ s.t.

$h \circ i = j$ ($V \xrightarrow{i} N \xrightarrow{h} M$). If h is $\neq Id$ then we can iterate the construction. $j(\kappa) > \kappa$ can be very much greater than $i(\kappa)$. (i, N) is a $(\kappa, \kappa+1)$-extender with is a sort of first order approximation of j. The Mitchell-Steel (κ, λ)-extenders are better approximations of j defined for $\kappa \leq \lambda \leq j(\kappa)$.

Using such techniques, Mitchell and Steel constructed inner L-like models $L[E]$ admitting LC *below* supercompact cardinals, witnessing that supercompactness is a turning point in the hierarchy of LC. Indeed Woodin proved that *no* Mitchell-Steel extender inner model $L[E]$ can accommodate a supercompact cardinal.

As emphasized by Woodin (ICM, 2010, p.505)

> "This program has been very successful, producing some of the most fundamental insights we currently have into the Universe of Sets."

The problem is that there is in general a theorem à la Scott (if $V = L$ there is no measurable cardinal) asserting that there cannot be in such a model $L[E]$ a large cardinal of a stronger type than κ. This seems to block the strategy.

> "Since it seems very unlikely that there could ever be a strongest large cardinal axiom, this methodology seems unable by its very nature to ever succeed in providing the requisite axiom for clarifying the conception of the Universe of Sets."

But around 2010 Woodin discovered a fundamental property of supercompact cardinals anabling to construct inner models $L[E]$ *escaping* the limit imposed by a L-like Scott's theorem

> "The situation has now (2010) changed dramatically and there is for the first time a genuine prospect for the construction of an ultimate enlargement of L. This arises not from the identification of a strongest large cardinal axiom but from the unexpected discovery that at a specific critical stage in the hierarchy of large cardinal axioms, the construction of an enlargement of L compatible with this large cardinal axiom must yield the ultimate enlargement of L. More precisely this construction must yield an enlargement which is compatible with all stronger large cardinal axioms." (p.505)

It is this "remarkable and completely surprising result" which I would now like to comment on. It enables to construct L-like inner models very close to V.

11.2 $HOD^{L(A,\mathbb{R})}$

To explore L-like inner models accommodating supercompact cardinals, Woodin studies the HOD hierarchy of V. Let's remember that HOD is the class of sets hereditary definable by (not necessarily first order) formulas with ordinal parameters. $X = \{y : \varphi(z, a_i), \text{with } a_i \in On\}$ (first order formulas would yield L).

Contrary to L which is constructed strictly bottom-up, HOD is global and the complexity of V is reflected in it. It is the greatest ZFC-inner model containing On and which can be (as L) *globally well-ordered*. But HOD is not canonical, not generically absolute. Every set X in V can become a set in HOD by forcing.

Under an appropriate LCA stronger than supercompactness there exists a *dichotomy* : either HOD is close to V, or HOD is far from V. A cardinal δ is *extendible* if for every $\alpha > \delta$ there exist a $\beta > \alpha$ and an elementary embedding $j : V_\alpha \to V_\beta$ with $\operatorname{crit}(j) = \delta$ s.t. $j(\delta) > \alpha$ is as far as you want (by construction $j(\delta) > \delta$).

Theorem (Woodin). If there exists an extendible cardinal δ, then

1. either every κ singular $> \delta$ is singular in HOD and $(\kappa^+)^{HOD} = \kappa^+$ (HOD is close to V),
2. or every κ regular $> \delta$ is measurable in HOD (HOD is far from V).

The key result is that this HOD dichotomy is *generically absolute*, i.e. if (1) or (2) is valid in V then it is valid in all generic extensions $V[G]$. Hence the relevance of the following conjecture :

Woodin's Conjecture. The case (1) is true.

To go further, Woodin investigated the structure of HOD in $L(\mathbb{R})$ and proved that under the PWC hypothesis $HOD^{L(\mathbb{R})}$ *cannot* be a Mitchell-Steel extender model and yields therefore a completely new type, "previously unknown", of inner model. He called it a "*srategic* extender model" :

> "I shoud emphasize that prior to the proof of Theorem 37, it was not known if strategic extender models could exist in any reasonable way." (ICM, p.525)

Then he generalized to $HOD^{L(A,\mathbb{R})}$ for A a universally Baire (UB) set. Here $L(A, \mathbb{R})$ is the transitive closure of $On \cup \mathbb{R} \cup \{A\}$. And finally he introduced the new axiom :

$V =$ **Ultimate-L axiom.** PCW and if φ is a formula valid in an initial segment of the cumulative hierarchy V, then there exists a UB set A s.t. φ is valid in an initial segment of $HOD^{L(A,\mathbb{R})}$.

In his 2017 “Midrasha Mathematicae Lectures” Woodin explains (p.2) :

> “This axiom strongly couples the width of the universe of sets to its height since in the context of the axiom $V =$ Ultimate-L, one cannot change the width using Cohen’s method of forcing without then changing the height. In particular, the axiom $V =$ Ultimate-L renders Cohen’s method of forcing completely useless as a method for establishing independence from the resulting conception of the universe of sets.”

As Venturi and Viale explained (p.34)

> “Under $V =$ Ultimate-L, Woodin offered a complete and detailed picture of the structure of the universe of all sets, much alike the picture one gets by analyzing the constructible universe L.”

The axiom implies $V = HOD$ and CH. It implies also the Ω-conjecture.

12 Conclusion

> Other approaches to the continuum problem in the set theoretical framework of $LCAs$ have been proposed. One of the most interesting alternative is provided by Matthew Foreman’s (2003) concept of *generic large cardinal* (GLC) defined by elementary embeddings $j : V \prec M$ of V in inner models M not of V itself but of generic extensions $V[G]$ of V . Such generic $LCAs$ can support rather CH than $\neg CH$.

All these results show what are the difficulties met in elaborating a “good” set theoretical determination of the continuum. They justify some sort of Gödel’s platonism introducing additional axioms as some kind of “physical hypotheses”, as if the realm of universes of sets has to be explored as a kind of “objective” and “empirical” reality. The nominalist antiplatonist philosophy of mathematics criticizing such axioms (in particular $LCAs$) as ontological naive beliefs is itself naive because it interprets platonism as the “ontological” thesis that *the* universe of sets must be unique and ZFC completely determined. It must be reconsidered and substituted by a *“conditional” platonism* in Woodin’s sense, a platonism which would be “conditional” to axioms which rigidify, freeze, seal the continuum and make its properties forcing-invariant.

Woodin concluded his 2010 ICM talk (p.526) saying :

> "This axiom (...) will arguably reduce all questions of Set Theory to axioms of strong infinity and so banish the specter of undecidability as demonstrated by Cohen's method of forcing."

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In my 1991, 1992 and 1995 papers on the continuum problem, I introduced the concept of "*transcendental platonism*". Classical platonism is a naive realist thesis concerning the ontological independence of mathematical idealities, and as such is always dialectically opposed to anti-platonist nominalism. Even to day, the debates concerning the status of mathematical idealities remain trapped into the realist/nominalist dialectic.[36]

The metaphysical antiplatonism denying any ontological content to ideal and abstract objects and structures is used to justify a "deflationist" ontological commitment concerning the axioms of existence which are admissible in set theory. As if constructivism was ontologically more secure! In the above cited papers, I have argued that

1. the nominalist thesis has no ontological relevance for mathematics;
2. it can't therefore be used as an argument against strongly non-constructive axioms of existence;
3. the problem of mathematical platonism does not concern ontological realism but *mathematical objectivity*, which has nothing to do with ontology: objectivity is the possibility of constituting and determining, using law-like procedures, some unknown properties (and not of discovering them as a preexisting reality);
4. for solving the problem, we need a philosophy of objectivity based on the difference between objectivity and ontology;
5. such a philosophy is afforded by transcendental philosophy;
6. in the transcendental perspective strong axioms of existence can be justified as far as they manifest a strong power of determination.

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[36] See e.g. Maddy [2005] on "naïve realism", "robust realism", "thin realism", etc.

It is for all these reasons that I coined the term of "transcendental platonism".

The main achievement of transcendentalism has been to overcome the scholastic antinomy between realism and nominalism and to show that mathematical and physical objectivity were neither ontological nor subjective. Objectivity is always *constituted* and therefore *conditional*, relative to eidetico-constitutive rules. A platonism defined in terms of objectivity and not ontology, is a transcendental platonism immune to the classical aporias of metaphysical transcendent platonism. As far as the question of the continuum is concerned, the eidetico-constitutive rules are the axioms of set theory and transcendental platonism means that the continuum problem can have a well determined solution in a rigid universe where $\mathbb{R}$ become *conditionally* generically absolute. I think that Woodin's conditional platonism can therefore be considered as a transcendental platonism relative to the continuum problem.

13 Two philosophies

We will meet in the sequel an alternative opposing two different types of philosophies.

1. Philosophies of the first type are "minimizing", "ontologically" deflationist (in the sense of restricting what axioms of existence are admissible), nominalist, and constructive. They consider that the only meaningful content of the continuum is the part which can be "conceptually" (that is symbolically) well determined and that the rest is "inherently vague". A celebrated representative of such a perspective is Solomon Feferman who considers that the continuum cannot be a definite mathematical object since some of its properties, such as Cantor's continuum hypothesis, are not expressible by definite propositions. But, as explained by John Steel (2004) in a criticism of Feferman:

   > "Taken seriously, this analysis leads us into a retreat to some much weaker constructivist language, a retreat which would toss out good mathematics in order to save inherently vague philosophy."

2. It is why, philosophies of the second type are, on the contrary, "maximizing", "ontologically" inflationist, platonist in a sophisticated sense, and highly non-constructive. They aim at modelling *inside* a ZFC-universe the transcendence of the intuitive continuum w.r.t. its logical symbolic mastery. Owing to this, they must introduce non-constructive axioms for higher infinite.

References

[1] Bagaria, J., 2021. "Large cardinals as principles of structural reflection", arXiv:2107.0158v1, 2021.

[2] Bagaria, J., Castells, N., Larson, P., 2005. "An Ω -logic Primer", *Set Theory*, CRM 2003-2004, Birkhauser (2006), 1-28.

[3] Becker, H., 1992. "Descriptive Set Theoretic Phenomena in Analysis and Topology", *STC [1992]*, 1-25.

[4] Bell, J., 2005. *The Continuous and the Infinitesimal in Mathematics and Philosophy*, Polimetrica, http://www.polimetrica.com/categories/02cat.html.

[5] Bellotti, L., 2005. "Woodin on the Continuum Problem: an overview and some objections", *Logic and Philosophy of Science*, III, 1.

[6] Benacerraf, P., Putnam, H. (eds.), 1964. *Philosophy of Mathematics: Selected Readings*, Prentice Hall, Englewood Ciffs, New-Jersey.

[7] Breysse, O., De Glas, M., 2007. "A New Approach to the Concepts of Boundary and Contact: Towards an Alternative to Mereotopology", *Fundamenta Informaticae*, 78, 2, 217-238.

[8] Chihara, Ch., S., 1990. *Constructibility and Mathematical Existence*, Clarendon Press, Oxford.

[9] Cohen, P., 1963-1964. "The independence of the Continuum Hypothesis", I, *PNAS*, 50 (1963) 1143-1148 and II, *PNAS*, 51 (1964) 105-110.

[10] Cohen, P. 1966. *Set Theory and the Continuum Hypothesis*, Benjamin, New York.

[11] Dehornoy, P., 1989. "La détermination projective d'après Martin, Steel et Woodin", *Séminaire Bourbaki*, #710.

[12] Dehornoy, P., 2003. "Progrès récents sur l'hypothèse du continu (d'après Woodin)", *Séminaire Bourbaki*, #915.

[13] Dehornoy, P., 2007. "Au-delà du forcing: la notion de vérité essentielle en théorie des ensembles", *Logique, dynamique et cognition*, (J.B. Joinet, ed.), Publications de la Sorbonne, Paris, 147–169.

[14] Easton, W. B., 1970. "Powers of regular cardinals", *AML*, 1 (1970), 139-178 (abridged version of the 1964 Ph.D. thesis).

[15] Farah, I., Larson, P. B., 2005. "Absoluteness for Universally Baire

Sets and the Uncountable, I", *Quaderni di Matematica*, 17 (2006), 47-92.

[16] Feferman S., 1989. "Infinity in Mathematics: Is Cantor Necessary?", *Philosophical Topics*, XVII, 2, 23-45.

[17] Feng, Q., Magidor, M., Woodin, W. H., 1992. "Universally Baire Sets of Reals", *STC 1992*, 203-242.

[18] Foreman, M., Magidor, M., Shelah, S., 1986. "$0^{\#}$ and Some Forcing Principles", *Journal of Symbolic Logic*, 51 (1986), 39-47.

[19] Foreman, M., 1998. "Generic large cardinals: new axioms for mathematics?", *ICM*, Vol. II, Berlin.

[20] Foreman, M., 2003. "Has the Continuum Hypothesis been settled?", http://www.math.helsinki.fi/logic/LC2003/presentations.

[21] Foreman, M., Magidor, M., 1997. "A Very Weak Square Principle, *Journal of Symbolic Logic*, 62, 1 (1997), 175-196.

[22] Gambetta, E., 2016. *Structural Reflection and the Ultimate L as the true noumenal universe of mathematics*, Thesis, Scuola Normale Superiore di Pisa.

[23] Gödel, K., 1938. "The consistency of the axiom of choice and the generalized continuum hypothesis", *PNAS*, 25 (1938), 556-557.

[24] Gödel, K., 1940. "The consistency of the axiom of choice and of the generalized continuum hypothesis with the axioms of set theory", *Annals of Math. Studies*, Princeton University Press, Princeton.

[25] Gödel K., 1947. "What is Cantor's Continuum Problem", *American Mathematical Monthly*, 54 (1947), 515-545 (reprinted in *Benacerraf-Putnam [1964]*, 470-485).

[26] Grigorieff, S., 1976. "Détermination des jeux boréliens d'après Martin", *Séminaire Bourbaki*, #478.

[27] Husserl, E., 1900-1901. *Logische Untersuchungen*, Max Niemeyer, Halle (1913).

[28] Jackson, S., 1989. "AD and the very fine structure of $L(\mathbb{R})$", *Bulletin of the American Mathematical Society*, 21, 1 (1989), 77-81.

[29] Jech, T., 1978. *Set Theory*, Academic Press, San Diego.

[30] Kanamori, A., 1995. *The Higher Infinite*, Perspectives in Mathematical Logic, Springer-Verlag, Berlin-Heidelberg-New York.

[31] Kanamori, A., 2011. "Large Cardinals with Forcing", *Handbook of the History of Logic: Sets and Extensions in the Twentieth Century* (Akihiro Kanamori, ed.), Elsevier, 2011.

[32] Kant, I., *Kants gesammelte Schriften*, Preussische Akademie der Wissenschaften, Georg Reimer, Berlin, 1911.

[33] Koellner, P., Woodin, W.H., 2010. "Large Cardinals from Determinacy", *Handbook of Set Theory*, (Foreman, M., Kanamori, A., eds), Springer, Dordrecht.

[34] Maddy, P., 1988. "Believing the Axioms I, II", *The Journal of Symbolic Logic*, 53, 2 (1988), 481-511 ; 53, 3 (1988), 736-764.

[35] Maddy, P., 2005. "Mathematical existence", *The Bulletin of Symbolic Logic*, 11, 3 (2005), 351-376.

[36] Martin, Donald, 1975. "Borel Determinacy", *Annals of Mathematics*, 102 (1975), 363-371.

[37] Martin, David, 2001. *Characterizations of* $0^{\#}$, Master Thesis, Carnegie Mellon University.

[38] Martin, D., Steel, J., 1989. "A Proof of Projective Determinacy", *Journal of the American Mathematical Society*, 2, 1 (1989), 71-125.

[39] Moschovakis, Y., 1980. *Descriptive Set Theory*, North-Holland.

[40] Panza, M., 1992. "De la Continuité comme Concept au Continu comme Objet", *Le Labyrinthe du Continu*, (J.-M. Salanskis, H. Sinaceur, eds.), Springer, Paris, 16-30.

[41] Panza, M., 2021. *Abstraction and Epistemic Economy.* arXiv:2107.01580v1

[42] Peirce, C. S., 1960. *Collected Papers*, (C. Hartshorne and P. Weiss, eds.), The Belknap Press of Harvard University Press, Cambridge, Mass.

[43] Peirce, C. S., 1976. *New elements of mathematics*, (C. Eisele, ed.), Mouton Publishers, The Hague.

[44] Petitot, J., 1979. "Infinitesimale", *Enciclopedia Einaudi*, VII, 443-521, Einaudi, Torino.

[45] Petitot J., 1989. "Rappels sur l'Analyse non standard", *La mathématique non-standard*, (H. Barreau, J. Harthong, eds.), Editions du CNRS, Paris, 187-209.

[46] Petitot J., 1991. "Idéalités mathématiques et Réalité objective. Approche transcendantale", *Hommage à Jean-Toussaint Desanti*, (G. Granel ed.), 213-282, Editions TER, Mauvezin.

[47] Petitot J., 1992. "Continu et Objectivité. La bimodalité objective du continu et le platonisme transcendantal", *Le Labyrinthe du Continu*, (J.-M. Salanskis, H. Sinaceur, eds.), Springer-Verlag, Paris, 239-263.

[48] Petitot J., 1994. "Phenomenology of Perception, Qualitative Physics and Sheaf Mereology", *Philosophy and the Cognitive Sciences*, (16th Wittgenstein Symposium, R. Casati, B. Smith, G. White, eds), Verlag Hölder-Pichler-Tempsky, Vienna, 387-408.

[49] Petitot J., 1995. "Pour un platonisme transcendantal", *L'objectivité mathématique. Platonisme et structures formelles*, (M. Panza, J-M. Salanskis, eds), 147-178, Masson, Paris.

[50] Schimmerling, E., 2001. "The ABC's of Mice", *The Bulletin of Symbolic Logic*, 7, 4 (2001), 485-503.

[51] Solovay, R. M., 1971. "Real-Valued Measurable Cardinals", *Axiomatic Set Theory*, (D. Scott ed.), Proceedings of Symposia in Pure Mathematics, Vol. XIII, Providence, AMS, 397-428.

[52] STC, 1992. *Set Theory of the Continuum*, (H. Judah, W. Just, H. Woodin, eds.), Springer, Berlin-New York.

[53] Steel, J., 2000. "Mathematics Needs New Axioms", http://math.berkeley.edu/˜steel.

[54] Steel, J., 2001. "Inner Model Theory", to appear in the *Handbook of Set Theory.*

[55] Steel, J., 2004. "Generic absoluteness and the continuum problem", http://math.berkeley.edu/˜steel/talks/Lectures.html.

[56] Stern, J., 1976. "Le problème des cardinaux singuliers d'après Jensen et Silver", *Séminaire Bourbaki*, #494.

[57] Stern, J., 1984. "Le problème de la mesure", *Séminaire Bourbaki*, #632.

[58] Thom, R., 1992. "L'antériorité ontologique du continu sur le discret", *Le Labyrinthe du Continu*, (J.-M. Salanskis, H. Sinaceur, eds.), Springer, Paris, 137-143.

[59] Venturi, G., Viale, M., 2018. "New Axioms in Set Theory", *Matematica, Cultura e Società - Rivista dell'Unione Matematica Italiana*, 3/3 (2018) 211–236, 2018.

[60] Weyl, H., 1918. *Das Kontinuum. Kritishe Untersuchungen über die Grundlager der Analysis*, Veit, Leipzig, (English translation, Lanham, University Press of America, 1987).

[61] Woodin, W. H., 1999. *The Axiom of Determinacy, Forcing Axioms, and the Nonstationary Ideal*, De Gruyter, Berlin.

[62] Woodin, W. H., 2000. "The Continuum Hypothesis", *Logic Colloquium*, Paris.

[63] Woodin, W. H., 2001a. The continuum hypothesis, I-II, *Notices Amer. Math. Soc.*, 48, 6 (2001), 567-576 and 48, 7 (2001) 681-690.

[64] Woodin, W. H., 2001b. "The Ω Conjecture", *Aspects of Complexity*, De Gruyter, Berlin, 155-169.

[65] Woodin, W. H., 2002. "Beyond Σ^2_1 Absoluteness", *ICM 2002*, Vol. I, 515-524.

[66] Woodin, W. H., 2003. "Set theory after Russell: the journey back to Eden", *One Hundred Years of Russell's Paradox*, (G. Link, ed.), De Gruyter, Berlin, 29-47 (2004).

[67] Woodin, W. H., 2004. "Is there really any evidence that the Continuum Hypothesis has no answer?", http://www.lps.uci.edu/home/conferences/ Laguna-Workshops/LagunaBeach2004/irv.pdf

[68] Woodin, W. H., 2010, "Strong Axioms of Infinity and the Search for

V", *Proceedings of the International Congress of Mathematicians 2010*, Hindustan Book Agency, New Delhi, 504-528.

[69] Woodin, W.H., 2017. "In search of Ultimate-L. The 19th Midrasha Mathematicae Lectures", *The Bulletin of Symbolic Logic*, 23/1 (2017) 1-109.