

A transcendental view on the continuum: Woodin's conditional platonism

Jean PETITOT*

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1 Introduction

One of the main general philosophical problem raised by the nature of the continuum is the conflict between two traditions: the older one, which can be called “neo-Aristotelian”, even if this sounds rather vague, and the now the classical one, namely the Cantorian tradition. According to the neo-Aristotelian tradition, the continuum is experienced and thought of as a non-compositional, cohesive, primitive, and intuitive datum. It can be segmented into parts but these parts are themselves continua and points are only their boundaries.

This point of view was very well defended by Kant. As soon as in his 1770 *Dissertatio*, he emphasized the fact that

“a magnitude is continuous when it is not composed out of simple elements” (AK, II, p. 399 ¹),

and explained that for the continuous “pure intuitions” of space and time

“any part of time is still a time, and the simple elements which are in time, namely the *moments*, are not parts but *limits* between which a time takes place” (AK, II, p. 399).

“space must necessarily be conceived of as a continuous magnitude, (...) and therefore simple elements in space are not parts but limits” (AK, II, p. 404).

*EHESS & CREA (Ecole Polytechnique), France. petitot@poly.polytechnique.fr.

¹AK refers to the collected works *Kants gesammelte Schriften*, Preussische Akademie der Wissenschaften.

He wrote also in the “Anticipations of perception” of the *Critic of Pure Reason*:

“Space and time are quanta continua because no part of them can be given without being enclosed into limits (points or moments) (...). Space is made up only of spaces and time of times. Points and moments are only limits, that is to say, simple places bounding space and time (...), and neither space nor time can be made up of simple places, that is of integral parts which would be given before space and time themselves” (AK, III, p. 154).

We see that this conception of the continuum is based on *mereological* properties, and especially on the concept of boundary (*Grenze*): points are boundaries and boundaries are dependent entities which cannot exist independently of the entities they bound. When there are no explicit boundaries, the continuum is characterized by the “fusion” of its parts.² Moreover, Leibniz’s principle of continuity holds (every function is continuous).

Even after the Cantor-Dedekind arithmetization, the “neo-Aristotelian” non-compositionality of the continuum kept on raising fundamental problems for some of the greatest philosophers, mathematicians, and psychologists, such as Peirce, Brentano, Stumpf (who elaborated the key concept of “fusion”: *Verschmelzung*), Husserl³, or Thom (for whom the continuum possessed an ontological primacy as a qualitative homogeneous Aristotelian “homeomer”). Gödel himself considered the real intuition of the continuum in this way and opposed it to its set theoretical idealisation.

Peirce developed a “synechology”, “syneche” being the greek term for “continuum”. Mathematically, he was also the first, as far as I know, to reject the continuum hypothesis CH ($2^{\aleph_0} = \aleph_1$) and to define the power \mathfrak{c} of the continuum as a *large cardinal*, namely an inaccessible cardinal (if $\kappa < \mathfrak{c}$ then $2^\kappa < \mathfrak{c}$). In some texts, Peirce even explained that the continuum could be so huge that it would fail to be a cardinal.

Husserl was the first, in the third *Logical Investigation*, to formalize the idea of a mereology, and, after him, Stanisław Lesniewski developed the theory between 1916 and 1921. But the definition of boundaries in a mereo(topo)logy

²It is very easy to construct a model of such a mereology. Let us take \mathbb{R} with its standard topology and posit that the only admissible parts U of \mathbb{R} are its open subsets $U \in \mathcal{P}_{\text{ad}}(\mathbb{R})$, that is the subsets for which the relation \in is *stable*. Let $U, V \in \mathcal{P}_{\text{ad}}(\mathbb{R})$. The complement $\neg U$ of U is the interior $\text{Int}(\overline{U})$ of its classical complement \overline{U} (which is a closed subset). Therefore $U \cap \neg U = \emptyset$ but $U \cup \neg U \neq \mathbb{R}$. Conversely; if $U \cup V = \mathbb{R}$ then $U \cap V \neq \emptyset$. This means that, for the open mereology, \mathbb{R} is indecomposable: if \mathbb{R} is a disjoint union $A + B$, then either A or B is equal to \mathbb{R} (topological connectedness). Topological boundaries ∂U are not admissible parts but only limits, and bound both U and $\neg U$. Heyting used this mereotopology for defining truth in intuitionist logic. The intuitionistic continuum is still more cohesive since $\mathbb{R} - \{0\}$ and even $\mathbb{R} - \mathbb{Q}$ are intuitionistically indecomposable.

³Concerning the concept of *Verschmelzung* in Stumpf and Husserl, see. Petitot [1994].

remained up to now highly problematic as it is argued, e.g., in Breysse-De Glas (2007).

All these conceptions develop the same criticism against the idea of *arithmetizing* the continuum. According to them, the continuum can be measured using systems of numbers, but no system can exhaust the substratum it measures. They consider:

1. that a point in the continuum is a discontinuity (a mark, a local heterogeneity, a boundary) which is like a singular individuated “atom” which can be referred to by a symbol;
2. that quantified sentences of an appropriate predicate calculus can be therefore interpreted in the continuum;
3. that systems of numbers can of course measure such systems of marks and enable their axiomatic control;
4. but that the arithmetization of the continuum postulates, what’s more, that the intuitive phenomenological continuum is reducible to such a set-theoretic system (Cantor-Dedekind);
5. and therefore that such a reductive arithmetization is unacceptable for it violates the original intuitive mode of givenness of the continuum.

Let us leave phenomenology and psychology for mathematics. Even if we adopt a set-theoretic perspective making the continuum a set, non-compositionality and cohesivity remain meaningful. They now mean in particular that the continuum cannot be identified with a set of well *individuated* points. It is the case in intuitionistic logic where the law of the excluded middle, which implies that two elements a and b of \mathbb{R} are different or equal, and the law of comparability, which implies $a = b$, or $a < b$, or $a > b$, are no longer valid. For Hermann Weyl (1918), this intrinsic lack of individuation and localization of points in \mathbb{R} characterizes the continuum as an intuitive datum.

Given the close link discovered by Bill Lawvere between intuitionistic logic and topos theory, it is not surprising that in many topoï the object \mathbb{R} is undecomposable and all the morphisms $f : \mathbb{R} \rightarrow \mathbb{R}$ are continuous (Leibniz’s principle, see e.g. John Bell’s works ⁴).

But, even in the realm of *classical* logic and classical set theory (that is *ZFC*: Zermelo-Fraenkel + axiom of choice) there exists a great deal of evidence on the transcendence of the continuum relatively to its symbolic logical control. Of course, they occur in non-standard models of \mathbb{R} , but these non-Archimedean

⁴See in particular Bell’s contribution to this volume.

models are not well-founded and will not be analyzed here.⁵ I will rather focus on their status in well-founded models V of ZFC .

So, the context of this paper will be the theory of the continuum in universes of sets satisfying ZFC . We will see that, even in this purely formal framework, the Kantian opposition between “intuitive” and “conceptual” remains operating, where “conceptual” now refers to the logical control and symbolic determination of the continuum (as it was already the case for Kant himself who considered arithmetic, contrary to geometry, as “conceptual” and “intellectual”, and algebra as a calculus on “symbolic constructions”).⁶ The key question remains the same: is it possible to determine completely the intuitive continuum using logical symbolic constructions?

We will meet in the sequel an alternative opposing two different types of philosophies.

1. Philosophies of the first type are “minimizing”, “ontologically” deflationist (in the sense of restricting what axioms of existence are admissible), nominalist, and constructive. They consider that the only meaningful content of the continuum is the part which can be “conceptually” (that is symbolically) well determined and that the rest is “inherently vague”. A celebrated representative of such a perspective is Solomon Feferman who considers that the continuum cannot be a definite mathematical object since some of its properties, such as Cantor’s continuum hypothesis, are not expressible by definite propositions. But, as explained by John Steel (2004) in a criticism of Feferman:

“Taken seriously, this analysis leads us into a retreat to some much weaker constructivist language, a retreat which would toss out good mathematics in order to save inherently vague philosophy.”

2. It is why, philosophies of the second type are, on the contrary, “maximizing”, “ontologically” inflationist, platonist in a sophisticated sense, and highly non-constructive. They aim at modelling *inside* a ZFC -universe the transcendence of the intuitive continuum w.r.t. its logical symbolic mastery. Owing to this, they must introduce non-constructive axioms for higher infinite.

2 Preliminaries

2.1 Axioms

We work in the theory ZFC . Among its classical axioms, two are the source of many difficulties:

⁵For an introduction to non-standard analysis, see Petitot [1979], [1989] and their bibliographies.

⁶*Critic of Pure Reason*, “Transcendental Methodology”, AK, III, p. 471.

Power set: the set $\mathcal{P}(X) = \{u \subseteq X\}$ of subsets of every set X exists,

Choice: every family of sets X_s , $s \in S$, has a choice function f associating to each $s \in S$ an element $f(s) \in X_s$ (this axiom of existence doesn't define any specific f and is highly non-constructive).⁷

It must be emphasized that these axioms reflect some fundamental aspects of the concept of set but are not to be considered as definitive. One of the main purpose of modern set theory is precisely to evaluate, modify and complete them, and this is an extremely difficult technical task.

2.2 Ordinals and cardinals

Two classes of sets are of fundamental importance for enumerate and characterize the size of sets. They are of a completely different nature.

On the one hand, *ordinals* are the sets that are \in -transitive ($y \in X$ implies $y \subset X$, that is $\bigcup X \subseteq X$ or $X \subset \mathcal{P}(X)$) and well-ordered by \in . All well-ordered sets are order-isomorphic to an ordinal. Every ordinal is a successor: $\alpha = \beta + 1$ or a limit ordinal $\alpha = \text{Sup}(\beta : \beta < \alpha)$ (and then $\forall \beta < \alpha$, $\beta + 1 < \alpha$). A limit ordinal is like an “horizon” for enumeration: it is impossible to reach its limit in a finite number of steps. The smallest limit ordinal is $\omega \simeq \mathbb{N}$. The sum of two ordinals is their concatenation (non-commutativity: $1 + \omega = \omega \neq \omega + 1$). The product of two ordinals $\alpha.\beta$ is β -times the concatenation of α (lexicographic order) (non-commutativity: $2.\omega = \omega \neq \omega.2 = \omega + \omega$). An ordinal α is a limit ordinal iff there exists β s.t. $\alpha = \omega.\beta$.

On the other hand, *cardinals* $|X|$ are equivalence classes of the equivalence relation of equipotence: $X \text{ eq } Y$ if there exists a bijective (i.e. one-to-one onto) map $f : X \rightarrow Y$. They highly depend upon the functions existing in the *ZFC*-universe under consideration.

Cantor theorem. $|X| < |\mathcal{P}(X)|$. □

Indeed, let $f : X \rightarrow \mathcal{P}(X)$. Then $Y = \{x \in X : x \notin f(x)\}$ exists (Comprehension axiom). But $Y \notin f(X)$ for if it would exist $z \in X$ with $f(z) = Y$, then $z \in Y \Leftrightarrow z \notin Y$. Contradiction. □

If $|A| = \kappa$, then $|\mathcal{P}(A)| = 2^\kappa$ since a subset $X \subseteq A$ is equivalent to its characteristic function $\chi_X : A \rightarrow \{0, 1\}$ and $\chi_X \in 2^\kappa$. Cantor theorem implies therefore $\kappa < 2^\kappa$.

For ordinals, cardinals *numbers* are the α s.t. $|\beta| < |\alpha|$ for every $\beta < \alpha$. They are the minimal elements in the equivalence classes of equipotent ordinals.

⁷The other axioms are (see e.g. Jech [1978]): *Extensionality*: sets are determined by their elements; *Pairing*: the pair $\{a, b\}$ exists for every sets a and b ; *Union*: the union $\bigcup X = \{u \in x \in X\}$ of every set X exists; *Comprehension* or *Separation* (axiom schema): if $\varphi(x)$ is a formula, the subset $\{x \in X : \varphi(x)\}$ exists for every set X ; *Replacement* (axiom schema): if $y = f(x)$ is function (i.e. a relation $\varphi(x, y)$ s.t. $\varphi(x, y)$ and $\varphi(x, z)$ imply $y = z$), then the image $\{f(x) : x \in X\}$ exists for every set X ; *Infinity*: there exists an inductive set, that is a set I s.t. $\emptyset \in I$ and if $x \in I$ then $x \cup \{x\} \in I$; *Regularity*: all sets have minimal \in -elements.

So every *infinite* cardinal number is a *limit* ordinal and a natural (but extremely hard) challenge is, for every infinite cardinal number κ with successor κ^+ to find bijections between κ and the ordinals $\kappa < \alpha < \kappa^+$. Every well-ordered set has a cardinal number for cardinal. These cardinal numbers are the *alephs* \aleph_α . Each \aleph_α has a successor, namely $\aleph_\alpha^+ = \aleph_{\alpha+1}$. If α is a limit ordinal, $\aleph_\alpha = \omega_\alpha = \sup_{\beta < \alpha} (\omega_\beta)$.

Theorem. For alephs, the sum and product operations are trivial: $\aleph_\alpha + \aleph_\beta = \aleph_\alpha \cdot \aleph_\beta = \max(\aleph_\alpha, \aleph_\beta)$ (the bigger takes all). \square

A consequence is that if $\alpha \leq \beta$ then $\aleph_\alpha^{\aleph_\beta} = 2^{\aleph_\beta}$. Indeed, $2^{\aleph_\beta} \leq \aleph_\alpha^{\aleph_\beta} \leq (2^{\aleph_\alpha})^{\aleph_\beta} = 2^{\aleph_\alpha \cdot \aleph_\beta} = 2^{\aleph_\beta}$. But other *exponentiations* raise fundamental problems.

Under the axiom of choice *AC*, every set can be well-ordered and therefore all cardinals are alephs. In particular 2^{\aleph_0} is an aleph \aleph_α .

3 The underdetermination of cardinal arithmetic in ZFC

Let V be a universe of set theory (i.e. a model of *ZFC*). We work in \mathbb{R} or in the isomorphic Baire space $\mathcal{N} = \omega^\omega$. The first limit we meet is that the axioms of *ZFC* are radically insufficient for determining the cardinal arithmetic of V as is clearly shown by the following celebrated result.⁸

3.1 Easton's theorem

For every ordinal α let $F(\alpha)$ be the power function defined by $2^{\aleph_\alpha} = \aleph_{F(\alpha)}$.⁹ One can show that:

1. F is a monotone increasing function: if $\alpha \leq \beta$ then $F(\alpha) \leq F(\beta)$;
2. *König's law*: $\text{cf}(\aleph_{F(\alpha)}) > \aleph_\alpha$, where the *cofinality* $\text{cf}(\alpha)$ of an ordinal α is defined as the smallest cardinality χ of a cofinal (i.e. unbounded) subset X of α (i.e. $\sup X = \alpha$). For instance, $\text{cf}(\omega + \omega) = \text{cf}(\aleph_{\alpha+\omega}) = \omega$. Of course, $\text{cf}(\alpha)$ is a limit ordinal and $\text{cf}(\alpha) \leq \alpha$. The cardinal κ is called *regular* if $\text{cf}(\kappa) = \kappa$ i.e. if, as far as we start with $\alpha < \kappa$, it is impossible to reach the horizon of κ in less than κ steps. In some sense the length of κ is equal to its “asymptotic” length and cannot be exhausted before reaching the horizon. As $\text{cf}(\text{cf}(\kappa)) = \text{cf}(\kappa)$, $\text{cf}(\kappa)$ is always regular. As $\text{cf}(\aleph_{\alpha+\omega}) = \omega$, $\aleph_{\alpha+\omega}$ is always singular.¹⁰

König's law is a consequence of a generalization of Cantor theorem which says that $1 + 1 + \dots$ (κ times) $< 2 \cdot 2 \cdot \dots$ (κ times): if $\kappa_i < \lambda_i \forall i \in I$, then $\sum_{i \in I} \kappa_i < \prod_{i \in I} \lambda_i$. Indeed, let $\kappa_i < 2^{\aleph_\alpha}$ and $\lambda_i = 2^{\aleph_\alpha}$ for $i < \omega_\alpha$. Then

⁸See e.g. Jacques Stern [1976] for a presentation.

⁹More generally one can consider the power function $(\lambda, \kappa) \mapsto \lambda^\kappa$ for each pair (λ, κ) of cardinals.

¹⁰So a cardinal κ is singular iff $\kappa = \bigcup_{i \in I} \alpha_i$ with $|I| < \kappa$ and $|\alpha_i| < \kappa \forall i \in I$.

$\sum_{i < \omega_\alpha} \kappa_i < \prod_{i < \omega_\alpha} \lambda_i = (2^{\aleph_\alpha})^{\aleph_\alpha} = 2^{\aleph_\alpha}$. Therefore with an ω_α sequence of $\kappa_i < 2^{\aleph_\alpha}$ it is impossible to get $\text{Sup}(\kappa_i) = 2^{\aleph_\alpha}$ since for infinite cardinals $\text{Sup} = \sum$.

An immediate corollary of König's law is

Theorem. *If κ is an infinite cardinal, $\kappa < \kappa^{\text{cf}(\kappa)}$ (compare with Cantor: $\kappa < 2^\kappa$).* \square

Indeed, if $\kappa_i < \kappa$ for $i < \text{cf}(\kappa)$ and $\kappa = \text{Sup}(\kappa_i) = \sum_{i < \text{cf}(\kappa)} \kappa_i$ then $\kappa = \sum_{i < \text{cf}(\kappa)} \kappa_i < \prod_{i < \text{cf}(\kappa)} \kappa = \kappa^{\text{cf}(\kappa)}$. \square

In fact one can prove that the essential cardinals for cardinal arithmetic are the 2^κ and the $\kappa^{\text{cf}(\kappa)}$ (Gimel function). They enable to compute all the $\aleph_\alpha^{\aleph_\beta}$:

Theorem.¹¹

1. If $\alpha \leq \beta$, then $\aleph_\alpha^{\aleph_\beta} = 2^{\aleph_\beta}$.
2. If $\alpha > \beta$ and $\exists \gamma < \alpha$ s.t. $\aleph_\gamma^{\aleph_\beta} \geq \aleph_\alpha$, then $\aleph_\alpha^{\aleph_\beta} = \aleph_\gamma^{\aleph_\beta}$.
3. If $\alpha > \beta$ and $\forall \gamma < \alpha$ we have $\aleph_\gamma^{\aleph_\beta} < \aleph_\alpha$ then
 - (a) if \aleph_α is regular or $\text{cf}(\aleph_\alpha) > \aleph_\beta$ then $\aleph_\alpha^{\aleph_\beta} = \aleph_\alpha$;
 - (b) if $\text{cf}(\aleph_\alpha) \leq \aleph_\beta < \aleph_\alpha$ then $\aleph_\alpha^{\aleph_\beta} = \aleph_\alpha^{\text{cf}(\aleph_\alpha)}$. \square

If the generalized continuum hypothesis (*GCH*) holds, König's law is trivial because $F(\alpha) = \alpha + 1$, every cardinal $\aleph_{\alpha+1}$ is regular and therefore $\text{cf}(\aleph_{\alpha+1}) = \aleph_{\alpha+1} > \aleph_\alpha$.

The fact that *ZFC* radically underdetermines cardinal arithmetic is particularly evident in Easton's striking result:

Easton's theorem. *For regular cardinals \aleph_α , one can impose via forcing in *ZFC* the power function $2^{\aleph_\alpha} = \aleph_{F(\alpha)}$ for quite every function F satisfying (i) and (ii).* \square

For regular cardinals κ , we have $\kappa^{\text{cf}(\kappa)} = \kappa^\kappa = 2^\kappa$ but for singular cardinals σ we have $\sigma^{\text{cf}(\sigma)} = \left(\begin{smallmatrix} \sigma \\ \text{cf}(\sigma) \end{smallmatrix} \right) 2^{\text{cf}(\sigma)}$ where $\left(\begin{smallmatrix} \kappa \\ \lambda \end{smallmatrix} \right)$ for $\kappa > \lambda$ is a sophisticated generalization of the binomial formula.

The proof of Easton's theorem uses iterated Cohen forcing.

3.2 Cohen forcing

Cohen forcing (1963)¹² allows to construct in a very systematic way “generic” extensions N of *inner models* M of *ZF* or *ZFC* (that is transitive \in -submodels $M \subset V$ of *ZF* or *ZFC* with $On \subseteq M$) where some desired properties become valid.

¹¹See Jech [1978], p. 49.

¹²Awarded a Fields Medal in 1966, Paul Cohen died on March 23, 2007.

Suppose for instance that, starting with a ground inner model M of ZFC in V , we want to construct another inner model N where ω_1^M (that is the cardinal $\omega_1 = \aleph_1$ as defined in M) collapses and becomes countable. We need to have at our disposal in N a surjection $f : \omega \rightarrow \omega_1^M$ that, by definition of ω_1^M , cannot belong to M . Suppose nevertheless that such an f exists. Then for every n the restriction $f \upharpoonright_n = (f(0), \dots, f(n-1))$ exists and is an element of the ground model M . Let us therefore consider the set $P = \{p\}$ of finite sequences $p = (\alpha_0, \dots, \alpha_{n-1})$ of countable ordinals $\alpha_i < \omega_1^M$ of M . Such p are called *forcing conditions* and must be interpreted as forcing $f \upharpoonright_n = p$. The set P exists, is well defined in M , and is endowed with a natural partial order “ $q \leq p$ iff $p \subseteq q$ ”¹³. If f exists, we can consider $G = \{f \upharpoonright_n\}_{n \in \mathbb{N}}$ which is a subset of P in V s.t. $\cup G = f$. But as $f \notin M$, G cannot be a subset of P in M .

If f exists, it is trivial to verify that G satisfies the following properties:

1. *Gluing and restriction conditions* (see topos theory): if $p, q \in G$, then p and q are initial segments of f and are compatible in the sense that $p \leq q$ or $q \leq p$ and therefore there exists a common smaller element $r \in G$ satisfying $r \leq p, r \leq q$.
2. for every $n \in \omega$, there exists $p \in G$ s.t. $n \in \text{dom}(p)$ (i.e. $\text{dom}(f) = \omega$).
3. for every M -countable ordinal $\alpha < \omega_1^M$, there exists $p \in G$ s.t. $\alpha \in \text{range}(p)$ (i.e. $\text{range}(f) = \omega_1^M$, it is the fundamental condition of surjectivity for the collapsing of ω_1^M in N).

Cohen’s idea is to construct sets G in V satisfying these properties and to show that extending the ground inner model M by such a G yields an appropriate inner model $N = M[G]$ which is the smaller inner model of V containing M and G .

So, one supposes that a partially ordered set of forcing conditions P is given. A subset of conditions $D \subseteq P$ is called *dense* if for every $p \in P$ there is a smaller $d \leq p$ belonging to D . One then defines *generic* classes $G \subseteq P$ of conditions. A subset of P , $G \notin M$, is generic over M iff is a *filter* for the order \leq ¹⁴ such that G intersects every dense set D of conditions $D \in M$ ¹⁵ (be careful: $D \in M$, $D \subseteq P \in M$, $G \subseteq P$, but $G \notin M$).

If G is generic, the properties (2) and (3) above are automatically satisfied since the sets of conditions $D_n = \{p \in P : n \in \text{dom}(p)\}$ for $n \in \omega$ and $E_\alpha = \{p \in P : \alpha \in \text{range}(p)\}$ for $\alpha < \omega_1^M$ are dense: (2) means that $G \cap D_n \neq \emptyset$ and (3) means that $G \cap E_\alpha \neq \emptyset$.

¹³That is $q < p$ means conventionally that q forces a better approximation of f than p (smaller is better).

¹⁴This means (i) that $p \in G$ and $p \leq q \in P$ implies $q \in G$, and (ii) that for every $p, q \in G$, there exists a common smaller $r \in G$ satisfying $r \leq p, r \leq q$,

¹⁵I.e. there exists $p \in D$ such that $p \in G$.

Cohen's main results say that M can be extended using G to an inner model $M[G]$ where properties "forced" by P become valid. The first key result concerns the existence of $M[G]$.

Cohen's theorem. *There exists a ZFC-model $\mathcal{A} = M[G]$ such that (1) M is an inner model of \mathcal{A} , (2) G is not a set in M but is a set in \mathcal{A} , (3) if \mathcal{A}' is another model satisfying (1) et (2), then there exists an elementary embedding $j : \mathcal{A} \prec \mathcal{A}'$ such that $j(\mathcal{A})$ is an inner model of \mathcal{A}' and $j|_M = \text{Id}(M)$, (4) \mathcal{A} is essentially unique.* \square

If $j : \mathcal{A} \prec \mathcal{A}'$ is an embedding of a model \mathcal{A} in a model \mathcal{A}' , *elementarity* means that \mathcal{A}' has exactly the same first-order theory as \mathcal{A} in the very strong language $\mathcal{L}_{\mathcal{A}}$ where there exist names for *every* element of \mathcal{A} (that is for every set x and first-order formula φ , $\mathcal{A} \models \varphi(x)$ iff $\mathcal{A}' \models \varphi(j(x))$). So, for first-order logic, \mathcal{A}' adds only indiscernible elements. A less constraining relation is *elementary equivalence*: \mathcal{A} and \mathcal{A}' are elementary equivalent, $\mathcal{A} \equiv \mathcal{A}'$, if they have the same first-order theory in the standard language \mathcal{L} used for talking about models such \mathcal{A} . In that case, elements of \mathcal{A} characterized by first-order sentences can be substituted for other elements of \mathcal{A}' . It is no longer the case for an elementary embedding.

An essential feature of forcing extensions is that it is possible to describe $M[G]$ using the language \mathcal{L}_G which is the language \mathcal{L} of M extended by a new symbol constant for G . As was emphasized by Patrick Dehornoy (2003), forcing is

“as a field extension whose elements are described by polynomials defined on the ground field”.

In particular, the validity of a formula φ in $M[G]$ can be coded by a *forcing relation* $p \Vdash \varphi$ defined in M . This is the second key result. The definition of $p \Vdash \varphi$ is rather technical but an excellent intuition is given by the idea of “localizing” truth, p being interpreted as a local domain (as an open set of some topological space), and $p \Vdash \varphi$ meaning that φ is “locally true” on p .

Forcing theorem. *For every generic $G \subseteq P$, $M[G] \models \varphi$ iff there exists a $p \in G$ s.t. $p \Vdash \varphi$.* \square

Using forcing, we can in particular add to \mathbb{R} (i.e. to $\mathcal{P}(\omega)$) new elements called *generic reals*. Let P be the partial order of binary finite sequences $p = (p(0), \dots, p(n-1))$. If $G \subseteq P$ is generic, $f = \cup G$ is a map $f : \omega \rightarrow \{0, 1\}$ which is the characteristic function $f = 1_A$ of a *new* subset $A \subseteq \omega$ and $A \notin M$. Indeed, if $g : \omega \rightarrow \{0, 1\}$ defines a subset $B \subseteq \omega$ which belongs to M , then the set of conditions $D_g = \{p \in P : p \not\subseteq g\} \in M$ is dense (if p is any finite sequence it can be extended to a sequence long enough to be different from g) and therefore $G \cap D_g \neq \emptyset$. But this means that $f \neq g$.

To prove the negation $\neg CH$ of CH , one adds to M a great number of generic reals. More precisely, one embeds ω_2^M into $\{0, 1\}^\omega$ (isomorphic to \mathbb{R}) using as

forcing conditions the set P of finite binary sequences of $\omega_2^M \times \omega$. If G is generic, then $f = \cup G$ is a map $f : \omega_2^M \times \omega \rightarrow \{0, 1\}$, that is an ω_2^M -family $f = \{f_\alpha\}_{\alpha < \omega_2^M}$ of generic reals $f_\alpha : \omega \rightarrow \{0, 1\}$. Using density arguments one shows that f yields an embedding $\omega_2^M \hookrightarrow \{0, 1\}^\omega$ in $M[G]$ and that ω_2^M *doesn't collapse* in $M[G]$ (because P is ω -saturated, i.e. there doesn't exist in P any infinite countable subset of incompatible elements). This implies immediately $\neg CH$.

Easton theorem is proved by *iterating* such constructions and adding to every regular \aleph_α as many new subsets as it is necessary to have $2^{\aleph_\alpha} = \aleph_{F(\alpha)}$.

3.3 Absoluteness

Many philosophers and logicians who are “deflationist” regarding mathematical “ontology” consider that the only sentences having a well determined truth-value are those the truth-value of which is the same in all models of ZFC , and that sentences the truth-value of which can change depending on the chosen model are “inherently vague”. Such an antiplatonist conception has drastic consequences. Indeed, contrary to first order arithmetic, which is ZF -absolute, that is invariant relative to extensions of the universe (Schönfield theorem), all structures and notions such as \mathcal{N} , \mathbb{R} , $\text{Card}(\chi)$, $x \rightarrow \mathcal{P}(x)$, $x \rightarrow |x|$, and second order arithmetic, *are not ZF-absolute*. They can vary widely from one model to another and can't have absolute truth value in ZF . This “vagueness” is one of the main classical arguments of antiplatonists against non-constructive set theories. But, it has been emphasized by Hugh Woodin in his 2003 paper *Set theory after Russell. The journey back to Eden* that vagueness is not an admissible argument against platonism and shows only that it is necessary to *classify* the different models of ZF and ZFC . As he explained also in his talk at the Logic Colloquium held in Paris in 2000 (quotation from Dehornoy, 2003, p. 23):

“There is a tendency to claim that the Continuum Hypothesis is inherently vague and that this is simply the end of the story. But any legitimate claim that CH is inherently vague must have a mathematical basis, at the very least a theorem or a collection of theorems. My own view is that the independence of CH from ZFC , and from ZFC together with large cardinal axioms, does not provide this basis. (...) Instead, for me, the independence results for CH simply show that CH is a difficult problem.”

In fact, the strong variability of the possible models of ZFC is an argument in favor of the *irreducibility* of the continuum to a set of points which can be “*individuated*” by a symbolic description, and platonism can be interpreted as a search for a certain type of absoluteness of the continuum w.r.t. some theories stronger than ZF and ZFC .

To tackle this problem, we will have to look at two opposed strategies, both introduced by Gödel, one “minimalist” (“ontologically” deflationist) and the other “maximalist” (“ontologically” inflationist). To explain this point we first introduce some classes of sets of reals studied by what is called “descriptive set theory”.

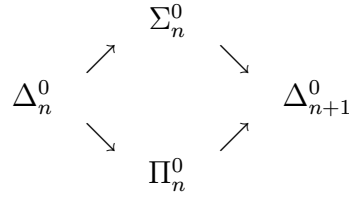
4 Borel and projective hierarchies

In descriptive set theory, one works in \mathbb{R} or in $\mathcal{N} = \omega^\omega$ or in $\{0,1\}^\omega$, and, more generally, on metric, separable, complete, perfect (closed without isolated points) spaces \mathcal{X} (Polish spaces). One considers in \mathcal{X} different “nicely” definable classes of subsets Γ . The first is the *Borel hierarchy* constructed from the open sets by iterating the set theoretic operations of complementation and of *countable* projection $\mathcal{X} \times \omega \rightarrow \mathcal{X}$. If $P \subseteq \mathcal{X} \times \omega$ (that is, if P is a countable family of subsets $P_n \subseteq \mathcal{X}$), one considers the subset of \mathcal{X} defined by $\exists^\omega P = \{x \in \mathcal{X} \mid \exists n P(x, n)\}$.¹⁶ It is the union $\bigcup_{n \in \omega} P_n$.

The Σ_1^0 are the open subsets of \mathcal{X} , the $\Pi_1^0 = \neg \Sigma_1^0$ are the closed subsets, $\Delta_1^0 = \Pi_1^0 \cap \Sigma_1^0$ the clopen subsets, and the Borel hierarchy B is defined by:

$$\Pi_n^0 = \{\neg \varphi \mid \varphi \in \Sigma_n^0\} = \neg \Sigma_n^0, \quad \Sigma_{n+1}^0 = \exists^\omega \neg \Sigma_n^0 = \exists^\omega \Pi_n^0, \quad \Delta_n^0 = \Pi_n^0 \cap \Sigma_n^0.$$

It can be shown that this hierarchy is *strict*:



One then defines the higher hierarchy of *projective* sets using a supplementary principle of construction, namely *continuous* projections $\mathcal{X} \times \mathcal{N} \rightarrow \mathcal{X}$, written $\exists^\mathcal{N}$. One gets a new hierarchy beginning with the class $\Sigma_1^1 = \exists^\mathcal{N} \Pi_1^0$ – the so called *analytic* subsets – and continuing with the classes:

$$\Pi_n^1 = \{\neg \varphi \mid \varphi \in \Sigma_n^1\} = \neg \Sigma_n^1, \quad \Sigma_{n+1}^1 = \exists^\mathcal{N} \neg \Sigma_n^1 = \exists^\mathcal{N} \Pi_n^1, \quad \Delta_n^1 = \Pi_n^1 \cap \Sigma_n^1.$$

For instance, $P \subseteq \mathcal{X}$ is Σ_1^1 if there exists a *closed* subset $F \subseteq \mathcal{X} \times \mathcal{N}$ such that: $P(x) \Leftrightarrow \exists \alpha F(x, \alpha)$. In the same way, $P \subseteq \mathcal{X}$ is Σ_2^1 if there exists an *open* subset $G \subseteq \mathcal{X} \times \mathcal{N} \times \mathcal{N}$ such that: $P(x) \Leftrightarrow \exists \alpha \forall \beta G(x, \alpha, \beta)$, etc.

More generally, one can define projective sets in V using the cumulative hierarchy of successive levels of V indexed by the class On of ordinals: $V_0 = \emptyset$, $V_{\alpha+1} = \{x : x \subset V_\alpha\}$ for a successor ordinal, and $V_\lambda = \bigcup_{\alpha < \lambda} V_\alpha$ for λ a limit

¹⁶ $P(x, n) = P_n(x)$. We identify predicates $\varphi(x)$, $P(x, n)$, etc. with their extensions.

ordinal. Then P is projective if it is definable with parameters over $(V_{\omega+1}, \in)$. More precisely, $P \subset V_{\omega+1}$ is Σ_n^1 if it is the set of sets x s.t. $(V_{\omega+1}, \in) \models \varphi(x)$ for a Σ_n formula $\varphi(x)$, that is a formula of the form $\varphi(x) = \exists x_1 \forall x_2 \dots \psi$ with n quantifiers and a ψ having only *bounded* quantifiers.¹⁷ In that sense, projective sets can be considered as the “reasonably” definable subsets of \mathbb{R} .

As the Borel hierarchy, the projective hierarchy is *strict* and it is a continuation of the Borel hierarchy according to:

Suslin theorem. $B = \Delta_1^1$. □

This theorem can be interpreted as a *construction principle*: it asserts that in the Δ_1^1 case, the complex operation of continuous projection can be reduced to an iteration of simpler operations of union and complementation.

There exist strict Π_n^1 and Σ_n^1 sets, which are very natural in classical analysis. For instance, in the functional space $C[0, 1]$ of real continuous functions on $[0, 1]$ endowed with the topology of uniform convergence, the subset

$$\{f \in C[0, 1] \mid f \text{ smooth}\}$$

is Π_1^1 (but not Δ_1^1). In the space $C[0, 1]^\omega$ of countable sequences (f_i) of functions, the subset:

$$\left\{ (f_i) \in C[0, 1]^\omega \mid \begin{array}{l} (f_i) \text{ converges for the topology} \\ \text{of simple convergence} \end{array} \right\}$$

is Π_1^1 , and the subset:

$$\left\{ (f_i) \in C[0, 1]^\omega \mid \begin{array}{l} \text{a sub-sequence converges for the} \\ \text{topology of simple convergence} \end{array} \right\}$$

is Σ_2^1 and every Σ_2^1 -subset of $C[0, 1]$ can be represented that way (Becker [1992]):

Becker representation theorem. *For every Σ_2^1 -set $S \subseteq C[0, 1]$ there exists a sequence (f_i) such that*

$$S = A_{(f_i)} = \left\{ g \in C[0, 1] \mid \begin{array}{l} \text{a sub-sequence of } (f_i) \text{ converges} \\ \text{towards } g \text{ for the topology} \\ \text{of simple convergence} \end{array} \right\}.$$

Another examples are given by the compact subsets $K \in \mathcal{K}(\mathbb{R}^n)$ of \mathbb{R}^n : for $n \geq 3$,

$$\{K \in \mathcal{K}(\mathbb{R}^n) \mid K \text{ arc connected}\}$$

is Π_2^1 , and for $n \geq 4$,

$$\{K \in \mathcal{K}(\mathbb{R}^n) \mid K \text{ simply connected}\}$$

is also Π_2^1 .

¹⁷Bounded quantifiers are of the form $\exists y \in z$ and $\forall y \in z$.

5 The “minimalist” strategy of the constructible universe

The first Gödelian strategy for constraining the structure of ZF -universes consisted in *restricting* the universe V . It is the strategy – referred to as $V = L$ – of *constructible* sets (Gödel 1938).

To define L , one substitutes, in the construction of the cumulative hierarchy V_α of V by means of a transfinite recursion on the $x \rightarrow \mathcal{P}(x)$ operation, the power sets $\mathcal{P}(x)$ – which are not ZF -absolute – with *smaller* sets $\mathcal{D}(x) = \{y \subseteq x \mid y \text{ elementary}\}$ (where “elementary” means definable by a first order formula over the structure $\langle x, \in \rangle$)¹⁸ – which are ZF -absolute. L is then defined as V using a transfinite recursion on ordinals: $L_0 = \emptyset$, $L_{\alpha+1} = \mathcal{D}(L_\alpha)$, $L_\lambda = \bigcup_{\alpha < \lambda} L_\alpha$ if λ is a limit ordinal, and $L = \bigcup_{\alpha \in On} L_\alpha$. The absoluteness of L comes from the fact that each level L_α is constructed using only unambiguous formulae and parameters belonging to the previous stages L_β , $\beta < \alpha$.

Gödel (1938, 1940) has shown that if $V = L$ it is possible to define a *global wellordering* on L , which is a very strong form of *global AC*. The wellorder relation is defined by a transfinite induction on the levels α . If x and y are of different levels their order is the order of their respective levels. If they are of the same level, their order is first that of the Gödel numbers of their minimal defining formulae, and then the order of their parameters (which are of lower order and therefore wellordered by the induction hypothesis). Gödel also proved the fundamental result that in ZF we have the generalized continuum hypothesis: $(V = L) \vdash GCH$.

L is in fact the *smallest* inner model of V :

- (i) $On \subset L$,
- (ii) L is *transitive*: if $y \in_V x$ and $x \in_L L$, then $y \in_L L$,
- (iii) (L, \in_L) is a model of ZF .

It can be defined in V by a sentence $L(x) = “x \text{ is constructible}”$ which is *independent* of V (ZF -absolute), and in that sense, it is a *canonical* model of ZFC .

Remark. It must be emphasized that the constructible universe L is not strictly constructive since it contains the class On of ordinals which is non-constructive. But the characteristic property of L is that it reduces non-constructivity exactly to On .

¹⁸This means that y is definable by a first-order formula $\varphi(z, a_i)$ with parameters $a_i \in x$: $z \in y \iff \varphi(z, a_i)$.

In the constructible universe L there exists a Δ_2^1 -wellorder relation $<$ on \mathbb{R} . But, according to a theorem due to Fubini, such a wellordering *cannot be Lebesgue measurable* and there exist therefore in L Δ_2^1 sets which, despite the fact they belong to the low levels of the projective hierarchy and are “simple” and “nice” to define, are nevertheless not Lebesgue measurable and therefore not well-behaved. This astonishing property is very troublesome.

With regards to CH , one uses the fact that the Δ_2^1 -wellorder relation $<$ on \mathbb{R} is a fortiori Σ_2^1 , and that the Σ_2^1 are the \aleph_1 -Suslin sets. If χ is an infinite cardinal, $P \subseteq \mathbb{R}$ is called a χ -Suslin set if it exists a closed subset $F \subseteq \mathbb{R} \times \chi^{\mathbb{N}}$ s.t. $P = \exists^{\chi^{\mathbb{N}}} F$ (i.e. P is the projection of F). The Σ_1^1 are, by definition, the \aleph_0 -Suslin sets. Indeed, if $\chi = \aleph_0$ then $P = \exists^{\mathbb{R}} F$ and therefore $P \in \Sigma_1^1$. A theorem of Martin says that $P \subseteq \mathcal{X}$ is an \aleph_n -Suslin set iff $P = \bigcup_{\xi < \aleph_n} P_\xi$ with

P_ξ Borelians.¹⁹ As the wellordering $<$ on \mathbb{R} is Σ_2^1 , according to a theorem of Schönfield, its ordinal is $< \aleph_2$ and CH is therefore valid.

In spite of its intrinsic limitations, L is a very interesting model of ZFC , which possesses a “fine structure” interpolating between the different L_α and very rich combinatorial properties investigated by Jensen. One of its main properties is the following. Let us first define what is a *club* (“closed unbounded” subset) $C \subseteq \alpha$ of a limit ordinal α : C is closed for the order topology (i.e. limits in C belong to C : if $\beta < \alpha$ and $\text{Sup}(C \cap \beta) = \beta$, then $\beta \in C$) and unbounded in α (for every $\beta < \alpha$ there exists an element $\gamma \in C$ s.t. $\beta < \gamma$). For a cardinal κ , let \square_κ be the property that there exists a sequence of clubs $C_\alpha \subseteq \alpha$ with limit ordinals $\alpha < \kappa^+$ s.t. C_α is of order type $\leq \kappa$ (and $< \kappa$ if $\text{cf}(\alpha) < \kappa$) and if λ is a limit point of C_α then $C_\alpha \cap \lambda = C_\lambda$. \square_κ is used to construct systematically and coherently bijections between κ and ordinals $\kappa \leq \alpha < \kappa^+$ by cofinalizing the α by clubs. As explain Matthew Foreman and Menachem Magidor (1987),

“[It] is useful for proving many combinatorial results, yielding a general method for passing singular limit cardinals in inductive constructions.”

In L , these constructions are always possible according to Jensen:

Theorem (Jensen, 1970). $V = L \models \forall \kappa \square_\kappa$. □

\square_κ constrains the structure of the *stationary* subsets S of κ^+ ($S \subset \kappa^+$ is stationary if $S \cap C \neq \emptyset$ for every club C in κ^+). They cannot reflect at some ordinal $\alpha < \kappa$ of cofinality $\text{cf}(\alpha) > \omega$, where “reflect” means “remaining stationary in α ”.

One can generalize the concept of constructibility in two ways.²⁰ First, if A is any set, one can relativize definability to A taking $\mathcal{D}_A(x) = \{y \subseteq x \mid y \text{ definable by a first order formula of the structure } \langle x, \in, A \cap x \rangle\}$. One gets that

¹⁹See Moschovakis [1980], p. 97.

²⁰See Kanamori [1994], p. 34.

way the universe, called $L[A]$, of constructible sets relative to A , which is the smallest inner model s.t. for every $x \in L[A]$ we have $A \cap x \in L[A]$. In $L[A]$ the only remaining part of A is $A \cap L[A] \in L[A]$. As L , $L[A]$ satisfies AC and is ZF -absolute. On the other hand, one can start the recursive construction of L not with $L_0 = \emptyset$ but with the transitive closure of $\{A\}$, $L_0(A)$. One gets that way $L(A)$ which is the smallest inner model containing On and A . If there is a wellordering on A (it the case if AC is valid), then $L(A)$ is globally wellordered for the same reasons as L . In particular, $L(\mathbb{R})$ is a good compromise between the non-constructibility of \mathbb{R} and the constructibility from \mathbb{R} of the rest of the universe.

In spite of its interest, the structure of L is rather pathological with regards to the continuum and many of the above results are in some sense counterintuitive. They result from the fact that the AC , which implies the existence of very complicated and irregular sets, remains valid in L and that the axiom of constructibility $V = L$ forces some of them to exist *inside* the projective hierarchy which should be composed only of relatively simple and regular sets: nicely definable sets are not necessarily well-behaved.

It is the reason why many specialists consider that the strategy $V = L$ is dramatically too restrictive and, moreover, not philosophically justifiable. For instance, John Steel (2000) claims:

“The central idea of descriptive set theory is that definable sets of reals are free from the pathologies one gets from a wellorder of the reals. Since $V = L$ implies there is a Δ_2^1 wellorder of the reals, under $V = L$ this central idea collapses low in the projective hierarchy, and after that there is, in an important sense, *no* descriptive set theory. One has instead infinitary combinatorics on \aleph_1 . This is certainly not the sort of theory that looks useful to Analysts.”

One could think that generalizations of constructibility such as $L(A)$ or $L[A]$ would overcome the problem. But it is not the case.

6 The “maximalist” strategy of large cardinals

It is therefore justified to reverse the strategy and to look for *additional* axioms, which could be thought of as “natural”, for ZF and ZFC , and to try to *generalize* to such augmented axiomatics the search of canonical models and fine combinatorial structures. As was emphasized by John Steel (2004):

“In extending ZFC , we are attempting to *maximize interpretative power*”.

And there is place for philosophy in such a maximizing strategy since the problem is not only to find a solution to the continuum problem but also to understand what “to be a solution” means. By the way, to study such “maximizing”

large cardinals models is perfectly compatible with a minimalist perspective: to retrieve L , one has only to relativize the theory to the constructible subuniverse L since $ZFC + “V = L” \vdash \varphi$ is equivalent to $ZFC \vdash \varphi^L$. As explained by Steel (2004), suppose that the philosopher A believes in L and the philosopher B in $L[G]$ with G forcing the adjunction of ω_2 reals to the model of \mathbb{R} in L . A believes in CH and B in $\neg CH$, but B can interpret the formulae φ of A as its own φ^L and A can interpret the formulae φ of B as forced φ (the truth of $\Vdash \varphi$ being definable in the ground model L , see above). There is therefore no real conflict.

Different “maximizing” strategies have been considered:

1. Iterate transfinitely theories $T_{\alpha+1} = T_\alpha + “consistency of T_α ”$
 starting from ZF or ZFC .
2. Postulate “good” regularity properties of projective sets, and therefore of the continuum.
3. Make the theory of the continuum “*rigid*”, that is define *under which conditions the properties of \mathbb{R} cannot be further modified by forcing*.

Strategy (3) tries to *reduce* – and even to neutralize – the variability induced by forcing. The ideal aim would be forcing invariance to make the theories of \mathbb{R} and $\mathcal{P}(\mathbb{R})$ in some sense as rigid as first order arithmetic. It is an extremely difficult program and we will first evoke some classical results concerning \mathbb{R} . CH concerns $\mathcal{P}(\mathbb{R})$ the forcing invariance of which is the object of more recent works of Woodin. But we first emphasize the fact that strategies (1), (2) converge towards the introduction of *large cardinal axioms (LCAs)* which express the existence of higher infinities. Indeed, it seems that every “maximizing” strategy is in some sense equivalent to a *LCA*. Look for instance at the *Proper Forcing Axiom PFA*. A forcing P is called *proper* if, for every regular uncountable cardinal λ , it preserves the stationary subsets of $[\lambda]^\omega$ (the set of countable subsets of λ).

Proper Forcing Axiom. If the forcing P is proper and if the D_α ’s are dense subsets indexed by the countable ordinals $\alpha < \omega_1$, then there exists a filter $G \subseteq P$ intersecting all the D_α ’s. (Compare with the definition of G being generic). \square

Many results are known for *PFA*. It implies $2^{\aleph_0} = \aleph_2$ (and therefore $\neg CH$), it implies projective determinacy *PD* (Woodin) and *AD* (Steel, 2007) for the inner model $L(\mathbb{R})$.²¹ As far as its consistency strength is concerned, it is known that²²

$$\text{Con}(\exists \kappa \text{ supercompact}) \Rightarrow \text{Con}(PFA) \Rightarrow \text{Con}(\exists \kappa \text{ measurable}).$$

²¹See below §§ 7.2 and 8 for a definition of *PD* and *AD*.

²²If A is an axiom, $\text{Con}(A)$ means “consistency of the theory $ZFC + A$ ”.

It is conjectured that in fact PFA is equiconsistent with “ $\exists \kappa$ supercompact” (see below for a definition of supercompactness).

It also seems that there exists a wellordering of LCA s which can be defined by the inclusion of their sets of Σ_2^1 (and even Π_1^0) consequences. As emphasized by John Steel (2000):

“It seems that the consistency strengths of all natural extensions of ZFC are wellordered, and the large cardinal hierarchy provides a sort of yardstick which enables us to compare these consistency strengths.”

Philosophically speaking, the nominalist confusion between a strong “quasi-ontology” for sets and a realist true ontology of abstract idealities has disqualified such axioms. But I think that such a dogmatic prejudice has been a great philosophical mistake. Indeed, one of the best philosophical formulation of incompleteness is precisely to say that a “good” theory of the continuum requires a very strong “quasi-ontology” for sets, a *maximal one*, not a minimal one. A good regularity of the continuum entails *for objective reasons* a strong platonist commitment concerning higher infinities. This key point has been perfectly emphasized by Patrick Dehornoy:

“properties which put into play objects as ‘small’ as sets of reals (...) are related to other properties which put into play very ‘huge’ objects which seem very far from them.”²³

Some specialists call “reverse descriptive set theory” this remarkable equivalence between properties of regularity of projective sets and LCA s.

There are many theorems showing that the platonist cost of a good theory is very high. Let us for instance mention one of the first striking theorems proved by Robert Solovay using forcing. Let CM be the axiom of existence of a measurable cardinal (see below for a definition).

Solovay theorem (1969). $ZFC + CM \vdash$ every Σ_2^1 is “regular” (where “regular” means properties such as Baire property, Lebesgue measurability, and perfect set property).²⁴ □

7 Regularity of projective sets

7.1 The regularity of analytic sets

The French school (Borel, Baire, Lebesgue) and the Russian and Polish schools (Suslin, Luzin, Sierpinski) initiated the study of the Borel and projective classes and achieved deep results concerning their *regularity* and their *representation*

²³Dehornoy [1989].

²⁴See Moschovakis [1980], p. 284.

where “regularity” means Lebesgue measurability, or the perfect set property (to be countable or to contain a perfect subset, i.e. a closed subset without isolated point), or the Baire property (to be approximated by an open subset up to a meager set, i.e. a countable union of nowhere dense sets).

The first regularity theorem is the celebrated:

Cantor-Bendixson theorem. *If $A \subseteq \mathbb{R}$ is closed, then A can be decomposed in a unique way as a disjoint union $A = P + S$ where P is perfect and S countable.* \square

As a perfect set P is of cardinality $|P| = 2^{\aleph_0}$, the *continuum hypothesis* CH holds for the closed sets Π_1^0 .

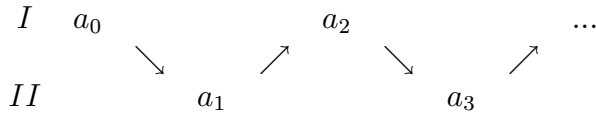
Another early great classical theorem of regularity is the:

Suslin theorem. *The analytic subsets Σ_1^1 shares the perfect subset property and CH is therefore true for the Σ_1^1 sets.* \square

In the same way, one can show that the Σ_1^1 share the Baire property and that the Σ_1^1 and Π_1^1 are Lebesgue measurable. But it is *impossible* to show in ZF that the Δ_1^1 and Σ_2^1 share the perfect set property and to show in ZFC that the Δ_2^1 share the Baire property. In fact – and it has been a revolutionary result – many of the “natural” properties of the projective sets go *far beyond* the demonstrative strength of ZF and ZFC . It is therefore methodologically and philosophically justified to look for additional axioms.

7.2 Projective determinacy and the “regularity” of the continuum

A very interesting regularity hypothesis is the so called *determinacy* property. One considers infinite games on sets X . Each player (I and II) plays in turn an element a of X :



At the end of the game we get a sequence $f \in X^{\mathbb{N}}$. Let $A \subset X^{\mathbb{N}}$. The player I (resp. II) wins the play f of the game $G = G_X(A)$ associated to A if $f \in A$ (resp. if $f \notin A$).

Definition. $A \subset X^{\mathbb{N}}$ is called determined (written $\text{Det}(A)$ or $\text{Det } G_X(A)$) if one player has a winning strategy. Therefore A is determined iff

$$\exists a_0 \forall a_1 \exists a_2 \dots (a_0, a_1, a_2, \dots) \in A.$$

Determinacy is a strong property of “regularity”. Indeed, for every $A \subset \mathbb{R}$ (\mathbb{R} being identified with $\mathcal{N} = \omega^\omega$, case where $X = \mathbb{N}$), $\text{Det}(A) \Rightarrow$ “ A satisfies the Baire and the perfect subset properties, and is Lebesgue measurable”.

The first theorem linking determinacy with the projective hierarchy has been the key result:²⁵

Gale-Stewart Theorem(1953). $ZFC \vdash$ the closed subsets A of $X^{\mathbb{N}}$ (the Π_1^0) are determined. \square

After many efforts, Donald Martin proved a fundamental theorem which concluded a first stage of the story:

Martin theorem (1975). $ZFC \vdash$ Borel sets (the $\Delta_1^1 \subset X^{\mathbb{N}}$) are determined. \square

This celebrated result shows that ZFC is a “good” axiomatic for the Borel subsets of \mathbb{R} . But, it is the *limit* of what is provable in ZFC . Indeed, ZFC cannot imply the determinacy of Σ_1^1 -sets since in the constructible model L of ZFC there exist Σ_1^1 -sets that don’t share the perfect set property. As for Π_1^1 -sets, their determinacy implies the measurability of the Σ_2^1 -sets, but in L there exists a Δ_2^1 -wellorder of \mathbb{R} , which, according to Fubini theorem, cannot be Lebesgue measurable.

So, in what concerns descriptive set theory, the conclusion is that ZFC correctly axiomatizes not general sets but only specific classes of sets such as Borelians. With ZFC we are therefore *very far* from a correct comprehension of the structure of the continuum.

8 The necessity of large cardinals and “reverse” descriptive set theory

To prove determinacy results for projective sets beyond Δ_1^1 , one must introduce additional axioms and many converging results show that the most natural are large cardinal axioms. The first example was introduced by Stan Ulam. If X is a set, a *filter* \mathcal{U} over X is a set of subsets of X , $\mathcal{U} \subseteq \mathcal{P}(X)$, s.t. the complementary set of \mathcal{U} in the Boolean algebra $\mathcal{P}(X)$ is an ideal.²⁶ \mathcal{U} is an *ultrafilter* if it is maximal, namely if for every $U \subseteq X$, either $U \in \mathcal{U}$ or $X - U \in \mathcal{U}$. For every $x \in X$, $\mathcal{U}_x = \{U \subseteq X : x \in U\}$ is an ultrafilter called “principal”. A non-principal ultrafilter is called “free”. Principal ultrafilters are in some sense trivial while free ultrafilters can explore infinite horizons. A typical exemple is the ultrafilter of the $U \subseteq \mathbb{N}$ s.t. $\mathbb{N} - U$ is finite.

Definition. A cardinal $\chi > \omega$ is measurable if it bears a free ultrafilter \mathcal{U} which is χ -complete (that is stable w.r.t. χ -infinite intersections $\bigcap_{\lambda < \chi} X_\lambda$ with $\lambda < \chi$). It is equivalent to say that χ bears a measure μ with range $\{0, 1\}$ (with $\mu(\chi) = 1$), diffuse (without atoms: $\forall \xi \in \chi$ we have $\mu(\{\xi\}) = 0$) and χ -additive. The equivalence is given by $\mu(A) = 1 \Leftrightarrow A \in \mathcal{U}$ and $\mu(A) = 0 \Leftrightarrow \chi - A \in \mathcal{U}$ \square

A measurable cardinal has “small” and “large” infinite subsets where “small” and “large” are as uncommensurable as finite and infinite in \mathbb{N} . A first typical

²⁵See Grigorieff [1976] and Moschovakis [1980], p. 288.

²⁶This means (i) $\emptyset \notin \mathcal{U}$, (ii) if $U \in \mathcal{U}$ and $U \subseteq V$ then $V \in \mathcal{U}$, (iii) if $U, V \in \mathcal{U}$ then $U \cap V \in \mathcal{U}$.

result was another theorem due to Donald Martin:

Martin theorem (1970). $ZFC + MC \vdash \text{Det}(\Sigma_1^1)$. \square

This remarkable result shows that the regularity of projective sets beyond the Borelians Δ_1^1 effectively depends upon LCA s.

Corollary: Solovay theorem (1969). $ZFC + MC \vdash$ “the Σ_2^1 -sets are ‘regular’ ”. \square

But Solovay also showed that $ZFC + MC \not\vdash PD$ (where PD is the axiom of *Projective Determinacy*: every projective $A \subseteq \mathbb{R}$ is determined, see below) since $ZFC + PD \vdash \text{Cons}(ZFC + MC)$ and therefore if $ZFC + MC \vdash PD$ we would have $ZFC + MC \vdash \text{Cons}(ZFC + MC)$, which would contradict Gödel theorem.

Scott theorem (1961). MC is false in $V = L$ and therefore $ZFC \not\vdash CM$. \square

The proof of Scott theorem uses the very important concept of an *ultrapower* $V^{\mathcal{U}}$ where \mathcal{U} is an ultrafilter on a set S . The elements of $V^{\mathcal{U}}$ are the maps $f : S \rightarrow V$, f and g being equivalent if they are equal almost everywhere (a.e.), that is if $\{s \in S : f(s) = g(s)\} \in \mathcal{U}$. Any element x of V is represented by the constant map $f(s) = x$ and this defines a canonical embedding $j : V \hookrightarrow V^{\mathcal{U}}$. If $\varphi(x_1, \dots, x_n)$ is a formula of the language of V , $\varphi(f_1, \dots, f_n)$ is valid in $V^{\mathcal{U}}$ ($V^{\mathcal{U}} \models \varphi(f_1, \dots, f_n)$) iff $\varphi(f_1, \dots, f_n)$ is valid a.e., that is if

$$\{s \in S : V \models \varphi(f_1(s), \dots, f_n(s))\} \in \mathcal{U}.$$

One shows that $V^{\mathcal{U}}$ is *well-founded* if the ultrafilter \mathcal{U} is ω_1 -complete and that there exists in that case an isomorphism between $\langle V^{\mathcal{U}}, \in_{\mathcal{U}} \rangle$ and $\langle M_{\mathcal{U}}, \in \rangle$ where $M_{\mathcal{U}}$ is an inner model (Mostowski collapsing lemma). A fundamental theorem of Łoś says that $j : V \prec M_{\mathcal{U}}$ is an elementary embedding (see above for the definition). If j is an elementary embedding $j : M \prec M^*$ of models of ZFC where M^* is an inner model of M and if $\alpha \in \text{On}(M)$ is an ordinal in M , one has $j(\alpha) \in \text{On}(M^*) \subset \text{On}(M)$ and, because of the elementarity of j , $\alpha < \beta \Leftrightarrow j(\alpha) < j(\beta)$. This implies $j(\alpha) \geq \alpha$. One shows that if j is not trivial there exists necessarily an ordinal α s.t. $j(\alpha) > \alpha$. Let χ be the smallest of these α . It is called the *critical ordinal* $\text{crit}(j)$ of j .

Theorem. If the free ultrafilter \mathcal{U} on the measurable cardinal χ is χ -complete, then $\text{crit}(j) = \chi$ and therefore $j(\chi) > \chi$. \square

Corollary: Scott theorem. \square

Indeed, suppose there exists a MC and let χ be the least MC . Now suppose that $V = L$. Elementarity implies $M_{\mathcal{U}} = L$ since $M_{\mathcal{U}}$ is an inner model satisfying the axiom of constructibility and therefore $L \subseteq M_{\mathcal{U}} \subseteq V = L$. Then in $M_{\mathcal{U}} = L = V$, $j(\chi)$ is the least MC , which contradicts $j(\chi) > \chi$. \square

Measurable cardinals χ are very large; such a χ is *regular* (there exists no unbounded $f : \lambda \rightarrow \chi$ with $\lambda < \chi$), *strongly inaccessible* ($\forall \lambda < \chi, 2^\lambda < \chi$), and preceded by χ strongly inaccessible smaller cardinals. But as large as they may be, MC s guarantee only the determinacy of the lowest post-Borelian level of

definable subsets of \mathbb{R} . To guarantee the determinacy of all projective subsets, one needs much stronger axioms, such as PD , which are not entailed by MC (see Solovay's remark above)

As we will see below, many specialists consider that Projective Determinacy is a “good” axiomatic for \mathbb{R} . Indeed, PD is “empirically complete” for the projective sets and $ZFC + PD$ “rigidifies” the properties of projective sets w.r.t. forcing: it makes them “forcing-absolute” or “generically absolute”. One can also consider the even stronger axiom (Woodin axiom) “ $L(\mathbb{R})$ satisfies AD ” where $L(\mathbb{R})$ is the constructible closure of \mathbb{R} (i.e. the smallest inner model containing On and \mathbb{R} , see above) and the *Axiom of Determinacy* AD means that *every* subset (not necessarily projective) of \mathbb{R} is determined. AD is incompatible with AC since, according to Fubini theorem, AC enables to construct non-Lebesgue measurable, and therefore non-determined, subsets of \mathbb{R} .

9 The transcendence of \mathbb{R} over L and the set $0^\#$ (0 sharp)

9.1 Indiscernible ordinals

Once we accept the relevance and the legitimacy of $LCAs$, we need some tools for measuring the transcendence of V over L . A first possibility is given by what are called *indiscernible* ordinals (Silver, 1966) which enable to construct the simplest canonical non-constructible real. We consider the levels of L of the form $\langle L_\lambda, \in \rangle$ with λ a limit ordinal. A set I of ordinals in this cumulative hierarchy L_λ of constructive sets up to level λ is called a set of indiscernibles if, for every n -ary first-order formula $\varphi(x_1, \dots, x_n)$, the validity of φ on I is independent of the choice of the x_i 's: that is for every sequences $c_1 < \dots < c_n$ and $d_1 < \dots < d_n$ in I

$$L_\lambda \models \varphi(c_1, \dots, c_n) \text{ iff } L_\lambda \models \varphi(d_1, \dots, d_n).$$

This means that the ordinals of I cannot be separated using first-order formulae.

When it exists, the set S of *Silver indiscernibles* is characterized by the following properties, which express that, for all *uncountable* cardinals κ , all the L_κ 's share essentially the same first-order structure:

1. $\kappa \in S$ (all uncountable cardinals of V are indiscernible in L).
2. $S_\kappa = S \cap \kappa$ is of order-type κ and S_κ is closed and unbounded (club, see above) in κ if κ is regular (S_κ is nicely distributed in κ up to its horizon).
3. $S_\kappa = S \cap \kappa$ is a set of indiscernibles for $\langle L_\kappa, \in \rangle$ (the relation between S_κ and L_κ increases nicely up to the limit relation between S and L).

4. S_κ generates L_κ in the sense that the *Skolem hull* of $S_\kappa = S \cap \kappa$ in L_κ is equal to L_κ : $\text{Hull}^{L_\kappa}(S_\kappa) = L_\kappa$, where the Skolem hull of $I \subset L_\kappa$ is constructed by adding for every $(n+1)$ -ary formula $\varphi(y, x_1, \dots, x_n)$ with $x_i \in I$ a Skolem term $t_\varphi(x_1, \dots, x_n)$ which is the smallest y (for the wellorder of L) s.t. $\varphi(y, x_1, \dots, x_n)$ if such an y exists and 0 otherwise. In other words every constructible element $a \in L_\kappa$ is *definable* by a definite description with indiscernibles parameters in S_κ .

This can be generalized to structures $\mathcal{M} = \langle M, E \rangle$ with a binary relation E looking like \in (that is, which are elementary equivalent to some $\langle L_\lambda, \in \rangle$ for λ a limit ordinal) and with $I \subset M$. In that case, we have $\text{Hull}^{\mathcal{M}}(I) \prec \mathcal{M}$ and in fact $\text{Hull}^{\mathcal{M}}(I)$ is the smallest elementary substructure of \mathcal{M} containing I . Let $\Sigma = \Sigma(\mathcal{M}, I)$ be the set of formulae φ which can be satisfied by \mathcal{M} on I . This defines particular sets of formulae called *EM-sets* (from Ehrenfeucht-Mostowski, 1956). The EM theorem says that if Σ is a theory having infinite models and if $\langle I, < \rangle$ is any total well-ordering of infinite order-type $\alpha \geq \omega$, then there exists a model \mathcal{M} of Σ containing I for which I is a set of indiscernibles, and moreover, \mathcal{M} can be chosen in such a way as to be the *Skolem hull* of I : $\mathcal{M} = \text{Hull}^{\mathcal{M}}(I)$. Such an (\mathcal{M}, I) is essentially unique and its transitive collapse (isomorphism with a structure where E becomes \in) is written $(\mathcal{M}(\Sigma, \alpha), I(\Sigma, \alpha))$ where α is the order type of I ($I(\Sigma, \alpha)$ is therefore a set of true \in -ordinals).

One can develop a theory of *EM-sets* and of their well-foundedness. If Σ is well-founded (i.e. if $\mathcal{M}(\Sigma, \alpha)$ is well-founded for every ordinal α) and if α is a limit ordinal, then $\mathcal{M}(\Sigma, \alpha)$ is isomorphic to a $\langle L_\lambda, \in \rangle$. Moreover, if $I(\Sigma, \kappa)$, with $\kappa > \omega$ an uncountable cardinal, is unbounded in the class of ordinals of $\mathcal{M}(\Sigma, \alpha)$ (and it is then the case for every ordinal $\alpha > \omega$), and if for every ordinal $\gamma < i_\omega$ (the ω -th element of $I(\Sigma, \kappa)$) we have $\gamma \in \text{Hull}^{\mathcal{M}}(\{i_n\})$ (and it is then the case for every ordinal $\alpha > \omega$), then $\mathcal{M}(\Sigma, \kappa) = \langle L_\kappa, \in \rangle$, $I(\Sigma, \kappa)$ is closed unbounded in κ and if $\tau > \kappa > \omega$ then $I(\Sigma, \tau) \cap \kappa = I(\Sigma, \kappa)$.

Let us return to Silver indiscernibles. If such a Σ exists, S is defined by

$$S = \bigcup \{I(\Sigma, \kappa) : \kappa \text{ uncountable cardinal}\}.$$

The uniqueness of S is a consequence of the unicity of such a Σ :

Theorem. *Such a Σ is unique and is the set of n -ary sentences φ s.t. $L_{\aleph_\omega} \models \varphi(\aleph_1, \dots, \aleph_n)$. It is called “zero sharp” and written $0^\#$ (see below). \square*

If there exists an uncountable limit cardinal κ s.t. $\langle L_\kappa, \in \rangle$ possesses an uncountable set I of indiscernibles, then S exists. The existence of S is also implied by large cardinal hypothesis as for instance:

Theorem. *If there exists a MC then S exists and moreover $L_\kappa \prec L_\lambda$ for every uncountable $\kappa < \lambda$. \square*

The existence of S under *MC* means that after the first uncountable level L_{\aleph_1} all the L_κ share essentially the same first-order theory. V transcends L ,

but in such a way that it makes L as simple as possible, the level $L_{\aleph_1} = \text{Hull}^{L_{\aleph_1}}(S \cap \aleph_1)$ determining the whole theory of L .

A deep consequence is that the *truth* in L becomes definable in V . Indeed, let $\varphi(x_1, \dots, x_n)$ be a formula. There exists an uncountable cardinal κ s.t.

$$\text{for all } (x_i) \in L_\kappa, L \models \varphi(x_i) \text{ iff } L_\kappa \models \varphi(x_i) .$$

As $L_\kappa \prec L_\lambda$ if $\kappa < \lambda$, we have

$$L \models \varphi(x_i) \text{ iff } L_\lambda \models \varphi(x_i) \text{ for all } \lambda \geq \kappa .$$

Now, we arithmetize the situation. Let $T = \{\ulcorner \varphi \urcorner : L_{\aleph_1} \models \varphi\}$ be the set of Gödel numbers of the φ valid in L_{\aleph_1} and therefore in all the L_κ (κ uncountable) by elementarity. Then

$$L \models \varphi \text{ iff } \ulcorner \varphi \urcorner \in T$$

defines the truth in L . This is not in contradiction with Gödel-Tarski incompleteness theorems since \aleph_1 and T are *not definable* in L and therefore the truth of L is not definable in L .

9.2 The set $0^\#$

As $L_{\aleph_\omega} \prec L$ and $\aleph_i \in S$ for $i > 0$ in ω , we can represent the indiscernibles in formulae by some of the \aleph_i 's and restrict to L_{\aleph_ω} , which contains all the \aleph_i 's. Then, $L \models \varphi(x_i)$ for $x_i \in S$ iff $L_{\aleph_\omega} \models \varphi(\aleph_i)$. Solovay called $0^\#$ (zero sharp) the set (if it exists) defined by

$$0^\# = \{\varphi : L_{\aleph_\omega} \models \varphi(\aleph_i)\}$$

which is the set of formulae true on the indiscernibles of L . Via Gödelization $0^\#$ becomes a set of integers (also written $0^\#$) and can therefore be coded by a real (also written $0^\#$).

We must emphasize the fact that, as $L_{\aleph_1} \prec L$, every constructible set $x \in L$ which is *definable* in L is *countable* since its definite description is valid in L_{\aleph_1} by elementarity and therefore $x \in L_{\aleph_1}$. More generally for every infinite constructible set $x \in L$ we have $|\mathcal{P}(x)|^L = |x|$. Since the existence of a measurable cardinal implies that $0^\#$ exists, we have:

Corollary. *If there exists a MC, the constructible continuum \mathbb{R}^L is countable.* \square

Via arithmetization through Gödel numbers, the non-constructible set $0^\#$ can be considered as a very special subset of $\omega = \mathbb{N}$ which does not belong to L , or as a very special real number coding the truth in L . Its existence implies that every uncountable cardinal κ of V is an indiscernible of L and shares *all* large cardinal axioms verified by L .

A property equivalent to the existence of $0^\#$ is the *non-rigidity* of L :

Theorem (Kunen). $0^\#$ exists iff there exists a non-trivial elementary embedding $j : L \prec L$ (this presuppose $V \neq L$ and j non-trivial, see below). \square

Indeed, as $\text{Hull}^L(S) = L$, for every $x \in L$ there exists a Skolem term t s.t. $x = t(i_{\alpha_1}, \dots, i_{\alpha_n})$, i_α being the α -th element of S . j is then simply defined by the *shift* on indiscernibles

$$j(x) = j(t(i_{\alpha_1}, \dots, i_{\alpha_n})) = t(i_{\alpha_1+1}, \dots, i_{\alpha_n+1}).$$

One shows that it is an elementary embedding and, as $j(i_0) = i_1 \neq i_0$, j is non-trivial. \square

By the way, this proves again that $V \neq L$ since another theorem of Kunen proves that

$$ZFC \vdash \text{there exists no } j : V \prec V.$$

The existence of $0^\#$ is a principle of transcendence of V over L expressing that V is very different from L . If $0^\#$ doesn't exist, then V looks like L (L is a good approximation of V) according to the result:

Covering lemma (Jensen). *If $0^\#$ doesn't exist, then if x is an uncountable set of ordinals there exists a constructible set $y \supseteq x$ of the same cardinality as x . So, every set x of ordinals can be covered by a constructible set $y \supseteq x$ of cardinality $|y| = |x| \cdot \aleph_1$.* \square

Corollary. *If $0^\#$ doesn't exist, the covering lemma implies that, for every limit singular cardinal κ of V , we have $(\kappa^+)^L = \kappa^+$, which shows that V and L are quite similar.* \square

Indeed (see Jech [1978], p. 358), if $\lambda = (\kappa^+)^L$ and if $\lambda < \kappa^+$, then $|\lambda| = \kappa$. But as κ is singular, we would have $\text{cf}(\lambda) < |\lambda|$ and this is impossible since λ is regular in L and $\lambda \geq \omega_2$. For, if x is an unbounded subset of λ of cardinal $|x| = \text{cf}(\lambda)$, it can be covered by a *constructible* subset $y \in L$ of λ of cardinal $|y| = |x| \cdot \aleph_1$ and, as λ is regular in L , $|y| = |\lambda|$. So, $|\lambda| = \aleph_1 \cdot \text{cf}(\lambda)$ and, as $\lambda \geq \omega_2$, $|\lambda| = \text{cf}(\lambda)$. \square

Corollary. *If GCH fails at a strong limit singular cardinal, then $0^\#$ exists.* \square

Indeed, if $0^\#$ doesn't exist, for such a singular cardinal κ we have $(\kappa^+)^L = \kappa^+$. As L satisfies GCH , $(2^\kappa)^L = (\kappa^+)^L = \kappa^+$.²⁷ Now, κ is a strong limit by hypothesis (i.e. $\lambda < \kappa \Rightarrow 2^\lambda < \kappa$) and this implies $\kappa^{\text{cf}(\kappa)} = 2^\kappa$ and moreover, since κ is singular and therefore $\text{cf}(\kappa) < |\kappa|$, $2^{\text{cf}(\kappa)} < \kappa$. Let $x \in A = [\kappa]^{\text{cf}(\kappa)}$ be a subset of κ of cardinal $\text{cf}(\kappa)$. It is covered by a constructible subset $y \in L$ of κ of cardinal $|y| = \lambda = \aleph_1 \cdot \text{cf}(\kappa)$. Then A can be covered by the union of the $[y]^{\text{cf}(\kappa)}$ for such Y . But $\left|[y]^{\text{cf}(\kappa)}\right| = \lambda^{\text{cf}(\kappa)} = (\aleph_1 \cdot \text{cf}(\kappa))^{\text{cf}(\kappa)} = 2^{\text{cf}(\kappa)}$, and by hypothesis $2^{\text{cf}(\kappa)} < \kappa$. Now, there exist at most $|\mathcal{D}(\kappa)|$ such y ²⁸ and $|\mathcal{D}(\kappa)| = (\kappa^+)^L = \kappa^+$.

²⁷As, if $0^\#$ exists, $(2^\kappa)^L = \kappa$ for every infinite cardinal since κ^+ is inaccessible in L , $(2^\kappa)^L = \kappa^+ > \kappa$ is a counter example when $0^\#$ doesn't exist.

²⁸Recall that $\mathcal{D}(x)$ is the set of constructible parts of x (see above).

All this implies $|A| = \kappa^{\text{cf}(\kappa)} = \kappa^+$. As $\kappa^{\text{cf}(\kappa)} = 2^\kappa$, we have $2^\kappa = \kappa^+$, that is GCH . \square

There are many ways for insuring that $0^\#$ exists, for instance PFA since PFA implies the failure of \square_κ for every κ (Todorćević, 1980). More generally, many results show that a failure of \square_κ is linked with $LCAs$. For instance:

- Jensen: if \square_κ fails for some singular κ , there exists an inner model M with a strong cardinal.
- Solovay: if κ is supercompact, then \square_λ fails for every $\lambda > \kappa$.

This is related to the fact that properties such as the covering between V and M allow to reflect \square_κ from M to V and therefore, if \square_κ fails in V for κ singular it is because there exist some LCs violating the covering lemma.

When $0^\#$ exists, a very interesting structure to look at is $L[0^\#]$. It can be shown²⁹ that $L[0^\#]$ has a fine structure, satisfies the Global Square property $\forall \kappa \square_\kappa$ and, for every singular cardinal κ of V , $(\kappa^+)^{L[0^\#]} = \kappa^+$. Iterating the sharp operation, one can get an increasing sequence of models M mildly transcendent over L which have also a fine structure, satisfy the Global Square property and, for every singular cardinal κ of V , $(\kappa^+)^M = \kappa^+$, and are smaller than the first inner model possessing a measurable cardinal. Under the hypothesis that $a^\#$ exists for every set of ordinals a ($a^\#$ is defined as $0^\#$ but in $L[a]$), one can even go beyond this limit, up to the existence of a supercompact cardinal.

9.3 $0^\#$ and the hierarchical structure of V beyond L .

The equivalence between the existence of $0^\#$ and the existence of a non-trivial elementary embedding $j : L \prec L$ enables to clarify the structure of V beyond L ³⁰. Let $\kappa = \text{crit}(j)$ and let \mathcal{U} be the set of subsets $u \subseteq \kappa$ s.t. $u \in L$ (i.e. $u \in L \cap \mathcal{P}(\kappa)$) and $\kappa \in j(u)$. \mathcal{U} is trivially a filter. It is an ultrafilter since for every $u \in L \cap \mathcal{P}(\kappa)$ either $\kappa \in j(u)$ or $\kappa \notin j(u)$. It is a free ultrafilter since if $u \subset \kappa$ is bounded, then $\kappa \notin u$, $j(u) = u$, $\kappa \notin j(u)$ and therefore $u \notin \mathcal{U}$. Moreover, it is κ -complete w.r.t. L since if $u_\alpha \in \mathcal{U} \cap L$ is a family with $\alpha < \beta < \kappa$ then $\bigcap_{\alpha < \beta} u_\alpha \in \mathcal{U}$. One can show that the ultrapower $L^\mathcal{U}$ is well-founded. Due to Łoś theorem, the embedding $i : L \prec L^\mathcal{U}$ is elementary. But the Mostowski collapsing lemma implies that $\langle L^\mathcal{U}, \in_\mathcal{U} \rangle \simeq \langle M_\mathcal{U}, \in \rangle$ for some transitive inner model $M_\mathcal{U}$. But necessarily $M_\mathcal{U} = L$ by minimality of L and via this isomorphism $i : L \prec L^\mathcal{U}$ becomes an elementary embedding $j : L \prec L$. It can be shown that, if $\lambda = (\kappa^+)^L$ then $\mathcal{M} = \langle L_\lambda, \in, \mathcal{U} \rangle$ is a model of $ZF - \{\text{Power Set axiom}\}$ where κ becomes the largest cardinal, \mathcal{U} remains a free ultrafilter κ -complete with

²⁹See e.g. Steel [2001].

³⁰See e.g. Schimmerling [2001].

“good” technical properties (“normality” and “amenability”). Such a procedure can be *iterated* on the ordinals. Starting from a $\mathcal{M}_0 = \langle L_{\lambda_0}, \in, \mathcal{U}_0 \rangle$, one gets a $\mathcal{M}_1 = \langle L_{\lambda_1}, \in, \mathcal{U}_1 \rangle$, etc. The successive \mathcal{M}_α yield a sequence of critical cardinals κ_α which are indiscernibles for L .

10 Determination and reflection phenomena

To measure the size of large cardinals, the best way is to use associated *reflection* phenomena which are of a very deep philosophical value.³¹ Intuitively, reflection means that the properties of the whole universe V are reflected in sub-universes. As was emphasized by Matthew Foreman (1998, p. 6):

“Any property that holds in the mathematical universe should hold of many set-approximations of the mathematical universe.”

Definition. A cardinal χ reflects a relation $\Phi(x, y)$ defined on ordinals if every solution $y \geq \chi$ parametrized by $x < \chi$ can be substituted for by a solution $y < \chi$:

$$\forall \alpha (\in On) < \chi \ [\exists \beta \geq \chi \ \Phi(\alpha, \beta) \Rightarrow \exists \beta^* < \chi \ \Phi(\alpha, \beta^*)].$$

Let j be an elementary embedding $j : M \prec M^*$. $\chi = \text{crit}(j)$ is a large – in fact at least measurable – cardinal, which increases indefinitely when M^* moves near to M , the limit $M^* = M$ being inconsistent according to Kunen theorem.

To see that it is a reflection phenomenon, let $\Phi(\alpha, \chi)$ be a relation that holds in M for $\alpha < \chi$. If M^* is sufficiently close to M for $\Phi(\alpha, \chi)$ to remain true in M^* , then $M^* \models \exists (x < j(\chi)) \Phi(\alpha, x)$ (it is sufficient to take $x = \chi$). But, according to the elementarity of the embedding j , this is equivalent to $M \models \exists (x < \chi) \Phi(\alpha, x)$.

To go beyond measurable cardinals, specialists use the following technique. Let V_α be the cumulative hierarchy of sets up to level α . For χ critical (and therefore measurable), one has $V_\chi^{M^*} = V_\chi^M$ (that is the equality of M and M^* up to level χ).

Definition. The cardinal χ is called *superstrong* in M if there exists an elementary embedding j s.t. $V_{j(\chi)}^{M^*} = V_{j(\chi)}^M$ (that is $V_{j(\chi)}^{M^*} \subset M$ and $M = M^*$ up to $j(\chi)$ and not only up to χ). \square

Between measurable and superstrong cardinals, Hugh Woodin introduced another class of large cardinals.

Definition. A cardinal δ is called a *Woodin cardinal* if for every map $F : \delta \rightarrow \delta$, there exists $\kappa < \delta$ and an elementary embedding j of critical ordinal κ s.t. $F|_\kappa : \kappa \rightarrow \kappa$ and $V_{j(F(\kappa))}^{M^*} = V_{j(F(\kappa))}^M$ (that is $M = M^*$ up to $j(F(\kappa))$). \square

Woodin has shown that:

³¹See Martin-Steel [1989], Patrick Dehornoy [1989].

1. if δ is a Woodin cardinal, there exist infinitely many smaller measurable cardinals $\chi < \delta$,
2. if λ is a superstrong cardinal, there exist infinitely many smaller Woodin cardinals $\delta < \lambda$.

A key result is the Martin-Steel theorem which evaluates exactly the “cost” of determinacy:

Martin-Steel theorem (1985). *If there exist n Woodin cardinals δ_i , $i = 1, \dots, n$, dominated by a measurable cardinal κ ($\kappa > \delta_i$ for all i), then $ZFC \vdash \text{Det}(\Pi^1_{n+1})$.* \square

The converse is due to Woodin.

Corollary. *If there exists a countable infinity of Woodin cardinals dominated by a measurable cardinal, in particular if there exists a superstrong cardinal λ , then Projective Determinacy is valid (all the projective subsets of \mathbb{R} are determined).* \square

Projective Determinacy is also valid under PFA (Woodin, see above).

It is for this reason that specialists consider that $ZFC + \text{Projective Determinacy}$ is a “good” axiomatic for \mathbb{R} in the sense of descriptive set theory. We must also emphasize the:

Martin-Steel-Woodin theorem (1987). *If there exists a countable infinity of Woodin cardinals dominated by a measurable cardinal, in particular if there exists a superstrong cardinal λ , then $L(\mathbb{R})$ (the smallest inner model of V containing the ordinals On and \mathbb{R} , see above) satisfies the axiom of complete determinacy AD : every $A \subseteq \mathbb{R}$ is determined. (This result is stronger than the previous one since $\mathcal{P}(\mathbb{R}) \cap L(\mathbb{R})$ is a larger class than the projective class.)* \square

AD is incompatible with AC since AC enables the construction of a non-determined well ordering on \mathbb{R} (see above).³²

But the most significative results concern perhaps the situation where no property of \mathbb{R} could be further modified in a *forcing* extension. In that case, the theory of the continuum becomes *rigid* w.r.t. forcing. Woodin and Shelah have shown that it is possible to approximate this ideal goal if there exists a *supercompact* cardinal κ . κ is γ -supercompact if there exists an elementary embedding $j : V \prec M$ s.t. $\text{crit}(j) = \kappa$, $\gamma < j(\kappa)$ and $M^\gamma \subseteq M$. κ is supercompact if it is γ -supercompact for every $\gamma \geq \kappa$ (κ is κ -supercompact iff it is measurable).³³ Such a deep result clarifies the nature of the axioms which are needed for a good theory of the continuum.

³²So, the inner model $L(\mathbb{R})$ of a ZFC -model V can violate AC .

³³See Dehornoy [2003].

11 Woodin's Ω -logic

Up to now, we looked for extensions of ZFC by large cardinal axioms ($LCAs$) which were “good” in the perspective of the descriptive set theory of the continuum. But if $LCAs$ can decide properties of regularity of \mathbb{R} , they *cannot* settle CH since a “small” forcing (adding \aleph_2 new subsets to ω) is sufficient to force $\neg CH$ from a CH -model and such a small forcing remains possible irrespective of what $LCAs$ are introduced (Levy-Solovay theorem). We need therefore a new strategy. As we have just seen, the most natural one is to try to make the properties of the continuum *immune* relatively to forcing, that is to make the continuum in some sense *rigid*. The deepest contemporary results in this perspective are provided by Woodin's extraordinary recent works on Ω -logic and the negation of CH .

We look for theories sharing *some absoluteness properties relatively to forcing*. This special kind of relative absoluteness is called “*conditional generic absoluteness*”.³⁴

The fragment of V where CH “lives” naturally is (H_2, \in) where (H_k, \in) is the set of sets x which are hereditary of cardinal $|x| < \aleph_k$. The fragment $(H_0, \in) = V_\omega$ is the set of hereditary finite sets and, with the axioms ZF minus the axiom of infinity, is equivalent to first order arithmetic $\langle \omega = \mathbb{N}, +, \cdot, \in \rangle$ with Peano axioms. In one direction, \mathbb{N} can be retrieved from H_0 using von Neumann's construction of ordinals and, conversely, H_0 can be retrieved from \mathbb{N} via Ackermann's trick: if p, q are integers, $p \in q$ iff the p -th digit in the binary extension of q is 1. For first order arithmetic, Peano axioms are “empirically” and practically complete in spite of Gödel incompleteness theorem. The following classical result expresses their rigidity:

Schönfield theorem. *H_0 is absolute and a fortiori forcing-invariant. Incompleteness cannot be manifested in it using forcing.* \square

We can therefore consider ZFC as a “good” theory for first order arithmetic. But it is no longer the case for larger fragments of V .

The fragment (H_1, \in) of V composed of countable sets of finite ordinals is isomorphic to $\langle \mathcal{P}(\omega) = \mathbb{R}, \omega, +, \cdot, \in \rangle$ and corresponds to second order arithmetic (i.e. analysis). The definable subsets $A \subseteq \mathcal{P}(\omega)$ are the projective subsets³⁵ and therefore H_1 can be considered as the fragment of V where the projective sets live. We have seen that to settle and “freeze” most of its higher order properties (regularity of projective sets) w.r.t. forcing, we need $LCAs$ and in particular projective determinacy PD .

As is emphasized by Woodin (2003, quoted in Dehornoy [2003]):

³⁴See Steel [2004]: “Generic absoluteness and the continuum problem”.

³⁵A set $A \subseteq \mathcal{P}(\omega)$ is definable in H_1 (with parameters in H_1) iff there exists a first-order formula $\varphi(x, y)$ and a parameter $b \in \mathcal{P}(\omega)$ s.t. $A = \{a \in \mathcal{P}(\omega) \mid H_1 \models \varphi(a, b)\}$. If $\pi : \mathcal{P}(\omega) \rightarrow [0, 1] \simeq \mathbb{R}$ is given by $\pi(a) = \sum_{i \in a} \frac{1}{2^i}$, then $X \subseteq [0, 1]$ is projective iff $A = \pi^{-1}(X)$ is definable in H_1 .

“*Projective Determinacy* settles (in the context of *ZFC*) the classical questions concerning the projective sets and moreover Cohen’s method of forcing *cannot* be used to establish that questions of second order number theory are formally unsolvable from this axiom. (...) I believe the axiom of *Projective Determinacy* is as true as the axioms of Number Theory. So I suppose that I advocate a position that might best be described as *Conditional Platonism*.”³⁶

We have also seen that under the *LCA* “there exists a proper class of Woodin cardinals” (*PCW*: for every cardinal κ there exists a Woodin cardinal $> \kappa$) we have:

Theorem (Woodin, 1984). *$ZFC + PCW \vdash H_1$ is immune relatively to forcing in the sense its properties are forcing-invariant.* \square

As *PCW* implies at the same time *PD* and forcing-invariance for H_1 , it can be considered as a “good” theory, “empirically” and “practically” complete (marginalizing incompleteness) for (H_1, \in) , that is for analysis (second order arithmetic). *PCW* implies the *generic completeness* result that all the $L[\mathbb{R}]$ of generic extensions $V[G]$ are elementary equivalent.

The idea is then to try to generalize properties of absoluteness relative to forcing. The general strategy for deciding that way *ZFC*-undecidable properties φ in a fragment H of V is described in the following way by Patrick Dehornoy (2003):

“every axiomatization freezing the properties of H relatively to forcing (i.e. neutralizing forcing at the level H) implies φ ”.

The main problem tackled by Woodin was to apply this strategy to the fragment (H_2, \in) of V which is associated to the set $\mathcal{P}(\omega_1)$ of countable ordinals. $\mathcal{P}(\omega_1)$ is not $\mathcal{P}(\mathbb{R})$ if $\neg CH$ is satisfied, but nevertheless it is possible to code CH by an H_2 -formula φ_{CH} s.t. $H_2 \models \varphi_{CH}$ is equivalent to CH .³⁷ The problem with H_2 is that “small” forcings preserve *LCAs*³⁸ and in particular (Levy-Solovay theorem, 1967) a small forcing of cardinal \aleph_2 that enables to violate CH by adding \aleph_2 subsets to \mathbb{N} preserves *LCAs*. Therefore H_2 *cannot* be rigidified by *LCAs*.³⁹ Whatever the large cardinal hypothesis A may be, there will be always generic extensions M and N of V both satisfying A such that $M \models CH$

³⁶We will return on this beautiful concept in our conclusion.

³⁷The point is rather technical. Woodin has shown that if $\neg CH$ is valid (i.e. $\mathbb{R} > \omega_1$), then $\mathcal{P}(\mathbb{R})$ doesn’t belong to H_2 and is already too big for freezing (neutralizing the effects of forcing) the fragments of V containing it.

³⁸Large cardinal axioms are axioms of the form $A = \exists \kappa \psi(\kappa)$ which share the property that if $V \models A$ then the cardinal κ is inaccessible and $\psi(\kappa)$ is forcing-invariant for every forcing extension $V[G]$ of forcing cardinal $< \kappa$ (“small” forcings).

³⁹See Dehornoy [2003].

and $N \models \neg CH$. As CH is equivalent to a Σ_2^1 formula, M and N cannot be elementary equivalent after the Σ_2^1 level.

Woodin's fundamental idea to overcome the dramatic difficulties of the problem at the H_2 level was to *strengthen logic* by restricting the admissible models and constructing a new logic adapted to forcing-invariance or "*generic invariance*". As he explains in his key paper on "The continuum hypothesis" (2001, p. 682):

"As a consequence (of generic invariance), any axioms we find will yield theories for $\langle H[\omega_2], \in \rangle$, whose 'completeness' is immune to attack by applications of Cohen's method of forcing, just as it is the case for number theory."

In a first step, he introduced the notion of Ω -validity \models_Ω also called at the beginning Ω^* -derivability \vdash_{Ω^*} .

Definition. *T being a theory in ZFC , we have $T \models_\Omega \varphi$ iff φ is valid in every generic extension where T is valid, that is iff for every generic extension $V[G]$ and every level α , $(V_\alpha)^{V[G]} \models T$ implies $(V_\alpha)^{V[G]} \models \varphi$. \square*

Of course \models implies \models_Ω . But the converse is trivially false: there exists Ω -valid formulae which are undecidable in ZFC , for instance $\text{Con}(ZFC)$. Indeed, if $(V_\alpha)^{V[G]} \models ZFC$ then $(V_\alpha)^{V[G]}$ is a model of ZFC and $(V_\alpha)^{V[G]} \models \text{Con}(ZFC)$. So, $ZFC \models_\Omega \text{Con}(ZFC)$, but of course (Gödel) $ZFC \not\models \text{Con}(ZFC)$.

It must be emphasized that Ω -validity *doesn't* satisfy the *compactness* property: there exist theories T and formulae φ s.t. we have $T \models_\Omega \varphi$ even if for every finite subset $S \subset T$ we have $S \not\models_\Omega \varphi$.⁴⁰

By construction, Ω -validity \models_Ω is itself forcing-invariant.⁴¹

Theorem ($ZFC + PCW$). *If $V \models "T \models_\Omega \varphi"$ then $V[G] \models "T \models_\Omega \varphi"$ for every generic extension of V . \square*

Woodin investigated deeply this new "strong logic". In particular he was able to show that, under suitable $LCAs$, CH rigidifies V at the Σ_1^2 -level (Σ_1 formulae for $V_{\omega+2}$):

Theorem (Woodin, 1984). *Under PCW_{meas} (there exists a proper class of measurable Woodin cardinals) and CH , Ω -logic is generically complete at the Σ_1^2 -level: for every φ of complexity Σ_1^2 either $ZFC+CH \models_\Omega \varphi$ or $ZFC+CH \models_\Omega \neg\varphi$. All generic extensions M and N of V satisfying both CH are Σ_1^2 elementary equivalent. \square*

The metamathematical meaning of this result of *conditional generic absoluteness* is that if a problem is expressed by a Σ_1^2 -formula φ then it is "settled by CH " and immunized against forcing under appropriate $LCAs$. But:

⁴⁰See Bagaria et al. [2005].

⁴¹See Woodin [2004].

Theorem (Abraham, Shelah). *This is false at the Σ_2^2 level. For every large cardinal hypothesis A there exist generic extensions M and N satisfying both CH s.t. in M there exist a Σ_2^2 -wellorder of \mathbb{R} while in N all the Σ_2^2 -subsets of \mathbb{R} are Lebesgue measurable.* \square

In a second step, Woodin interpreted the Ω -validity $T \models_\Omega \varphi$ as the semantic validity for an Ω -logic whose syntactic derivation $T \vdash_\Omega \varphi$ had to be defined. His idea was to witness the Ω -proofs by particular sets that, under PCW , generalize the projective sets and can be interpreted without ambiguity in every generic extension. It is the most difficult part of his work, not only at the technical level but also at the philosophical level. The definition (under PCW) is the following:

Definition (PCW). $T \vdash_\Omega \varphi$ iff there exists a universally Baire (UB) set $A \subseteq \mathbb{R}$ s.t. for every A -closed countable transitive model (ctm) M of T we have $M \models \varphi$ (in other words $M \models "T \models_\Omega \varphi"$). \square

A subset $A \subseteq \mathbb{R}$ of \mathbb{R} is called UB if for every continuous map $f : K \rightarrow \mathbb{R}$ with source K compact Hausdorff, $f^{-1}(A)$ has the Baire property (there exists an open set U s.t. the symmetric difference $f^{-1}(A) \Delta U$ is meager). If $A \subseteq \mathbb{R}$ is UB , it is interpreted canonically in every generic extension $V[G]$ as $A_G \subseteq \mathbb{R}^{V[G]}$. This is due to the fact that there exists a *tree presentation* of A . One identifies \mathbb{R} with ω^ω and one considers trees $T \subset (\omega \times \gamma)^\omega$ and the projections $p[T]$ on ω^ω of their infinite branches:

$$p[T] = \{x \in \omega^\omega \mid \exists z \in \gamma^\omega \text{ with } (x \upharpoonright_n, z \upharpoonright_n) \in T, \forall n \in \omega\}.$$

A subset $A \subseteq \mathbb{R}$ is UB iff there exist trees T and S s.t. $p[T] = A$ and $p[S] = \omega^\omega - A$ in every generic extension $V[G]$. $p[T]$ yields a canonical interpretation of A in every generic extension $V[G]$. A ctm M is called A -closed if, for every ctm $N \supseteq M$, $A \cap N \in N$, in particular for every generic extension $V[G]$ and $N = M[G]$ we have $A \cap M[G] \in M[G]$. If A is Borelian, every ctm is always A -closed.⁴² But it is no longer the case for general UB sets.

As far as, in the definition of $T \vdash_\Omega \varphi$, the class of admissible models is *restricted* to A -closed $ctms$, logic becomes strengthened. Of course, $T \vdash \varphi$ implies $T \vdash_\Omega \varphi$, but the converse is false for the same reasons as for \models_Ω . Indeed, $ZFC \vdash_\Omega \text{Con}(ZFC)$ because every suitable ctm provides a model of ZFC and validates therefore $\text{Con}(ZFC)$.

More technically, what is really needed for the definition $T \vdash_\Omega \varphi$ are UB sets $A \subseteq \mathbb{R}$ sharing the following two properties:

1. $L(A, \mathbb{R}) \models AD^+$, where AD^+ is a strengthening of the axiom of determinacy saying that not only all $A \subseteq \mathbb{R} \simeq \omega^\omega$ are determined, but also all the $\pi^{-1}(A)$ for all maps $\pi : \lambda^\omega \rightarrow \omega^\omega$ with an ordinal $\lambda < \mathfrak{c}^+$;

⁴²If M is A -closed for every A of Π_1^1 -complexity, then M is well-founded.

2. every $A \subset \mathcal{P}(\mathbb{R}) \cap L(A, \mathbb{R})$ is UB .⁴³

PCW implies these two properties and is therefore a good hypothesis.

It must be emphasized that this definition of Ω -provability is very original. As explain Joan Bagaria, Neus Castells and Paul Larson in their “ Ω -logic primer”:

“The notion of Ω -provability differs from the usual notions of provability, e.g., in first-order logic, in that there is no deductive calculus involved. In Ω -logic, the same UB set may witness the Ω -provability of different sentences. For instance, all tautologies have the same *proof* in Ω -logic, namely \emptyset . In spite of this, it is possible to define a notion of height of proof in Ω -logic.”

As Patrick Dehornoy explained to me (private communication), in Ω -logic a proof $\vdash_{\Omega} \varphi$ is a certificate of some property of the formula φ . This witnessing is no longer a derivation iterating syntactic rules but a UB subset of \mathbb{R} . What is common to classical and Ω -logics is that a very “small” object endowed with a precise internal structure warrants the validity of φ in a lot of immensely large models.

Woodin proved that Ω -logic is *sound*: if $T \vdash_{\Omega} \varphi$ then $T \models_{\Omega} \varphi$, i.e. (under PCW) if $\vdash_{\Omega} \varphi$ then $\models \varphi$ in all ZFC -models $(V_{\alpha})^{V[G]}$. He then formulated the main conjecture:

Ω -conjecture (1999). *Ω -logic is complete: if $\models_{\Omega} \varphi$ then $\vdash_{\Omega} \varphi$.* □

As he emphasized in Woodin (2002, p. 517):

“If the Ω -conjecture is true, then generic absoluteness is equivalent to absoluteness in Ω -logic and this in turn has significant metamathematical implications”.

Indeed (Dehornoy, 2007), the Ω -conjecture means that any formula φ valid in a lot of immensely large models satisfying LCA s are certified by UB subsets of \mathbb{R} . The key fact proved by Woodin is the link of the concept of Ω -provability with the existence of *canonical models* for LCA s (that is models which are in a certain way minimal and universal, as L for $ZFC + CH$). The Ω -conjecture expresses essentially the hypothesis that every LCA admits a canonical model.

Theorem. *A sentence φ is Ω -provable ($\vdash_{\Omega} \varphi$) iff $ZFC + A \vdash \varphi$ for some large cardinal axioms A admitting a canonical model.* □

Now, the key point is that when H_2 is rigidified, CH becomes automatically *false*.

⁴³See Woodin [2000].

“If the theory of the structure $\langle \mathcal{P}(\omega_1), \omega_1, +, \cdot, \in \rangle$ is to be resolved on the basis of a good axiom then necessarily CH is false.”

The idea is that if the theory T of $\mathcal{P}(\omega_1)$ is completely unambiguous in the sense that there exists an axiom A s.t. $T \models \varphi$ iff $A \models_{\Omega} “T \models \varphi”$, then CH is necessarily false since the theory of $\mathcal{P}(\mathbb{R})$ cannot share this property.

Woodin theorem (2000, under PCW). (i) For every “solution” for H_2 (that is axioms freezing the properties of H_2 w.r.t. forcing) based on an Ω -complete axiom A (i.e. for every $\varphi \in H_2$, either $ZFC + A \vdash_{\Omega} “(H_2, \in) \models \varphi”$ or $ZFC + A \vdash_{\Omega} “(H_2, \in) \models \neg\varphi”$), CH is false. (ii) If the Ω -conjecture is valid, every “solution” for H_2 is based on an Ω -complete axiom and therefore CH is false. \square

The proof uses Tarski results on the impossibility of defining truth and is quite interesting (Woodin 2001, p. 688). Let

$$\Gamma = \{ \ulcorner \varphi \urcorner : ZFC + A \vdash_{\Omega} “(H_2, \in) \models \varphi” \}$$

be the (extremely complicated) set of Gödel numbers of the sentences Ω -valid in H_2 . By hypothesis, Γ is Ω -recursive in the sense there exists a UB set B s.t. Γ is definable and recursive in $L(B, \mathbb{R})$. Now, PCW implies that Γ being Ω -recursive, it is definable in $(H(\mathfrak{c}^+), \in)$. If CH would be valid, then $\mathfrak{c} = \omega_1$, $H(\mathfrak{c}^+) = H_2$ and Γ would be definable in H_2 , which would violate Tarski theorem.

It is in that sense Woodin (2001, p. 690) can claim:

“Thus, I now believe the Continuum Hypothesis is solvable, which is a fundamental change in my view of set theory”.

12 Conclusion

Hugh Woodin has already proved a great part of the Ω -conjecture.

Other approaches to the continuum problem in the set theoretical framework of $LCAs$ have been proposed. One of the most interesting alternative is provided by Matthew Foreman’s (2003) concept of *generic large cardinal* (GLC) defined by elementary embeddings $j : V \prec M$ of V in inner models M not of V itself but of generic extensions $V[G]$ of V . Such generic $LCAs$ can support rather CH than $\neg CH$.

But all these results show what are the difficulties met in elaborating a “good” set theoretical determination of the continuum. They justify some sort of Gödel’s platonism introducing additional axioms as some kind of “physical hypotheses”, as if the realm of universes of sets has to be explored as a kind of “objective” and “empirical” reality. The nominalist antiplatonist philosophy of mathematics criticizing such axioms (in particular $LCAs$) as ontological naive

beliefs is itself naive because it interprets platonism as the “ontological” thesis that *the* universe of sets must be unique and *ZFC* completely determined. It must be reconsidered and substituted by a “conditional” *platonism* in Woodin’s sense, a platonism which would be “conditional” to axioms which rigidify the continuum and make its properties forcing-invariant.

In my 1991, 1992 and 1995 papers on the continuum problem, I introduced the concept of “*transcendental platonism*”. Classical platonism is a naive realist thesis concerning the ontological independence of mathematical idealities, and as such is always dialectically opposed to anti-platonist nominalism. Even to day, the debates concerning the status of mathematical idealities remain trapped into the realist/nominalist dialectic.⁴⁴

The metaphysical antiplatonism denying any ontological content to ideal and abstract objects and structures is used to justify a “deflationist” ontological commitment concerning the axioms of existence which are admissible in set theory. As if constructivism was ontologically more secure! In the above cited papers, I have argued that

1. the nominalist thesis has no ontological relevance for mathematics;
2. it can’t therefore be used as an argument against strongly non-constructive axioms of existence;
3. the problem of mathematical platonism does not concern ontological realism but *mathematical objectivity*, which has nothing to do with ontology: objectivity is the possibility of constituting and determining, using law-like procedures, some unknown properties (and not of discovering them as a preexisting reality);
4. for solving the problem, we need a philosophy of objectivity based on the difference between objectivity and ontology;
5. such a philosophy is afforded by transcendental philosophy;
6. in the transcendental perspective strong axioms of existence can be justified as far as they manifest a strong power of determination.

It is for all these reasons that I coined the term of “transcendental platonism”.

The main achievement of transcendentalism has been to overcome the scholastic antinomy between realism and nominalism and to show that mathematical and physical objectivity were neither ontological nor subjective. Objectivity is always *constituted* and therefore *conditional*, relative to eidetico-constitutive

⁴⁴See e.g. Maddy [2005] on “naïve realism”, “robust realism”, “thin realism”, etc.

rules. A platonism defined in terms of objectivity and not ontology, is a transcendental platonism immune to the classical aporias of metaphysical transcendent platonism. As far as the question of the continuum is concerned, the eidetico-constitutive rules are the axioms of set theory and transcendental platonism means that the continuum problem can have a well determined solution in a rigid universe where \mathbb{R} become *conditionally* generically absolute. I think that Woodin's conditional platonism can therefore be considered as a transcendental platonism relative to the continuum problem.

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