Structuralism in Physics and Non Commutative Geometry

Jean PETITOT[*]

7 March 2005

Abstract.

Contents

1. Introduction

I think that structuralism can be considered as a direct consequence of a general transcendental perspective on modern physics based on the three following principles:

1. Physics concerns only phenomena. Phenomena are relational entities which are inseparable of their conditions of observation, and the accessibility conditions (observation, measure, extraction of information, etc.) are constitutive of the very concept of physical object. In that sense physical objectivity cannot be the ontology of a mind-independent substantial reality and any ontological realism has to be rejected.

2. But "ontological" concepts still have a theoretical function. To be transformed in objects, phenomena must be conceptually "legalized" via a categorial structure. The first philosophical thematization of this principle was Kant's *Metaphysische Anfangsgründe der Naturwissenschaft* explaining how the four categorial blocks: phoronomy, dynamics, mechanics, and phenomenology, act in Newtonian mechanics.

---

[*]CREA, Ecole Polytechnique, petitot@poly.polytechnique.fr

1

3. The essential feature of physics is the mathematical interpretation of the categorial concepts which transforms them into *algorithms* for the *mathematical reconstruction* of phenomena. This is the main point. Physics has to solve an *inverse problem*, namely the inverse problem of the abstraction problem. Conceptual *analysis* must be complemented with a *computational synthesis* of phenomena. In Kant's, computational synthesis if based first on the schematization and second on the "construction" of categories.

The consequence of (3) is that physics is necessarily structural since computational synthesis is mathematical and mathematics are structural.

In the *Metaphysische Anfangsgründe der Naturwissenschaft* the categorial moments of classical Mechanics can be summarized in the following way (in modern terms).

1. **Phoronomy**: Euclidean metric as background structure (a priori), Galilean relativity.

2. **Dynamics**: Physical dynamics has to be described by differential entities (velocities, accelerations, etc.) varying covariantly (link with phoronomy). Physics must be a differential geometry.

3. **Mechanics**: the category of substance is reinterpreted as the principle of conservation laws, the category of causality as forces, the category of community as interaction.

4. **Phenomenology** (modality): due to relativity, movement is not real (it is a purely relational ungrounded phenomenon). Position and velocity are not observable properties whose values could individuate dynamical states. The body S "has" such position and such velocity in the sense of "having a property" is not a physical judgement.

A modern striking example of (3) is given by the constitutive role of *symmetries*. In general relativity and non abelian gauge theories, the drastic enlargement of the symmetry groups enables to mathematically construct the physical content of the categories of force and interaction *from their relativity principles*. I think that this thesis is akin to Van Fraassen's hypothesis that symmetry groups are devices allowing problem solving.

For me, as for my colleague Daniel Bennequin (a specialist of symplectic geometry and string theory) this is a deep manifestation of the *Galoisian* essence of modern physics: *the symmetries which express*

entities which cannot be physical observables are at the same time principles of determination of the physical obervables.

In the evolution of modern physics one can observe a great stability of the categorial structures coupled with a great variability of their successive mathematical interpretations. I think that this variability is by no means an argument against a transcendental approach. For instance, in Kant, the a priori nature of space and time means essentially that the Galilean structure of space-time is a background structure for Mechanics. And this is perfectly true. In GR metric is no longer a background structure and becomes a dynamical feature of the theory. The Diff(M)-invariance as gauge-invariance implies that localization becomes purely relational and that points lack any physical content. But I think that this background independence doesn't refute transcendentalism at all. I have developped the hypothesis that the *differentiable* structure of space-time with the *cohomology* of differential forms remains a background structure.

In this talk, I want to comment on a new technical example of mathematical reinterpretation of the categorial structures of physics in relation with John Baez' requisite (less radical than Lee Smolin's) of background independence of any *geometric* structure. The problem is rather difficult, especially in what concerns Quantum Gravity. How eliminate background geometry in QM while maintaining at the same time the computational efficiency of geometry?

I think that the most interesting answer to this problem comes from *Non Commutative Geometry* and I want to present here an example, namely how in NCG metric can be reinterpreted in purely spectral terms using the formalism of Clifford algebras and Dirac operators and how a pure non-commutative generalization yields a natural interpretation of the Higgs phenomenon.

Philosophically, the breakthrough of NCG is to start from QM and to "quantize" all classical geometrical concepts. The conflict between geometry and QM disappears from the outset since quantum concepts are no longer subordinated to any backgroung geometrical structure.

2.   Gelfand theory

To understand NC Geometry we must first come back to Gelfand theory of commutative C^*-algebras.

Recall that a C^*-algebra $\mathcal{A}$ is a Banach algebra (i.e. normed and complete) with an involution $x \to x^*$ s.t. $\|x\|^2 = \|x^*x\|$. $\|x\|^2$ is the spectral radius of the ≥ 0 element x^*x, that is the Sup of the spectral values of x^*x:

$$\|x\|^2 = \text{Sup} \left\{ |\lambda| : x^*x - \lambda \text{ is not invertible} \right\}.$$

In a C^*-algebra the norm becomes therefore a purely *spectral* concept.

In this classical setting, the mathematical interpretation of the fundamental (categorial) concepts of

1. space of states,

2. observable,

3. measure,

is the following:

1. the space of states is a smooth manifold: the phase space M;

2. the observables are functions $f : M \to \mathbb{R}$ (interpreted as $f : M \to \mathbb{C}$ s.t. $f = \bar{f}$);

3. the measure of f in the state $x \in M$ is the evaluation $f(x) = \delta_x(f)$ (where δ_x is the Dirac distribution at x).

The observables constitute a commutative C^*-algebra $\mathcal{A}$ and Gelfand theory explains that the geometry of the manifold M can be retreived from the algebraic structure of $\mathcal{A}$. *It is an excellent example of the structuralist thesis that the structures are only indivituated up to isomorphism.*

Let M be a topological space and let $\mathcal{A} = \mathcal{C}(M)$ be the algebra of continuous functions $f : M \to \mathbb{C}$. It is a C^*-algebra under general conditions (e.g. if M is compact).

The possible values of f — that is the possible results of a measure of f — can be defined in a purely algebraic way as the *spectrum* of f that is

$$\operatorname*{sp}_{\mathcal{A}}(f) = \{c : f - c \text{ is not invertible in } \mathcal{A}\}.$$

Indeed, if $f(x) = c$ then $f - c$ is not invertible in $\mathcal{A}$.

The main point is that the evaluation process $f(x)$ — that is the measure — can be interpreted as a *duality* $\langle f, x \rangle$ between the space M and the algebra $\mathcal{A}$. Indeed, to a point x of M we can associate the *maximal ideal* of $\mathcal{A}$:

$$\mathfrak{M}_x = \{f : f(x) = 0\}.$$

But the maximal ideals $\mathfrak{M}$ of $\mathcal{A}$ constitute a space — the *spectrum* of the algebra $\mathcal{A}$. The $\mathfrak{M}$ can be considered as the *kernels* of the *characters* of $\mathcal{A}$, that is of the morphisms (multiplicative linear forms) $\chi : \mathcal{A} \to \mathbb{C}$,

$$\mathfrak{M} = \chi^{-1}(0).$$

A character is a procedure for evaluating the elements $f \in \mathcal{A}$. The evaluation $\chi(f)$ is also a *duality* $\langle \chi, f \rangle$.and $\chi(f) \in \mathrm{sp}_{\mathcal{A}}(f)$.

As *distributions*, the characters correspond to the Dirac distributions δ_x.

The *spectrum* of the C^*-algebra $\mathcal{A}$ is by definition the space of characters $\mathrm{Sp}(\mathcal{A}) = \{\chi\}$. If $f \in \mathcal{A}$ is an element of $\mathcal{A}$, we associate to it a function $\tilde{f}$ on $\mathrm{Sp}(\mathcal{A})$

$$\begin{aligned} \tilde{f} : \mathrm{Sp}(\mathcal{A}) &\rightarrow \mathbb{C} \\ \chi &\mapsto \tilde{f}(\chi) = \chi(f) = \langle \chi, f \rangle. \end{aligned}$$

We get that way a map

$$\begin{aligned} \tilde{\ } : \mathcal{A} &\rightarrow \mathcal{C}(\mathrm{Sp}(\mathcal{A})) \\ f &\mapsto \tilde{f} \end{aligned}$$

which is called the *Gelfand transform*.

For every f we have

$$\tilde{f}(\mathrm{Sp}(\mathcal{A})) = \mathrm{sp}_{\mathcal{A}}(f).$$

The key result is then:

Gelfand-Neimark theorem. If $\mathcal{A}$ is a commutative C^*-algebra, the Gelfand transform $\tilde{\ }$ is an *isomorphism* between $\mathcal{A}$ and $\mathcal{C}(\mathrm{Sp}(\mathcal{A}))$.

Gelfand theory shows that in the classical case of Halmiltonian Mechanics and commutative C^*-algebras there is an *equivalence* between the geometric and the algebraic perspectives.

In Quantum Mechanics (Von Neumann, Gelfand, Neimark, Segal) the basic structure is that of the non commutative C^*-algebras of observables. It is therefore natural to wonder if there could exist a *geometric correlate* of this non commutative algebraic setting. It is the origin of Connes' *Non Commutative Geometry* also called *Spectral Geometry* or *Quantum Geometry*.

3.   NCG AND DIFFERENTIAL FORMS

The most fascinating aspect of the research program of NCG is how Alain Connes succeeded in *reinterpreting* all the basic structures of classical geometry in the framework of NC C^*-algebras operating on Hilbert spaces. *The basic concepts remain the same but there mathematical content is complexified*, their classical content becoming a commutative limit. We meet here a new very deep example of the conceptual invariance through mathematical enlargements as in GR or QM.

Connes reinterpreted (in an extremely deep and technical way) the six classical levels:

1. Measure theory;

2. Algebraic topology and topology (K-theory);

3. Differentiable structure;

4. Differential forms and De Rham cohomology;

5. Fiber bundles, connections, covariant derivations, Yang-Mills theories;

6. Riemannian manifolds and metric structures.

Let us take as a first example the reinterpretation of the differential calculus.

3.1.   A universal and formal differential calculus.   Let $\mathcal{A}$ be a NC C^*-algebra. We want first to define *derivations* $D : \mathcal{A} \to \mathcal{E}$ that is $\mathbb{C}$-linear maps satisfying the *Leibniz rule* (which is the universal formal rule for derivations):

$$D(ab) = (Da)b + a(Db)$$

For that, $\mathcal{E}$ must be endowed with a structure of $\mathcal{A}$-bimodule (right and left products of elements of $\mathcal{E}$ by elements of $\mathcal{A}$). A consequence of Leibniz formula is

$$D(1) = D(1.1) = D(1)1 + 1D(1) = 2D(1)$$

and therefore $D(1) = 0$, which implies $D(c) = 0 \ \forall c \in \mathbb{C}$.

Let $\mathrm{Der}(\mathcal{A}, \mathcal{E})$ be the $\mathbb{C}$-vector space of such derivations. It is a Lie algebra since $[D_1, D_2]$ is a derivation if D_1, D_2 are derivations. In $\mathrm{Der}(\mathcal{A}, \mathcal{E})$ there exist very particular elements, the interior derivatives, associated with the elements m of $\mathcal{E}$:

$$D(a) = \mathrm{ad}(m)(a) = ma - am.$$

Indeed,

$$
\begin{aligned}
\mathrm{ad}(m)(a).b + a.\,\mathrm{ad}(m)(b) &= (ma - am)b + a(mb - bm) \\
&= mab - abm \\
&= \mathrm{ad}(m)(ab).
\end{aligned}
$$

Now, we stress the fact that there exists *a universal derivation structure* depending only upon the structure of $\mathcal{A}$. It is given by

$$
\begin{aligned}
d : \mathcal{A} &\to \mathcal{A} \otimes_{\mathbb{C}} \mathcal{A} \\
a &\mapsto da = 1 \otimes a - a \otimes 1.
\end{aligned}
$$

Let $\Omega^1 \mathcal{A}$ be the sub-bimodule of $\mathcal{A} \otimes_{\mathbb{C}} \mathcal{A}$ generated by the *adb*. It is the bimodule of universal 1-forms on $\mathcal{A}$. Universality means that

$$\mathrm{Der}(\mathcal{A}, \mathcal{E}) \simeq \mathop{\mathrm{Hom}}_{\mathcal{A}} \left(\Omega^1 \mathcal{A}, \mathcal{E} \right)$$

If $D : \mathcal{A} \to \mathcal{E}$ is an element of $\mathrm{Der}(\mathcal{A}, \mathcal{E})$, the associated morphism $\tilde{D} : \Omega^1 \mathcal{A} \to \mathcal{E}$ is defined as

$$a \otimes b \mapsto aD(b).$$

So $da = 1 \otimes a - a \otimes 1 \mapsto 1.D(a) - a.D(1) = D(a)$ (since $D(1) = 0$).

We can generalize this construction to universal n-forms which will have the symbolic form

$$a_0 da_1 ... da_n.$$

The differential is then

$$\begin{aligned} d : \Omega^n \mathcal{A} &\to \Omega^{n+1} \mathcal{A} \\ a_0 da_1 ... da_n &\mapsto da_0 da_1 ... da_n. \end{aligned}$$

It is easy to verify the fundamental property $d^2 = 0$.

3.2. Non commutative differential calculus or "quantized" calculus. We suppose now that the C^*-algebra $\mathcal{A}$ acts upon an Hilbert space $\mathcal{H}$ and we want to interpret in this representation the universal, formal, and purely symbolic differential calculus of the previous section. For that we must interpret the df of the elements $f \in \mathcal{A}$, these f being now *operators* on $\mathcal{H}$. Connes main idea is to use the well known formula of QM

$$\frac{df}{dt} \propto [F, f]$$

where F is the Hamiltonian of the system and f any obervable.

We interpret the symbol df as

$$df = [F, f]$$

for an appropriate self-adjoint operator F. We want of course $d^2 f = 0$. But $d^2 f = [F^2, f]$ and the simplest solution is given by $F^2 = 1$.

The main constraint is that the df must be *infinitesimal*. We have therefore *to reinterpret the classical concept of infinitesimal in the NC framework*. Connes definition is that an operator T is infinitesimal if it is *compact*, that is if the eigenvalues $\mu_n(T)$ of $|T| = (T^*T)^{1/2}$ converge to 0 that is if for every $\epsilon > 0$ the size of T is $< \epsilon$ outside a subspace of *finite* dimension. If $\mu_n(T) \underset{n \to \infty}{\to} 0$ as $\frac{1}{n^k}$ then T is an infinitesimal of order k.

We interpret therefore the differential calculus in the NC framework through triples $(\mathcal{A}, \mathcal{H}, F)$ where $[F, f]$ is compact for every $f \in \mathcal{A}$. Such a structure is called *a Fredholm module*.

The differential forms $a_0 da_1 ... da_n$ are now interpreted as operators

$$a_0 \left[F, a_1\right] ... \left[F, a_n\right]$$

In the classical case of the commutative C^*-algebra $\mathcal{A} = \{f : \mathbb{R} \to \mathbb{R}\}$ (*f non* necessarilly differentiable) acting by multiplication upon the Hilbert space $\mathcal{H} = L^2(\mathbb{R})$, the Fredholm operator F is the *Hilbert transform*

$$F\xi(s) = \frac{1}{i\pi} \int \frac{\xi(t)}{s - t} dt$$

and $df = [F, f]$ is (up to $\frac{1}{i\pi}$) the operator defined by the kernel $k(s, t) = \frac{f(s) - f(t)}{s - t}$

$$df(\xi)(s) = \int k(s, t) \xi(t) dt$$

df is compact iff f has a vanishing mean oscillation, that is if

$$\operatorname*{Sup}_{|I| \leq a} \frac{1}{|I|} \int_I |f - M_I(f)| \underset{a \to 0}{\to} 0$$

$M_I(f)$ being the mean value of f on the interval I.

It must be emphasized that the NC generalization of differential calculus is a wide and wild generalization since it enables a differential calculus on fractals!

4. NC RIEMANNIAN GEOMETRY, CLIFFORD ALGEBRAS, AND DIRAC OPERATOR

Another great success of Alain Connes was the complete and deep reinterpretation of the ds^2 in Riemannian geometry. Classically, $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$. But in the NC framework, the dx must be interpreted as $dx = [F, x]$ where $(\mathcal{A}, \mathcal{H}, F)$ is a Fredholm module and the $(g_{\mu\nu})$ as an element of the $n \times n$ matrix algebra $M_n(\mathcal{A})$. The ds^2 must become an operator

$$G = [F, x^\mu]^* g_{\mu\nu} [F, x^\nu]$$

4.1. A redefinition of distance. Connes' idea is to reinterpret the classical notion of distance $d(p, q)$ between two points p, q of a Riemannian manifold M as the Inf of the length $L(\gamma)$ of the paths $\gamma : p \to q$

$$d(p, q) = \operatorname*{Inf}_{\gamma : p \to q} L(\gamma)$$

$$L(\gamma) = \int_p^q ds = \int_p^q (g_{\mu\nu} dx^\mu dx^\nu)^{1/2}.$$

An elementary computation shows that this definition of the distance is equivalent to the dual algebraic definition using only concepts concerning the C^*-algebra $\mathcal{A}$

$$d(p,q) = \text{Sup} \{|f(p) - f(q)| : \|\text{grad}(f)\|_\infty \leq 1\}$$

where $\|...\|_\infty$ is the L^∞ norm, that is the Sup on $x \in M$ of the norms on the tangent spaces $T_x M$.

4.2. Clifford algebras. Now the core of the NC definition of distance uses *the Clifford algebra* of the Riemannian manifold M.

Recall that the formalism of Clifford algebras relates *the differential forms and the metric* in Riemannian manifolds. In the classical case of the Euclidean space $\mathbb{R}^n$, the main idea is to encode the isometries $O(n)$ in an algebra structure. As every isometry is a product of reflections (Cartan), we can associate to any vector $v \in \mathbb{R}^n$ the reflection $\bar{v}$ relative to the orthogonal hyperplane $v^\perp$ and introduce a multiplication $v.w$ which is nothing else than the composition $\bar{v} \circ \bar{w}$.

We are then naturally led to the anti-commutation relations

$$\{v, w\} = -2(v, w)$$

More generally, let V be a $\mathbb{R}$-vector space endowed with a quadratic form g. Its Clifford algebra $\text{Cl}(V, g)$ is its tensor algebra quotiented by the relations

$$v \otimes v = -g(v), \ \forall v \in V$$

In $\text{Cl}(V, g)$ the tensorial product $v \otimes v$ becomes a product $v.v$. It must be stressed that there exists always in $\text{Cl}(V, g)$ the constants $\mathbb{R}$ which correspond to the 0th tensorial power of V.

Using the scalar product

$$2g(v, w) = g(v + w) - g(v) - g(w)$$

associated with g, one gets the *anti-commutation* relations

$$\{v, w\} = -2g(v, w)$$

Elementary examples are given by the $\text{Cl}_n = \text{Cl}(\mathbb{R}^n, g_{\text{Euclid}})$.

- $\text{Cl}_0 = \mathbb{R}$.

- $Cl_1 = \mathbb{C}$ ($V = i\mathbb{R}$, $i^2 = -1$, $Cl_1 = \mathbb{R} \oplus i\mathbb{R}$).

- $Cl_2 = \mathbb{H}$ ($V = i\mathbb{R} + j\mathbb{R}$, $ij = k$, $Cl_2 = \mathbb{R} \oplus i\mathbb{R} \oplus j\mathbb{R} \oplus k\mathbb{R}$).

- $Cl_3 = \mathbb{H} \oplus \mathbb{H}$.

- $Cl_4 = \mathbb{H}[2]$ (2×2 matrices with entries in $\mathbb{H}$).

- $Cl_5 = \mathbb{C}[4]$.

- $Cl_6 = \mathbb{R}[8]$.

- $Cl_7 = \mathbb{R}[8] \oplus \mathbb{R}[8]$.

- $Cl_{n+8} = Cl_n \otimes \mathbb{R}[16]$ (Bott periodicity theorem).

If $g(v) \neq 0$ (which would always be the case for $v \neq 0$ if g is non degenerate) v is invertible in this algebra structure and

$$v^{-1} = -\frac{v}{g(v)}.$$

The multiplicative Lie group $Cl^\times(V, g)$ of the invertible elements of $Cl(V, g)$ act by *inner automorphisms* on $Cl(V, g)$. This yields the *adjoint representation*

$$\begin{aligned} \mathrm{Ad} : Cl^\times(V, g) &\rightarrow \mathrm{Aut}\,(Cl(V, g)) \\ v &\mapsto \mathrm{Ad}_v : w \mapsto v.w.v^{-1}. \end{aligned}$$

But

$$\begin{aligned} v.w.v^{-1} &= -v.w.\frac{v}{g(v)} = -(-w.v - 2g(v, w))\frac{v}{g(v)} \\ &= w.\frac{v^2}{g(v)} + \frac{2g(v, w)v}{g(v)} \\ &= -w + \frac{2g(v, w)v}{g(v)}. \end{aligned}$$

Hence

$$-\mathrm{Ad}_v(w) = w - \frac{2g(v, w)}{g(v)}v$$

The derivative ad of the adjoint representation enables to retreive the Lie bracket of the Lie algebra $cl^\times(V, g) = Cl(V, g)$ of the Lie group $Cl^\times(V, g)$

$$\begin{aligned} \mathrm{ad} : cl^\times(V, g) = Cl(V, g) &\rightarrow \mathrm{Der}\,(Cl(V, g)) \\ v &\mapsto \mathrm{ad}_v : w \mapsto [v, w] \end{aligned}$$

Now there exists a fundamental relation between the Clifford algebra $\mathrm{Cl}(V, g)$ of V and its exterior algebra $\Lambda^* V$. If $g = 0$ and if we interpret $v.w$ as $v \wedge w$, the anti-commutation relations become simply $\{v, w\} = 0$, that is the antisymmetry $w \wedge v = -v \wedge w$. Therefore

$$\Lambda^* V = \mathrm{Cl}(V, 0).$$

In fact, $\mathrm{Cl}(V, g)$ can be considered as a way of quantifying $\Lambda^* V$ using the metric g to get *non trivial* anti-commutation relations.

Due to the relations $v^2 = -g(v)$ which decrease the degree of a product by 2, $\mathrm{Cl}(V, g)$ is no longer a $\mathbb{Z}$-graded algebra but only a $\mathbb{Z}/2$-graded algebra, the $\mathbb{Z}/2$-gradation corresponding to the even/odd elements. But we can reconstruct a $\mathbb{Z}$-graded algebra $\mathcal{C} = \bigoplus\limits_{k=0}^{k=\infty} C^k$ associated to $\mathrm{Cl}(V, g)$, the C^k being the homogeneous terms of degree k: $v_1. \cdots .v_k$.

Theorem. The map of graded algebras $\mathcal{C} = \bigoplus\limits_{k=0}^{k=\infty} C^k \rightarrow \Lambda^* V = \bigoplus\limits_{k=0}^{k=\infty} \Lambda^k$ given by $v_1. \cdots .v_k \rightarrow v_1 \wedge \cdots \wedge v_k$ is a *linear isomorphism*.

We consider now 2 operations on the exterior algebra:

1. The exterior multiplication $\varepsilon(v)$ by $v \in V$:

$$\varepsilon(v) \left(\bigwedge_i u_i \right) = v \wedge \left(\bigwedge_i u_i \right).$$

   We have $\varepsilon(v)^2 = 0$ since $v \wedge v = 0$.

2. The contraction (interior multiplication) $\iota(v)$ induced by the metric g:

$$\iota(v) \left(\bigwedge_i u_i \right) = \sum_{j=1}^{j=k} (-1)^j g(v, u_j) u_1 \wedge \cdots \wedge \widehat{u_j} \wedge \cdots u_k.$$

   We have also $\iota(v)^2 = 0$.

One shows that the anti-commutations relations obtain:

$$\{\varepsilon(v), \iota(w)\} = -g(v, w)$$

Let $c(v) = \varepsilon(v) + \iota(v)$. We get the anti-commutation relations of the Clifford algebra

$$\{c(v), c(w)\} = -2g(v, w)$$

4.3. Spin groups. The isometry group $O(n)$ (or $O(p,q)$ if the signature of g is different) is canonically embedded in $\mathrm{Cl}(V,g)$. If we try to characterize its elements in $\mathrm{Cl}(V,g)$ in terms of the Clifford multiplication we use the two properties:

1. $g.g^c = 1$ (where $(v_1.\cdots.v_k)^c = (-1)^k (v_k.\cdots.v_1)$ is the Clifford conjugation,

2. $g.V.g^t \subseteq V$ (where $(v_1.\cdots.v_k)^t = (v_k.\cdots.v_1)$ is the transposition).

But these properties characterize a larger group, the *pin group* $\mathrm{Pin}(n)$, which is a 2-fold covering of $O(n)$. If we take into account the orientation and restrict to $SO(n)$ the 2-fold covering becomes the *spin group* $\mathrm{Spin}(n)$. By restriction of the Clifford multiplication and of the adjoint representation $w \mapsto v.w.v^{-1}$ to $\mathrm{Spin}(n)$, we get therefore a representation γ of $\mathrm{Spin}(n)$ in the spinor space $\mathbb{S} = \mathrm{Cl}(V,g)$.

4.4. Dirac equation. We can use the Clifford algebra to change the exterior derivative of differential forms.

$$d = \varepsilon\,(dx_k)\,\frac{\partial}{\partial x_k}.$$

We define the Dirac operator as

$$
\begin{aligned}
D &= c\,(dx_k)\,\frac{\partial}{\partial x_k} \\
&= \gamma_k \frac{\partial}{\partial x_k}
\end{aligned}
$$

where c is the Clifford multiplication.. $D^2 = -\Delta$ (the Laplacian) and D acts on the spinor space $\mathbb{S}$.

4.5. NC distance and Dirac operator. More generally, if M is a Riemannian manifold the previous construction can be done for every tangent spage $T_x M$ endowed with the quadratic form g_x. We get that way a *bundle* of Clifford algebras $\mathrm{Cl}(TM,g)$. If S is a spinor bundle, a bundle of $\mathrm{Cl}(TM)$-modules s.t. $\mathrm{Cl}(TM) \simeq \mathrm{End}(S)$, with a covariant derivative ∇, we associate to it the Dirac operator $D : \mathcal{S} = \Gamma(S) = C^\infty(M,S) \to \Gamma(S)$ which is a first order elliptic operator interpretable as the "square root" of the Laplacian Δ, which interprets itself the metric in operatorial terms. D can be extended from the $C^\infty(M)$-module $\mathcal{S} = \Gamma(S)$ to the Hilbert space $\mathcal{H} = L^2(M,S)$. In general, due to chirality, S will be the direct sum of an even and an odd part,

$S = S^+ \oplus S^-$ and D will have the characteristic form

$$D \;=\; \begin{bmatrix} 0 & D^- \\ D^+ & 0 \end{bmatrix}$$
$$D^+ \;:\; \Gamma(S^+) \to \Gamma(S^+)$$
$$D^- \;:\; \Gamma(S^-) \to \Gamma(S^-)$$

D^+ and D^- being adjoint operators.

The metric g induces canonical isomorphisms $T_x M \widetilde{\to} T_x^* M$ between the tangent and cotangent spaces and therefore

$$\mathrm{Cl}(TM, g) \widetilde{\to} \mathrm{Cl}(T^*M, g^{-1})$$

and the 1-forms $\alpha \in \Omega^1(M) = \Gamma(T^*M)$ act on $\mathcal{S} = \Gamma(S)$ via the spin representation γ. They satisfy the anti-commutation relations

$$\{\gamma(\alpha), \gamma(\beta)\} = -2g^{-1}(\alpha, \beta)$$

and if (x_k) are local coordinates, $\gamma_k = \gamma(dx_k)$ are elements of $\mathcal{L}(S)$.

Let us look now at the connections. There exists first the *Levi-Civita connection* on M:

$$\nabla^g : \Omega^1(M) \to \Omega^1(M) \underset{C^\infty(M)}{\otimes} \Omega^1(M)$$

with the Leibniz rule for $\alpha \in \Omega^1(M)$ and $f \in C^\infty(M)$:

$$\nabla^g(f\alpha) = f\nabla^g(\alpha) + df \otimes \alpha.$$

There exists also the *spin connection* on S

$$\nabla^S : \Gamma(S) \to \Omega^1(M) \underset{C^\infty(M)}{\otimes} \Gamma(S)$$

with the Leibniz rule for $\psi \in \Gamma(S)$ and $f \in C^\infty(M)$:

$$\nabla^S(f\psi) \;=\; f\nabla^S(\psi) + df \otimes \psi$$
$$\nabla^S(\gamma(\alpha)\psi) \;=\; \gamma(\nabla^g(\alpha))\,\psi + \gamma(\alpha)\nabla^S(\psi).$$

The Dirac operator on $\mathcal{H} = L^2(M, S)$ is then defined as

$$D = \gamma \circ \nabla^S$$

In local coordinates terms

$$D \;=\; \gamma(dx_k)\nabla^S_{\partial/\partial x_k}$$
$$\left(= \gamma_\mu \partial_{x_\mu}\right)$$

It then easy to compute the bracket $[D, f]$ for $f \in C^\infty(M)$. If $\psi \in \Gamma(S)$, we have

$$
\begin{aligned}
D\,(f\psi) &= \gamma\left(\nabla^S(f\psi)\right) \\
&= \gamma\left(f\nabla^S(\psi) + df \otimes \psi\right) \\
&= f\gamma\left(\nabla^S(\psi)\right) + \gamma\left(df \otimes \psi\right) \\
&= fD(\psi) + \gamma\,(df)\,\psi
\end{aligned}
$$

and therefore $[D, f](\psi) = \gamma\,(df)\,\psi$, that is

$$
[D, f] = \gamma\,(df)\,.
$$

In the standard case where $M = \mathbb{R}^n$ and $S = \mathbb{R}^n \times V$, V being a Cl_n-module of spinors ($\mathrm{Cl}_n = \mathrm{Cl}\,(\mathbb{R}^n, g_{\mathrm{Euclid}})$), we have seen that D is a differential operator with constant coefficients taking its values in V.

$$
D = \sum_{k=1}^{k=n} \gamma_k \frac{\partial}{\partial x_k}
$$

with the constant matrices $\gamma_k \in \mathcal{L}(V)$ satisfying the anti-commutation relations

$$
\left\{\gamma_j, \gamma_k\right\} = -2\delta_{jk}
$$

The classical Dirac matrices γ^μ are the $-i\gamma_\mu$ for $\mu = 0, 1, 2, 3$. They satisfy the anti-commutation relations

$$
\{\gamma^\mu, \gamma^v\} = 2\delta^{\mu\nu}.
$$

The fundamental point is that the γ_k are associated with the basic 1-forms dx_k through the isomorphism

$$
c : \mathcal{C} = \Lambda^*(M) \to \mathrm{gr}\,(\mathrm{Cl}(TM))
$$

$$
[D, f] = \gamma\,(df) = c(df)
$$

and $\|[D, f]\|$ is the norm of the Clifford action of df on the space of spinors $L^2(M, S)$. But

$$
\begin{aligned}
\|c(df)\|^2 &= \operatorname*{Sup}_{x \in M} g_x^{-1}\left(d\bar{f}(x), df(x)\right) \\
&= \operatorname*{Sup}_{x \in M} g_x\left(\operatorname*{grad}_x \bar{f}, \operatorname*{grad}_x f\right) \\
&= \|\operatorname{grad}(f)\|_\infty^2
\end{aligned}
$$

Whence the definition:

$$d(p, q) = \text{Sup} \{|f(p) - f(q)| : f \in \mathcal{A}, \|[D, f]\| \leq 1\}$$

In this reinterpretation, ds corresponds to *the propagator of the Dirac operator* D. If F is the Fredholm operator defining the differentials df, D can be symbolically interpreted as

$$D = \frac{F}{ds}$$

and therefore $[D, f]$ interprets $\frac{df}{ds}$. As operator acting on the Hilbert space $\mathcal{H}$, D is an unbounded self-adjoint operator such that $[D, f]$ is bounded $\forall f \in A$ and such that its resolvent $(D - \lambda)^{-1}$ is compact $\forall \lambda \notin \text{Sp}(D)$ (which corresponds to the fact that ds is infinitesimal). In terms of the operator $G = [F, x^{\mu}]^* g_{\mu\nu} [F, x^{\nu}]$, $G = D^{-2}$.

5.   Yang-Mills theory of a NC coupling between 2 points and the Higgs mechanism

An extraordinary example of pure NC physics is given by Connes' interpretation of the Higgs phenomenon.

5.1.   Symmetry breaking and classical Higgs mechanism.   Let us first recall the classical Higgs mechanism. Consider e.g. a φ^4 theory for 2 scalar real fields φ_1 and φ_2. The Lagrangian is

$$\mathcal{L} = \frac{1}{2} \left(\partial_{\mu} \varphi_1 \partial^{\mu} \varphi_1 + \partial_{\mu} \varphi_2 \partial^{\mu} \varphi_2 \right) - V \left(\varphi_1^2 + \varphi_2^2 \right)$$

with the potential

$$V \left(\varphi^2 \right) = \frac{1}{2} \mu^2 \varphi^2 + \frac{1}{4} |\lambda| \left(\varphi^2 \right)^2$$

It is $SO(2)$-invariant.

For $\mu^2 > 0$ the minimum of V (the quantum vacuum) is non degenerate: $\varphi_0 = (0, 0)$ and the Lagrangian $\mathcal{L}_{os}$ of small oscillations in the neighborhood of φ_0 is the sum of 2 Lagrangians of the form:

$$\mathcal{L}_{os} = \frac{1}{2} \left(\partial_{\mu} \varphi \partial^{\mu} \varphi \right) - \frac{1}{2} \mu^2 \varphi^2$$

describing particles of mass μ^2.

But for $\mu^2 < 0$ the situation becomes completely different. Indeed the potential V has a full circle (an $SO(2)$-orbit) of minima

$$\varphi_0^2 = -\frac{\mu^2}{|\lambda|} = v^2$$

and the vacuum state is highly degenerate.

One must therefore *break the symmetry* to choose a vacuum state. Let us take for instance $\varphi_0 = \begin{bmatrix} v \\ 0 \end{bmatrix}$ and translate the situation to φ_0:

$$\varphi = \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix} = \begin{bmatrix} v \\ 0 \end{bmatrix} + \begin{bmatrix} \xi \\ \eta \end{bmatrix}$$

The oscillation Lagrangian at φ_0 becomes

$$\mathcal{L}_{os} = \frac{1}{2} \left(\partial_\mu \eta \partial^\mu \eta + 2\mu^2 \eta^2 \right) + \frac{1}{2} \left(\partial_\mu \xi \partial^\mu \xi \right)$$

There exist 2 particles:

1. a particle η of mass $m = \sqrt{2} \, |\mu|$ which corresponds to radial oscillations,

2. a particule ξ of mass $m = 0$ connecting the vacuum states. ξ is the *Goldstone boson*.

As is well known, the Higgs mechanism consists in using a cooperation between the gauge boson and the Goldstone boson to confer a mass to the gauge boson.

Let $\varphi = \frac{1}{\sqrt{2}} \left(\varphi_1 + i\varphi_2 \right)$ the complex field of the previous model. the Lagrangian is

$$\mathcal{L} = \partial_\mu \overline{\varphi} \partial^\mu \varphi - \mu^2 \, |\varphi|^2 - |\lambda| \, |\varphi|^4 \, .$$

It is of course invariant by the global internal symmetry $\varphi \to e^{i\theta} \varphi$. If we localize the global symmetry by $\varphi(x) \to e^{iq\alpha(x)} \varphi(x)$ and if we take into account the coupling with an electro-magnetic field deriving from the vector potential A_μ, we get

$$\mathcal{L} = \nabla_\mu \overline{\varphi} \nabla^\mu \varphi - \mu^2 \, |\varphi|^2 - |\lambda| \, |\varphi|^4 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

with the covariant derivative

$$\nabla_\mu = \partial_\mu + iq A_\mu$$

and the force field

$$F_{\mu\nu} = \partial_\nu A_\mu - \partial_\mu A_\nu.$$

The Lagrangian remains invariant if we compensate the localization of the global internal symmetry with a change of gauge

$$A_\mu \to A'_\mu = A_\mu - \partial_\mu \alpha(x).$$

For $\mu^2 > 0$, $\varphi_0 = 0$ and we get 2 scalar particles φ and $\overline{\varphi}$ and a photon A_μ.

For $\mu^2 > 0$, if we take $\varphi_0 = \frac{v}{\sqrt{2}}$ and write

$$\varphi = \frac{1}{\sqrt{2}}(v + \eta + i\xi) \approx \frac{1}{\sqrt{2}}e^{i\frac{\xi}{v}}(v + \eta) \text{ if } \xi \text{ and } \eta \text{ are small,}$$

we get for the oscillation Lagrangian:

$$\mathcal{L}_{os} = \frac{1}{2}\left(\partial_\mu\eta\partial^\mu\eta + 2\mu^2\eta^2\right) + \frac{1}{2}\left(\partial_\mu\xi\partial^\mu\xi\right) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + qvA_\mu\left(\partial_\mu\xi\right) + \frac{q^2v^2}{2}A_\mu A^\mu.$$

1. The field η (radial oscillations) has mass $m = \sqrt{2}\,|\mu|$.

2. The boson A_μ acquires a mass due to the term $A_\mu A^\mu$ and interacts with the Goldstone boson ξ.

The terms containing the gauge boson A_μ and the Goldstone boson ξ write

$$\frac{q^2v^2}{2}\left(A_\mu + \frac{1}{qv}\partial_\mu\xi\right)\left(A^\mu + \frac{1}{qv}\partial^\mu\xi\right)$$

and are therefore generated by the gauge change

$$\begin{aligned}\alpha &= \frac{\xi}{qv}\\ A_\mu &\to A_\mu + \partial_\mu\alpha\end{aligned}$$

We see that we can use the gauge transformations

$$A_\mu \to A'_\mu = A_\mu + \frac{1}{qv}\partial^\mu\xi$$

for fixing the vacuum state. The transformation corresponds to the phase rotation

$$\varphi \to \varphi' = e^{-i\frac{\xi}{v}}\varphi = \frac{v + \eta}{\sqrt{2}}$$

In this new gauge where the Goldstone boson ξ disappears, the vector particule A'_μ acquires a mass qv. The Lagrangian writes

$$\mathcal{L}_{os} = \frac{1}{2}\left(\partial_\mu\eta\partial^\mu\eta + 2\mu^2\eta^2\right) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{q^2v^2}{2}A'_\mu A'^\mu$$

The Goldstone boson connecting the degenerate vacuum states is in some sense "captured" by the gauge boson and transformed in mass.

5.2.   NC Yang-Mills theory of 2 **points.**   Let $\mathcal{A} = \mathcal{C}(Y) = \mathbb{C} \oplus \mathbb{C}$ the C^*-algebra of the space Y composed of 2 points a and b. Its elements $f = \left\{ \begin{bmatrix} f(a) & 0 \\ 0 & f(b) \end{bmatrix} \right\}$ act by multiplication on the Hilbert space $\mathcal{H} = \mathcal{H}_a \oplus \mathcal{H}_b$. We take for Dirac operator an operator of the form

$$D = \begin{bmatrix} 0 & M^* = D^- \\ M = D^+ & 0 \end{bmatrix}$$

and introduce the "chirality" $\gamma = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ (the γ_5 of the standard Dirac theory). In this discrete situation we define df as

$$df = [D, f] = \Delta f \begin{bmatrix} 0 & M^* \\ -M & 0 \end{bmatrix}$$

with $\Delta f = f(b) - f(a)$. Therefore

$$\|[D, f]\| = |\Delta f| \, \lambda$$

where $\lambda = \|M\|$ is the greatest eigenvalue of M.

If we apply now the formula for the distance:

$$\begin{aligned} d(a, b) &= \operatorname{Sup} \{|f(a) - f(b)| : f \in \mathcal{A}, \|[D, f]\| \leq 1\} \\ &= \operatorname{Sup} \{|f(a) - f(b)| : f \in \mathcal{A}, |f(a) - f(b)| \lambda \leq 1\} \\ &= \frac{1}{\lambda} \end{aligned}$$

we see that the distance $\frac{1}{\lambda}$ between the two points a and b has a *spectral* content and is measured by the Dirac operator.

We consider now the 2 idempotents (projectors) e and $1 - e$ defined by

$$\begin{aligned} e(a) &= 1, e(b) = 0 \\ (1 - e)(a) &= 0, (1 - e)(b) = 1 \end{aligned}$$

Every $f \in \mathcal{A}$ writes $f = f(a)e + f(b)(1 - e)$, and therefore

$$\begin{aligned} df &= f(a)de + f(b)d(1 - e) \\ &= (f(a) - f(b)) \, de \\ &= -(\Delta f) de \\ &= -(\Delta f)ede + (\Delta f)(1 - e)d(1 - e) \end{aligned}$$

ede and $(1 - e)d(1 - e) = -(1 - e)de$ provide a natural basis of the space of 1-forms $\Omega^1 \mathcal{A}$. Let

$$\begin{aligned} \omega &= \lambda ede + \mu(1 - e)d(1 - e) \\ &= \lambda ede - \mu(1 - e)de \end{aligned}$$

a 1-form. ω is represented by

$$\omega = (\lambda e - \mu(1 - e)) [D, e]$$

But $[D, e] = - \begin{bmatrix} 0 & M^* \\ -M & 0 \end{bmatrix}$ and therefore

$$\omega = \begin{bmatrix} 0 & -\lambda M^* \\ -\mu M & 0 \end{bmatrix}$$

Let us now construct the *Yang-Mills theory*. A vector potential V — a connection in the sense of gauge theories — is a self-adjoint 1-form and has the form

$$\begin{aligned} V &= -\overline{\varphi}e + \varphi(1 - e)de \\ &= \begin{bmatrix} 0 & \overline{\varphi}M^* \\ \varphi M & 0 \end{bmatrix}. \end{aligned}$$

Its curvature is the 2-form

$$\theta = dV + V \wedge V$$

and a computation gives

$$\theta = - (\varphi + \overline{\varphi} + \varphi\overline{\varphi}) \begin{bmatrix} -M^*M & 0 \\ 0 & -MM^* \end{bmatrix}.$$

The Yang-Mills action is the integral of the curvature 2-form, that is the *trace* of θ:

$$YM(V) = \text{Trace} \left(\theta^2 \right)$$

But as $\varphi + \overline{\varphi} + \varphi\overline{\varphi} = |\varphi + 1|^2 - 1$ and

$$\text{Trace} \left(\begin{bmatrix} -M^*M & 0 \\ 0 & -MM^* \end{bmatrix}^2 \right) = 2\,\text{Trace} \left((M^*M)^2 \right)$$

we get

$$YM(V) = 2 \left(|\varphi + 1|^2 - 1 \right)^2 \text{Trace} \left((M^*M)^2 \right).$$

5.3. Higgs mechanism. This Yang-Mills action manifests *a pure Higgs phenomenon of symmetry breaking*. The minimum of $YM(V)$ is reached on all the circle $|\varphi + 1|^2 = 1$ and the gauge group $U(1) \times U(1)$ of the unitary elements of $\mathcal{A}$ acts on it by

$$V \to uVu^* + udu^*$$

where $u = \begin{bmatrix} u_1 & 0 \\ 0 & u_2 \end{bmatrix}$ with $u_1, u_2 \in U(1)$.

The field φ is a Higgs bosonic field corresponding to a gauge connection on a NC space of 2 points.

6.   The NC derivation of the Glashow-Weinberg-Salam standard model

A remarkable achievement of this NC approach of Yang-Mills theories is given by Connes NC derivation of the Glashow-Weinberg-Salam standard model. It is easy to reinterpret in the NC framework classical gauge theories where M is a spin manifold, $\mathcal{A} = \mathcal{C}^\infty(M)$, D is the Dirac operator and $\mathcal{H} = L^2(S)$ is the space of L^2 sections of the spinor bundle S. $\text{Diff}(M) = \text{Aut}(\mathcal{A}) = \text{Aut}(\mathcal{C}^\infty(M))$ is the relativity group (the gauge group) of the theory.

In $\text{Aut}(\mathcal{A})$ there exists the subgroup $\text{Inn}(\mathcal{A})$ of inner automorphisms acting by conjugation $f \to ufu^{-1}$. It is trivial in the commutative case and is one of the main feature of the non commutative case.

In $\text{Inn}(\mathcal{A})$ there exists the *unitary* group $\mathcal{U}(\mathcal{A})$ of unitary elements $u^* = u^{-1}$ acting by $f \to ufu^*$.

In the NC framework we can easily formulate standard Yang-Mills theories. A vector potential V is a self-adjoint operator interpreting a 1-form

$$V = \sum_j a_j[D, b_j]$$

and the force is the curvature 2-form

$$\theta = dV + V^2$$

The unitary group $\mathcal{U}(\mathcal{A})$ acts on V by gauge transformations

$$\begin{aligned} V &\to uVu^* + udu^* = uVu^* + u[D, u^*] \\ \theta &\to u\theta u^*. \end{aligned}$$

The interest of NC geometry is the possibility of coupling such classical gauge theories with purely NC ones introducing Higgs fields. Connes main result is:

Connes theorem. The Glashow-Weinberg-Salam standard model can be entirely reconstructed from the C^*-algebras

$$\mathcal{A} = \mathcal{C}^\infty(M) \otimes (\mathbb{C} \oplus \mathbb{H} \oplus M^3(\mathbb{C})).$$

where the "internal algebra $\mathbb{C} \oplus \mathbb{H} \oplus M^3(\mathbb{C})$ has for unitary group the symmetry group

$$U(1) \times SU(2) \times SU(3).$$

The first step is to construct the model which is the product $\mathcal{C}^\infty(M) \otimes (\mathbb{C} \oplus \mathbb{C})$ of the classical Dirac fermionic model $(\mathcal{A}_1, \mathcal{H}_1, D_1, \gamma_5)$ and the previous purely NC 2-points model $(\mathcal{A}_2, \mathcal{H}_2, D_2, \gamma)$ with $D_2 = \begin{bmatrix} 0 & M^* \\ M & 0 \end{bmatrix}$:

$$\begin{aligned} \mathcal{A} &= \mathcal{A}_1 \otimes \mathcal{A}_2 \\ \mathcal{H} &= \mathcal{H}_1 \oplus \mathcal{H}_2 \\ D &= D_1 \otimes 1 + \gamma_5 \otimes D_2 \end{aligned}$$

The second step is to complexify the model and to show that it enables to derive the complete GWS Lagrangian

$$\mathcal{L} = \mathcal{L}_G + \mathcal{L}_f + \mathcal{L}_\varphi + \mathcal{L}_Y + \mathcal{L}_V.$$

1. $\mathcal{L}_G$ is the Lagrangian of the gauge bosons

$$
\begin{aligned}
\mathcal{L}_G &= \frac{1}{4}\left(G_{\mu\nu a}G_a^{\mu\nu}\right) + \frac{1}{4}\left(F_{\mu\nu}F^{\mu\nu}\right) \\
G_{\mu\nu a} &= \partial_\mu W_{\nu a} - \partial_\nu W_{\mu a} + g\varepsilon_{abc}W_{\mu b}W_{\nu c},\ W_{\mu a}\ SU(2)\ \text{gauge field} \\
F_{\mu\nu} &= \partial_\mu B_\nu - \partial_\nu B_\mu,\ B_\mu\ SU(1)\ \text{gauge field.}
\end{aligned}
$$

2. $\mathcal{L}_f$ is the fermionic kinetic term

$$
\begin{aligned}
\mathcal{L}_f = &-\sum \overline{f_L}\gamma^\mu\left(\partial_\mu + ig\frac{\tau_a}{2}W_{\mu a} + ig'\frac{Y_L}{2}B_\mu\right)f_L + \\
&\overline{f_R}\gamma^\mu\left(\partial_\mu + ig'\frac{Y_R}{2}B_\mu\right)f_R
\end{aligned}
$$

with $f_L = \begin{bmatrix} \nu_L \\ e_L \end{bmatrix}$ left fermion fields of hypercharge $Y_L = -1$ and $f_R = (e_R)$ right fermion fields of hypercharge $Y_R = -1$.

3. $\mathcal{L}_\varphi$ is the Higgs kinetic term

$$\mathcal{L}_\varphi = -\left|\left(\partial_\mu + ig\frac{\tau_a}{2}W_{\mu a} + i\frac{g'}{2}B_\mu\right)\varphi\right|^2$$

with $\varphi = \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix}$ a $SU(2)$ pair of scalar complex fields.

4. $\mathcal{L}_Y$ is a Yukawa coupling between the Higgs fields and the fermions

$$\mathcal{L}_Y = -\sum\left(H_{ff'}\left(\overline{f_L}.\varphi\right)f_R' + H_{ff'}^*\overline{f'_R}\left(\varphi^+.f_L\right)\right)$$

where $H_{ff'}$ is a coupling matrix.

5. $\mathcal{L}_V$ is the Lagrangian of the self-interaction of the Higgs fields

$$\mathcal{L}_V = \mu^2\left(\varphi^+\varphi\right) - \frac{1}{2}\lambda\left(\varphi^+\varphi\right)^2.$$

7.   Quantum gravity and NCG

Recently Connes realized a breakthrough in Quantum Gravity by coupling such models with General Relativity. In NCG QG can be thought of in a principled way and not as a "bricolage". Indeed it becomes possible to introduce in the model a purely gravitational Hilbert-Einstein action.

The general strategy is to find a C^*-algebra $\mathcal{A}$ s.t. $\mathrm{Inn}(\mathcal{A})$ is the gauge group and $\mathrm{Out}(\mathcal{A})$ plays the role of $\mathrm{Diff}(M)$ in a gravitational theory.

8. THE NC IDENTIFICATION OF 2 POINTS AND THE GELFAND-NEIMARK-SEGAL CONSTRUCTION

Let us explain how the NC identification of 2 points is different from the classical identification.

Let $Y = \{a, b\}$ a set of two points. The (commutative) algebra $\mathcal{C}(Y)$ of functions on Y is

$$\mathcal{A} = \mathcal{C}(Y) = \mathbb{C} \oplus \mathbb{C} = \left\{ \left[\begin{array}{cc} f(a) & 0 \\ 0 & f(b) \end{array} \right] \right\}$$

There are two completely different algebraic ways of defining the quotient space $X = Y/(a \sim b)$ in terms of $\mathcal{C}(Y)$.

1. In the classical commutative way, one considers the sub-algebra of $\mathcal{C}(Y)$ of the f s.t. $f(a) = f(b)$.

2. In the non commutative way, one considers on the contrary $\mathcal{C}(Y)$ as a sub-algebra of the non commutative algebra $\mathcal{B} = M^2(\mathbb{C})$. An element $f = \left[\begin{array}{cc} f_{aa} & f_{ab} \\ f_{ba} & f_{bb} \end{array} \right] \in \mathcal{B}$ is therefore interpreted as a function not over $\{a, b\}$ but *over an identification diagram*

$$\circlearrowleft a \rightleftarrows b \circlearrowright$$

To understand why this is a true identification we must return to the Gelfand-Neimark-Segal construction in the NC case. The problem is to reverse the classical construction of QM where the basic structure is the Hilbert space of states H and the observables are (self-adjoint) operators on $\mathcal{H}$. On the contrary, in the Gelfand-Neimark-Segal construction one starts with a C^*-algebra $\mathcal{A}$ and define states as secondary construed structures. Normally, if $\pi : \mathcal{A} \to \mathcal{L}(\mathcal{H})$ is a representation of the C^*-algebra $\mathcal{A}$ in the Hilbert space $\mathcal{H}$ and if $\xi \in \mathcal{H}$ is a state, $\omega_\xi : \mathcal{A} \to C$ defined by the scalar product $\omega_\xi(f) = (\xi, \pi(f)\xi)$ is a positive ($\omega_\xi(f^*f) \geq 0 \; \forall f \in \mathcal{A}$) normalized linear form on $\mathcal{A}$. This suggests to *define* a state ω as a positive normalized linear form on $\mathcal{A}$. If $f \in \mathcal{A}$ is an observable, $\omega(f)$ is interpreted as the *expectation* of f in state ω.

The space of states $\mathcal{S}(\mathcal{A})$ is *convex* and its extremal elements are the *pure* states.

We use then the formalism of *Hilbertian algebras.* If $\omega \in \mathcal{S}(\mathcal{A})$ is a state one can use it to define a *scalar product* $(f, g) = \omega(f^*g)$ on $\mathcal{A}$. In the case of $\omega = \omega_\xi$, we have

$$\omega_\xi(f^*g) = (\xi, \pi(f^*g)\xi) = (\xi, \pi(f)^*\pi(g)\xi) = (\pi(f)\xi, \pi(g)\xi)$$

$(f, g) = \omega(f^*g)$ defines a preHilbertian structure. If we take the quotient by the ideal $\mathcal{N}_\omega$ of "null norm" elements (i.e. by the f s.t. $\omega(f^*f) = 0$), the *completion* of $\mathcal{A}$ w.r.t. (f, g) becomes an Hilbert space $\mathcal{H}_\omega$ on which

$\mathcal{A}$ acts by multiplication. Moreover there exists a "cyclic" element $\xi_\omega \in \mathcal{H}_\omega$ having the property that

$$\omega(f) = (\xi_\omega, f\xi_\omega) \, \forall f \in \mathcal{A}$$

ξ_ω is simply 1 *modulo* $\mathcal{N}_\omega$ since $(1, f1) = \omega(1^* f) = \omega(f)$. A state ω is *pure* if and only if the representation of $\mathcal{A}$ in $\mathcal{H}_\omega$ is *irreducible*.

In our case, we look at the 2 pure states $\omega_a(f) = f_{aa}$ and $\omega_b(f) = f_{bb}$ giving the values of the f at the 2 points a and b.

We have

$$f^* f = \left[\begin{array}{cc} |f_{aa}|^2 + |f_{ba}|^2 & \overline{f_{aa}} f_{ab} + \overline{f_{ba}} f_{bb} \\ \overline{f_{ab}} f_{aa} + \overline{f_{bb}} f_{ba} & |f_{ab}|^2 + |f_{bb}|^2 \end{array} \right]$$

Therefore $\omega_a(f^* f) = |f_{aa}|^2 + |f_{ba}|^2$ and the "null norm" condition means $f_{aa} = f_{ba} = 0$:

$$\mathcal{N}_a = \mathcal{N}_{\omega_a} = \left\{ \left[\begin{array}{cc} 0 & f_{ab} \\ 0 & f_{bb} \end{array} \right] \right\}$$

The associated Hilbert space is then

$$\mathcal{H}_a = \mathcal{H}_{\omega_a} = \left\{ \left[\begin{array}{cc} f_{aa} = x_1 & 0 \\ f_{ba} = x_2 & 0 \end{array} \right] \right\} \simeq \mathbb{C}^2 = \left\{ X = \left[\begin{array}{c} x_1 \\ x_2 \end{array} \right] \right\}$$

with the scalar product $(X, X') = \overline{x_1} x_1' + \overline{x_2} x_2'$.

The elements of $\mathcal{B} = M^2(\mathbb{C})$ act on the elements of $\mathcal{H}_a$ by multiplication

$$gX = \left[\begin{array}{cc} g_{aa} x_1 + g_{ab} x_2 & 0 \\ g_{ba} x_1 + g_{bb} x_2 & 0 \end{array} \right]$$

and a cyclic vector is the identity matrix $\left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right]$ modulo $\mathcal{N}_a$ that is the vector $\xi_a = \left[\begin{array}{c} 1 \\ 0 \end{array} \right]$.

In exactly the same way $\omega_b(f^* f) = |f_{ab}|^2 + |f_{bb}|^2$ and the "null norm" condition means $f_{ab} = f_{bb} = 0$:

$$\mathcal{N}_b = \mathcal{N}_{\omega_b} = \left\{ \left[\begin{array}{cc} f_{aa} & 0 \\ f_{ba} & 0 \end{array} \right] \right\}$$

The associated Hilbert space is then

$$\mathcal{H}_b = \mathcal{H}_{\omega_b} = \left\{ \left[\begin{array}{cc} 0 & f_{ab} = y_1 \\ 0 & f_{bb} = y_2 \end{array} \right] \right\} \simeq \mathbb{C}^2 = \left\{ Y = \left[\begin{array}{c} y_1 \\ y_2 \end{array} \right] \right\}$$

with the scalar product $(Y, Y') = \overline{y_1} y_1' + \overline{y_2} y_2'$.

The elements of $\mathcal{B} = M^2(\mathbb{C})$ act on the elements of $\mathcal{H}_b$ by multiplication

$$gY = \begin{bmatrix} 0 & g_{aa} y_1 + g_{ab} y_2 \\ 0 & g_{ba} y_1 + g_{bb} y_2 \end{bmatrix}$$

and a cyclic vector is the identity matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ modulo $\mathcal{N}_b$ that is the vector $\xi_b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Now the point is that the operator $U = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ establishes an equivalence between these 2 irreducible representations of $\mathcal{B} = M^2(\mathbb{C})$ which exchanges their cyclic vectors (in fact there exists only one irreducible representation of $\mathcal{B} = M^2(\mathbb{C})$, the standard one on $\mathbb{C}^2$).

It is this equivalence which expresses the NC identification of the 2 points a and b.