The structural link between “external” metric and “internal” gauge symmetries in noncommutative geometry

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1 Introduction

In this talk I would like to discuss the propositions of Alain Connes' noncommutative geometry (NCG) for tackling the problem of unifying general relativity (GR) and quantum field theory (QFT) into a background free theory. This doesn't solve the problem of Quantum Gravity but indicates a principled way towards a solution.

The problem can be roughly formulated in the following way: the invariance groups are respectively the group of diffeomorphisms $\mathrm{Diff}\,(M)$ for GR (M being the space-time manifold) and the gauge group $\mathcal{G} = C^{\infty}(M, G)$ for QFT where G is the (non abelian) group of internal symmetries. So the total invariance group is the semidirect product

$$\mathfrak{G} = \mathcal{G} \rtimes \mathrm{Diff}\,(M)$$

where $\mathrm{Diff}\,(M)$ acts on the base space M of the fiber bundle $P \to M$ used for describing the fields through $\varphi\,(f)\,(x) = f\left(\varphi^{-1}(x)\right)$ for $\varphi \in \mathrm{Diff}\,(M)$ and $f \in \mathcal{G}$. The group $\mathcal{G} = C^{\infty}(M, G)$ is a *normal* subgroup of $\mathfrak{G}$ (normal = stable by conjugation). Hence the most straightforward route for unification would be to look for a manifold N such that $\mathfrak{G} = \mathrm{Diff}\,(N)$.

As Alain Connes emphasized it:

"Complete geometrization of the Standard Model coupled with gravity (...) means turning the whole coupled theory into pure gravity on a suitable space." ([9], p. 165)

But this is *impossible* due to a fundamental theorem saying that $\mathfrak{G}$ is a *simple* group (i.e. without any non trivial normal subgroup) and, therefore, cannot be a semidirect product.

So, there exists an obstruction to extending the principle of general covariance to a unified theory of GR and QFT within the framework of commutative geometry. But the obstruction can be overcome in the framework of NCG and it is what Alain Connes explained.

2 Structuralism, general covariance, and background independence

Philosophers of physics who have promoted a structuralist point of view have strongly underlined the "active" interpretation of diffeomorphisms $\varphi \in \mathrm{Diff}\,(M)$ in GR.

2.1 Invariance under diffeomorphisms as a constitutive transcendental principle

In his beautiful book *The Reign of Relativity* [24], Thomas Ryckman explains very well the principle of general covariance as a "regulative idea" requiring to redefine space-time in *dynamical* terms. General covariance is more than the mathematical requirement of describing physical entities with tensorial structures having an intrinsic (coordinate-independent) content. As a principle of invariance w.r.t. to active diffeomorphisms, it is physically constitutive and leads to the conclusion that space-time points have no physical content and only enable to implement a *relational* dynamics. In particular, fields have no longer values and properties at such points.

In GR space-time has no separate existence outside the metric $g_{\mu\nu}$ defined by Einstein's equations:

> "There is no such thing as an empty space, i.e., a space without a field. Space-time does not claim existence on its own, but only as a structural quality of the field." (Ryckman [24], p. 18)

Coordinates x^{μ} are labels useful for computing but have no metrical, hence physical, meaning.

> "The points of the space-time manifold (...) do not inherit their individuality, hence physical existence, from the underlying differential-topological structure of the manifold." (Ryckman [24], p. 21)

2.2 Transcendental ideality of space and time

In *Dynamics of Reason*, Michael Friedman [11] has shown how transcendental principles can be generalized to an historical dynamics.

> "What we end up with (...) is thus a relativized and dynamical conception of a priori mathematical-physical principles, which change and developp along with the development of the mathematical and physical sciences themselves,

> but which nevertheless retain the characteristically Kantian constitutive function of making the empirical natural knowledge thereby structured and framed by such principles first possible." (Friedman [11], p. 31)

Indeed, philosophically speaking, GR is an astonishing renewal and revenge of Kant's thesis of *transcendental ideality* of space and time. In my 1992 paper *Actuality of Transcendental Aesthetics for Modern Physics* [22] (see also my 2009 paper [23] in *Constituting Objectivity*), I developped the idea that GR is a radical extension of transcendental aesthetics as Kant used it in his *Metaphysische Anfangsgründe der Naturwissenschaft* [15], which shifts the transcendental moment of "Mechanics" (the third group of "dynamical" categories of relation and the "Analogies of Experience" that correspond to conservation laws, causal physical forces, and interactions) towards an *enlargement* of the transcendental moment of "Phoronomy" or Kinematics (the first group of "mathematical" categories and the "Axioms of Intuition" that correspond to the Euclidean metric of space and Galilean relativity).

2.3 Background independence

To take as constitutive principle

> "the requirement that dynamical laws must be formulated without a "background" space and time." (Ryckman, [24], p. 7)

implies deep consequences. Indeed, the symmetry group of a theory defines the elements of structure "acting but not acted upon", i.e. which are

> "unaffected by dynamical laws and so not among the set of state variables distinguishing differents physical states of affairs." (Ryckman [24], p. 22)

When the symmetry group is $\mathrm{Diff}\,(M)$ then all metric elements become background free and the background structure is reduced to the differentiable structure, which lacks any physical content. A consequence is that, in GR, observables no longer have values at points of M, and are therefore necessarily non local. But to find a natural set of such observables is a highly non trivial challenge.

Of course, one can construct a lot of $\mathrm{Diff}\,(M)$-invariant global observables of the type $\int_M F\,(K)\,\sqrt{g}d^4x$ with $F\,(K)$ an invariant function of the Riemann curvature K. But, as Connes and Marcolli [9] emphasized

> "While in principle [such] a quantity is observable, it is in practice almost impossible to evaluate, since it involves the knowledge of the entire space-time and is in that way highly non-localized." ([9], p. 184)

Hence the practical necessity of finding an other type of dynamical variables.

3 An obstruction to the unification GR-QFT

In gauge theories, there are two sorts of fields:

1. Matter fields, which are interpreted as sections of fiber bundles P on the base space-time M. The fiber coordinates express internal degrees of freedom and their symmetry group G expresses the global internal symmetries.

2. Gauge fields, which are interaction fields represented by connections (potential vectors) on these fiber bundles.

The Yang-Mills Lagrangian is associated with the curvature of the connections and is invariant under the gauge group.

Gauge theories construct interactions by making global internal symmetries depend upon spatio-temporal positions. When one localizes internal symmetries and maintains the requirement of invariance of the field equations, one must introduce corrective terms and the "miracle" is that these supplementary terms are exactly interaction terms.

The simplest case (Hermann Weyl) is that of the minimal coupling between an electron and the electromagnetic field $\mathcal{F}$. Let ψ be the wave function of the electron. Its evolution is driven by Dirac equation. The Dirac Lagrangien $\mathcal{L}_D = \overline{\psi}(x)(i\gamma^\mu\partial_\mu - m)\psi(x)$ is invariant under the global internal symmetry $\psi \to \exp(-ie\theta)$ where e is the electric charge of the electron and θ is a phase). The group of internal symmetries is therefore the phase group $U(1)$.

On the other hand, the Maxwell Lagrangian $\mathcal{L}_M = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - J^\mu A_\mu$ is invariant under a gauge transformation $A \to A + d\Lambda$, where A is a vector potential for the field $\mathcal{F}$ (A being interpreted as a connection on a fiber bundle E over M), and Λ a function on M. If one makes the phase factor θ depend upon the spatio-temporal position $x \in M$, then $\mathcal{L}_D$ is no longer invariant. But the corrective term can be exactly compensated by the gauge transformation $A \to A + d\theta$.

The relativity group is now $C^\infty(M, G) \rtimes \mathrm{Diff}(M)$, which is much larger than $\mathrm{Diff}(M)$.

The problem is that in such gauge theories the metric of M acts as a background structure. How is it possible to eliminate this background metric while, at the same time, keeping the power of geometrical tools and the ideal of geometrizing fundamental physics?

The objective is to enlarge the symmetry principles of unification, whose "explanatory role", as explained by Katherine Brading and Elena Castellani in their Introduction to *Symmetries in Physics* [3],

> "arises from their place in the hierarchy of the structure of physical theory, which in turn derives from their generality."
> ([3], p. 12)

But, as we have seen, $C^\infty(M,G) \rtimes \mathrm{Diff}(M)$ *cannot* be a group of diffeomorphisms of a larger manimold N since $\mathrm{Diff}(N)$ must be *simple*.

4 Simplicity of groups of diffeomorphisms $\mathrm{Diff}(M)$

Let us spend some time on this deep result. It is the consequence of the *loss of interdependence* between *local* and *global* properties at the C^∞ level of structure. Let us give a simple example.

4.1 Local VS global for C^∞maps

Let $f : \mathbb{R} \to \mathbb{R}$ be a real C^∞ function. The best local approximation of f at x by polynomials of degree k is given by the k-jet:

$$j^k f(x)(h) = f(x) + hf'(x) + \ldots + h^k \frac{f^{(k)}(x)}{k!}$$

where $f^{(k)}$ is the k-th derivative of f. For every $x \in \mathbb{R}$ one can consider the formal series $T_f(x)$ (Taylor expansion)

$$T_f(x)(h) = f(x) + hf'(x) + \ldots + h^k \frac{f^{(k)}(x)}{k!} + \ldots$$

For a polynomial P of degree k, it is evident that:

(i) all Taylor expansions stop at order k, *i.e.* $T_P(x) = j^k P(x)$;

(ii) P is locally equal at every point to its Taylor expansion, *i.e.* $P(x+h) = T_P(x)(h) = j^k P(x)(h)$;

(iii) P is equal at every point to each of its Taylor expansions, *i.e.* the approximation at order k is not only local but also global: the local structure at any point determines the global structure.

The situation is completely different for C^∞ functions.

(i) When a C^∞ function f is everywhere equal to its Taylor expansion $T_f(x)$ at a single point x, f is an *entire* function. But a C^∞ fonction is not entire in general.

(ii) When f is equal in a neighborhood of every point $x \in \mathbb{R}$ to its Taylor expansion $T_f(x)$ at x, f is (real) *analytic*. But a C^∞ fonction is not analytic in general.

(iii) Often, f is not even representable by its Taylor expansion: either $T_f(x)$ is a divergent series which cannot represent any function at all, or $T_f(x)$ is convergent but has a value different from $f(x)$.

So, at the C^∞ level of structure, there is no interdependence between local and global structures, and the behavior is completely different from that of analytic maps where the local structure determines through *analytic continuation* the global structure.

A striking manifestation of this local "plasticity" is the existence of C^∞ *partitions of unity*: if $\mathcal{U} = (U_i)_{i \in I}$ is any open covering of a C^∞ manifold M, there exists *localized* C^∞ functions $\varphi_i(x)$ with support in U_i s.t., at every point $x \in M$, $\sum_i \varphi_i(x) = 1$. This means that it is possible to *fragment* the constant function 1 into *local* fragments.

Due to the local/global independence, the topology of functional spaces of C^∞ maps can be rather complex. For instance, in the case of $\mathcal{F} = C^\infty(M, N)$ (with M, N manifolds), the topology is the *Whitney* topology of uniform convergence of the maps and all their derivatives on compact subsets of M, with stationarity at infinity. For the Whitney topology, such functional spaces are *Fréchet* manifolds locally modeled on Fréchet vector spaces which are limits for $k \to \infty$ of decreasing sequences of Banach spaces (normed complete vector spaces) corresponding to the C^k classes.

4.2 The hole argument

Let us emphasize the fact that the local plasticity of diffeomorphisms is basic for the "hole argument" in GR. In a hole U where the tensor $T_{\mu\nu}$ vanishes: $T_{\mu\nu} \equiv 0$, you can use a *local* diffeomorphism φ with $\mathrm{supp}(\varphi) \subset U$ to change the values $g_{\mu\nu}(x)$ of the metric tensor inside U without changing the coordinates. This seems to imply a violation of causality since the tensor $T_{\mu\nu}(x)$ remains constant while, according to Einstein's equation, it must determine the metric. The response to this remark is of course that the two situations $(M, g_{\mu\nu}, T_{\mu\nu})$ and $(M, \varphi^*(g_{\mu\nu}), T_{\mu\nu})$ are physically identical. The argument will not be conclusive for analytic automorphisms of M, which, due to the global coherence of analytic continuation, would necessarily change also $T_{\mu\nu}$.

4.3 The case of homeomorphisms (Anderson)

The story begins with a result of Ulam and von Neumann who proved in 1947 [26] that the group $\mathrm{Homeo}(\mathbb{S}^2)$ of homeomorphisms of the sphere (preserving orientation) is simple.

Then, in 1961, Anderson [1] proved that the group $\mathrm{Homeo}_0(M)$ of homeomorphisms g of a manifold M that are isotopic to Id (i.e. there exists a path $\mathrm{Id} \to g$ in $\mathrm{Homeo}_0(M)$) is simple. The key of the proof is that homeomorphisms g satisfy a property of "fragmentation". Let $g \in \mathrm{Homeo}_0(M)$ and $\mathcal{U} = (U_i)_{i \in I}$ be an open covering of M. Then, there exist local homeomorphisms g_i with $\mathrm{supp}(g_i) \subset U_i$ s.t. $g = \prod_i g_i$. This property implies that $\mathrm{Homeo}_0(M)$ is the *smallest* normal subgroup of $\mathrm{Homeo}(M)$ and hence is simple.

Stephen Smale (Fields Medal 1966 for the proof of Poincaré's conjecture in dimension ≥ 5) then asked if it is also the case for diffeomorphisms.

4.4 The case of diffeomorphisms of the torus (Herman)

The first deep result for diffeomorphisms was proved by Michael Herman in 1973 [12] for the torus $\mathbb{T}^n$. The proof is much more difficult because when one tries to prove smoothness properties one comes up against problems of "small denominators" discovered by Poincaré and lying at the core of the celebrated KAM theorem (Kolmogorov-Arnold-Moser).

Let $G = \mathrm{Diff}_0(\mathbb{T}^n)$ be the group of diffeomorphisms of the n-torus that are isotopic to Id (G is the connected component of Id inside $\mathrm{Diff}(\mathbb{T}^n)$). The first move is to show that G is *perfect*, i.e. equal to its commutator $[G, G]$ ($[G, G]$ is the subgroup composed by products of commutators $[f, g]$; it is normal because it is stable by conjugation: $h^{-1}[f, g]h = [h^{-1}fh, h^{-1}gh]$). To be perfect is the opposed property of being abelian $[G, G] = 0$.

$\mathbb{T}^n$ is embedded in G through the *translations* $R_\gamma : x \mapsto x + \gamma$. Translations are products of commutators and belong to $[G, G]$.

Let $\Phi_\gamma : G \times \mathbb{T}^n \to G$ be the map

$$(\psi, \lambda) \mapsto R_\lambda \psi^{-1} R_\gamma \psi = R_{\lambda+\gamma} R_\gamma^{-1} \psi^{-1} R_\gamma \psi = R_{\lambda+\gamma}[R_\gamma, \psi]$$

Of course, $\Phi_\gamma(\mathrm{Id}, 0) = R_\gamma$. As $R_{\lambda+\gamma}$ is a product of commutators, $\Phi_\gamma(\psi, \lambda)$ is also a product of commutators and belongs to $[G, G]$.

The key technical lemma says that, *under a Diophantine condition*, Φ_γ is locally *surjective* onto a neighborhood of R_γ. More precisely:

Lemma. If there exist $\beta > 0$ and $C > 0$ s.t. for every $k = (k_i) \neq 0 \in \mathbb{Z}^n$ we have $\|\langle k, \gamma \rangle\| \geq C(\sup|k_i|)^{-\beta}$ (arithmetic condition of KAM type meaning that γ is badly approximated by rational points), then there exists a *section* of Φ_γ, $s : V \to G \times \mathbb{T}^n$, $\Phi_\gamma \circ s = \mathrm{Id}$, over a neighborhood V of R_γ.

In other words, if a diffeomorphism is sufficiently close to a *Diophantine* translation R_γ it can be conjugated to it up to a small translation

R_λ.

The technical lemma implies that the smallest normal subgroup generated by $\mathbb{T}^n$ in G is G itself. But $\mathbb{T}^n \subset [G,G]$, hence $G = [G,G]$ and G is perfect.

Herman uses then a fundamental result, proved by Epstein in 1970 [10], saying that under general conditions of fragmentation and of transitivity of the action of G on the open subsets of M, the commutator $[G,G]$ is *simple*. As $G = [G,G]$, G itself is simple.

The proof of the lemma is rather delicate. The guiding idea is

1. to linearize $\Phi_\gamma(\psi,\lambda)$ in the neighborhood of $(\mathrm{Id},0)$ (whose image is $\Phi_\gamma(\mathrm{Id},0) = R_\gamma$),

2. to show that the linear tangent map $D\Phi_\gamma$ is invertible, and

3. to apply a theorem of local inversion.

The proof of (3) is not trivial because the functional spaces involved are not Banach spaces but Fréchet spaces and one has to use the Nash-Moser-Hamilton theorem of local inversion saying that if the tangent map $D\Phi_\gamma(\psi,\lambda)$ is invertible in a *full neighborhood* of $(\mathrm{Id},0)$ then Φ_γ is locally invertible at $(\mathrm{Id},0)$.

It is for proving (2) that the Diophantine condition becomes essential. Indeed to solve $D\Phi_\gamma(\psi,\lambda)\left(\widehat{\psi},\widehat{\lambda}\right) = \widehat{\eta}$ for $\widehat{\psi}$, $\widehat{\lambda}$ tangent vectors to G and $\mathbb{T}^n$ respectively at ψ and λ, and $\widehat{\eta}$ a tangent vector to G at $\Phi_\gamma(\psi,\lambda) = R_\lambda\psi^{-1}R_\gamma\psi = R_{\lambda+\gamma}[R_\gamma,\psi]$, one uses Fourier expansions where differentiability is controled by the rapidity of decrease at infinity of the Fourier coefficients. The Diophantine condition assures that, if the coefficients of the data are rapidly decreasing at infinity, then the coefficients of the solutions are also rapidly decreasing at infinity.

4.5 The general case (Thurston, Mather)

Let now $G = \mathrm{Diff}_0(M)$ and $G_U = \mathrm{Diff}_0(U)$ for any open subset U of M. One shows first that G satisfies the fragmentation property in the sense that, if $g \in M$ and if $\mathcal{U} = (U_i)_{i\in I}$ is an open covering of M, then g is a product of g_i with $\mathrm{Supp}(g_i) \subset U_i$. This implies that, if the local groups G_U are perfect, then the global group G is perfect, hence simple according to Epstein's theorem.

To show that the G_U are perfect, it is sufficient to prove it in the torus case. Now, according to Herman's theorem, any $g \in G_U$ is a product of commutators in G. So, the last technical move is to prove that $g \in G_U$ is also a product of commutators in G_U.

One can use a type of homology, called "cubic" homology, of the torus, and the key result that, if $\widetilde{G}$ is the universal covering of G, then

$$\frac{\widetilde{G}}{\left[\widetilde{G},\widetilde{G}\right]} = H_1\left(G\right)$$

which means that $H_1\left(G\right)$ is the abelianization of $\widetilde{G}$.

Now, Thurston [25] and Mather [21] (in his papers, Mather proved also the cases C^r for $r \neq n+1$) proved the *deformation theorem* saying that the embedding $G_U \hookrightarrow G$ induces un *isomorphism* $H_1\left(G_U\right) \simeq H_1\left(G\right)$.

So: (i) $\widetilde{G}$ is perfect iff $H_1\left(G\right) = 0$, (ii) but for the torus, $H_1\left(G_U\right) \simeq H_1\left(G\right)$, but (iii) G is perfect for the torus, therefore (iv) $H_1\left(G\right) = 0$, therefore (v) $H_1\left(G_U\right) = 0$ and (vi) $\widetilde{G_U}$ is perfect for the torus, which implies (vii) G_U is perfect for the torus, hence (viii) G_U is perfect for M, and so (ix) G is perfect and simple for M.

5 The relevance of noncommutative geometry (NCG)

Let us emphasize now the relevance of NCG in this context. Technicalities being very complex we can present here only guiding ideas (for details, see my chapter [23] in *Constituting Objectivity*).

5.1 Gelfand's theory

The starting point is *Gelfand's theory* establishing a perfect *duality* between locally compact Hausdorff topological spaces X and *commutative* C^*-algebra $\mathcal{A} = C\left(X\right)$. If X is compact, $\mathcal{A}$ is unital. In a C^*-algebra $\mathcal{A}$, that is a Banach algebra (i.e. normed and complete) endowed with an involution $a \to a^*$ s.t. $\left\|a\right\|^2 = \left\|a^*a\right\|$, $\left\|a\right\|^2$ is the spectral radius of the ≥ 0 element a^*a and the norm becomes a purely *spectral* concept ($a \in \mathcal{A}$ is self-adjoint if $a = a^*$, normal if $aa^* = a^*a$, and unitary if $a^{-1} = a^*$ ($\left\|a\right\| = 1$)).

The points $x \in X$ are associated with the maximal ideals $\mathfrak{M}_x = \{f \in \mathcal{A} : f(x) = 0\}$, which constitute the *spectrum* of $\mathcal{A}$ and are the kernels of the *characters* (multiplicative linear forms $\chi : \mathcal{A} \to \mathbb{C}$) of $\mathcal{A}$: $\mathfrak{M} = \chi^{-1}(0)$.

The main philosophical point is that the existence of geometrically *individuated points* $x \in X$ is fundamentally linked with the *commutativity* of the algebra $\mathcal{A} = C\left(X\right)$.

5.2 Spectral geometry

Now, coming back to the commutative C^*-algebra $\mathcal{A} = C^\infty\left(M\right)$, we must point out that the diffeomorphisms $\varphi \in \mathrm{Diff}\left(M\right)$ are essentially the same thing as the group $\mathrm{Aut}(\mathcal{A})$ of $*$-*automorphisms* of $\mathcal{A}$ through $\varphi(f)(x) = f(\varphi^{-1}(x))$ for $f \in \mathcal{A}$.

So the obstruction resulting from the simplicity of $\mathrm{Diff}(M)$ comes from the fact that $\mathrm{Aut}(\mathcal{A}) = \mathrm{Aut}(\mathcal{C}^{\infty}(M))$ cannot be a semidirect product for this kind of *commutative* algebra.

But in the NC case, there exists in $\mathrm{Aut}(\mathcal{A})$ the normal subgroup $\mathrm{Inn}(\mathcal{A})$ of *inner* automorphisms acting by conjugation $a \to uau^{-1}$. In $\mathrm{Inn}(\mathcal{A})$ there exists the *unitary* group $\mathcal{U}(\mathcal{A})$ of unitary elements $u^* = u^{-1}$ acting by $\alpha_u(a) = uau^*$. $\mathrm{Inn}(\mathcal{A})$ is *trivial* in the commutative case.

The main idea of NCG is to start with NC C^*-algebras and to look at the generalized NC spaces with which they are in duality. These NC spaces are "without" points because their NC algebras of coordinates have no maximal ideals and go therefore far beyond the classical concept of "localization". In a sense, as *Spectral Geometry* or *Quantum Geometry*, NCG starts from the outset with QM and "quantizes" in algebraic terms all classical geometrical concepts of differential geometry, Riemannian geometry and Cartan geometry.

We can say that Alain Connes realized that before enlarging the ideal of geometrizing physics up to a unification of GR and QFT, one must aim first at quantizing geometry. As explains Daniel Kastler [18]:

> "Alain Connes' noncommutative geometry (...) is a systematic quantization of mathematics parallel to the quantization of physics effected in the twenties. (...) This theory widens the scope of mathematics in a manner congenial to physics."

In particular, in NCG, one can give a deep *spectral reinterpretation of metric* using the formalism of Dirac operators and Clifford algebras.

5.3 Dirac operators and metric

In the classical case of the vector space $V = \mathbb{R}^n$ endowed with the Euclidean scalar product g, the Dirac operator $D = \sum_{\mu=1}^{\mu=n} \gamma^{\mu} \frac{\partial}{\partial x^{\mu}}$ is of the form $D = \sum_{\mu=1}^{\mu=n} c(dx^{\mu}) \frac{\partial}{\partial x^{\mu}}$ where c is an endomorphism of the exterior algebra $\Lambda^* V$ satisfying the non trivial *anti-commutation* relations $\{c(v), c(w)\} = -2g(v, w)$.

The $c(v)$ (identified with v) generate into $\mathrm{End}_{\mathbb{R}}(\Lambda^* V)$ the Clifford algebra $\mathrm{Cl}(V, g)$, which is the quotient of the tensor algebra of V by the relations $\{v, w\} = -2g(v, w)$. $\mathrm{Cl}(V, g)$ is isomorphic to the exterior algebra $\Lambda^* V$ as a vector space but not as a $\mathbb{R}$-algebra. Indeed, the antisymmetry $w \wedge v = -v \wedge w$ corresponds to $\{v, w\} = 0$, i.e. to the totally degenerate metric $g = 0$. We can say that $\mathrm{Cl}(V, g)$ uses the metric g to "twist" the exterior product of differential forms.

This can be generalized first to any metric g and then to cotangent bundles $V_x = (T_x^* M, g^{-1})$ of Riemannian manifolds M. The most interesting situation is when M is a Riemannian spin manifold (the gauge group $SO(n)$ can be extended to $\mathrm{Spin}(n)$) and S is a spinor bundle (bundle of $\mathrm{Cl}(TM)$-modules s.t. $\mathrm{Cl}(TM) \simeq \mathrm{End}(S)$). D can be extended from the $C^\infty(M)$-module $\Gamma(S)$ of sections of S to the Hilbert space $\mathcal{H} = L^2(M, S)$.

One shows that for $f \in C^\infty(M)$, considered as an operator of multiplication in $\mathcal{H} = L^2(M, S)$, one get the fondamental formula

$$[D, f] = c\,(df)$$

Moreover, one can prove that the classical definition of distance $d(p,q) = \operatorname*{Inf}_{\gamma : p \to q} L(\gamma)$ (with $L(\gamma) = \int_p^q ds = \int_p^q (g_{\mu\nu} dx^\mu dx^\nu)^{1/2}$ = the length of the curves) is equivalent to the *dual* definition:

$$d(p,q) = \mathrm{Sup}\,\{|f(p) - f(q)| : f \in \mathcal{A}, \|[D, f]\| \leq 1\}$$

6 The NC derivation of the GWS standard model (Connes-Lott)

6.1 Differential and metric NC geometry

Relying on these basic results, NCG defines a NC metric geometry as a *spectral triple* $(\mathcal{A}, \mathcal{H}, D)$, where $\mathcal{A}$ is a NC C^*-algebra, $\mathcal{H}$ a Hilbert space endowed with a representation of $\mathcal{A}$, and D an unbounded operator whose inverse D^{-1} defines the line element ds.

The metric is no longer induced by an underlying Riemannian manifold on which $\mathcal{A}$ would be an algebra of functions, but by an *independent operator* on the Hilbert $\mathcal{H}$ where $\mathcal{A}$ is represented. In the NC framework, the metric concerns a NC space correlative to a NC algebra of observables including non commutant coordinates with which ds doesn't commute.

As Alain Connes [7] emphasized:

> "It is precisely this lack of commutativity between the line element and the coordinates on a space [between $ds = D^{-1}$ and the $a \in \mathcal{A}$] that will provide the measurement of distance."

This remark is philosophically fundamental. The commutativity between the coordinates x^μ and the line element ds implies two key consequences in the classical setting:

1. the fact that the x^μ can have well defined values: points have a physical meaning;

2. the fact that ds is localized w.r.t. the x^μ.

> "When the hypothesis of commutativity is dropped it is no longer the case that the line element ds needs to be localized and in fact it is precisely the lack of commutation of ds with the coordinates that makes it possible to measure distances." (Connes, Marcolli [9], p. 168)

With such spectral triples one can develop a generalized NC differential geometry. The main constraint is that differentials df must be *infinitesimal*. Connes defines the property for an operator T on $\mathcal{H}$ to be infinitesimal by the property of being *compact*: the eigenvalues $\mu_n(T)$ of $|T| = (T^*T)^{1/2}$ converge to 0. If $\mu_n(T) \underset{n\to\infty}{\to} 0$ as $\frac{1}{n^\alpha}$, then T is an infinitesimal of order α.

If T is an infinitesimal of order 1, its trace $\mathrm{Trace}\,(T) = \sum_n \mu_n(T)$ presents a logarithmic divergence. The basic tool for *NC integration* is the *Dixmier trace* extracting the logarithmic divergence. It vanishes for infinitesimals of order > 1.

To get a "good" NC differential calculus, one assumes that all $[D, a]$ are compact for $a \in \mathcal{A}$. Then differentials can be defined by $da = [D, a]$ for every observable $a \in \mathcal{A}$. It must be emphasized that the differentials are defined through the Dirac operator which does not belong to the algebra $\mathcal{A}$.

The *dimension* n of the NC space is encoded in the fact that $ds = D^{-1}$ is an infinitesimal of order $\frac{1}{n}$ (i.e. the eigenvalues of $|ds|$ decrease to 0 as $\frac{1}{\sqrt[n]{k}}$ when $k \to \infty$) and therefore that $ds^n = D^{-n}$ is an infinitesimal of order 1.

In classical geometry there exist commutation relations: $[[D, a], b] = 0$, for every $a, b \in \mathcal{A}$. So, according to Jones and Moscovici [14]:

> "while ds no longer commutes with the coordinates, the algebra they generate does satisfy non trivial commutation relations."

Moreover, there exists a *real structure*, i.e. an anti-linear isometry $J : \mathcal{H} \to \mathcal{H}$ retrieving the $*$-involution via algebraic conjugation: $JaJ^{-1} = a^*$ for every $a \in \mathcal{A}$.

In the noncommutative case, the real structure $JaJ^{-1} = a^*$ is more sophisticated. Let $b^0 = Jb^*J^{-1}$, then $[a, b^0] = 0$ for every $a, b \in \mathcal{A}$ and $\mathcal{H}$ becomes a $\mathcal{A} \otimes \mathcal{A}^\circ$-module (where $\mathcal{A}^\circ$ is the opposed algebra of $\mathcal{A}$) or a (left-right) $\mathcal{A}$-bimodule through

$$\left(a \otimes b^0\right) \xi = aJb^*J^{-1}\xi \text{ or } a\xi b = aJb^*J^{-1}\xi.$$

The universal commutation relations $[[D,f],g]=0$ become $[[D,a],b^\circ]=0$, for every $a,b\in\mathcal{A}$ (which is equivalent to $[[D,b^\circ],a]=0$ since a and b° commute).

6.2 Gauge theory and NCG

Connes and Lott derived *all the terms* of the Glashow-Weinberg-Salam standard model using a quite simple NC gauge theory. It was an extraordinary "tour de force". Here is the GWS Lagrangian.

- Gauge bosons: A_μ, $W^\pm$, Z^0_μ, g^a_μ
- Quarks: u^κ_j, d^κ_j collectively denoted q^κ_j (the index κ is the generation (flavor) index, the index j is the color index).
- Leptons: e^κ, ν^κ.
- Higgs fields: H, ϕ^0, ϕ^+, ϕ^-.
- Masses: m^κ_d, m^κ_u, m^κ_e, m_h (Higgs mass) and M (mass of the W).
- c_w and s_w: cosine and sine of the weak mixing angle θ_w.
- $C_{\lambda\kappa}$: Cabibbo-Kobayashi-Maskawa mixing matrix.
- Coupling constants: g (with $\alpha=\frac{s_w^2 g^2}{4\pi}$ the fine structure constant), g_s (coupling of the strong force), $\alpha_h=\frac{m_h^2}{4M^2}$ (Higgs scattering parameter).
- β_h: tadpole constant.
- f^{abc}: structure constants of the Lie algebra of $SU(3)$.
- λ^a_{ij}: Gell-Mann matrices.
- Ghosts: G^a, X^0, X^+, X^-, Y.
- The choice of gauge fixing is the Feynman gauge for all gauge fields except the $SU(2)$ ones and the Feynman-'t Hooft gauge for the $SU(2)$ gauge fields.

As we have seen, the main problem for reconciling QFT with GR, is to mix non abelian gauge theories which are non commutative at the level of their *internal* quantum variables with the geometry of the *external* space-time M whose diffeomorphism group $\mathrm{Diff}(M)$ is the automorphism group $\mathrm{Aut}\left(C^\infty(M)\right)$ of a *commutative* C^*-algebra $C^\infty(M)$.

In classical Yang-Mills theories one works with a fiber bundle $P\to M$ with a non Abelian Lie group G acting upon the fibers P_x, $\mathfrak{g}$ its Lie

$$
\begin{gathered}
\mathcal{L}_{SM} = -\tfrac{1}{2}\partial_\nu g^a_\mu \partial_\nu g^a_\mu - g_s f^{abc}\partial_\mu g^a_\nu g^b_\mu g^c_\nu - \tfrac{1}{4}g_s^2 f^{abc}f^{ade}g^b_\mu g^c_\nu g^d_\mu g^e_\nu - \partial_\nu W^+_\mu \partial_\nu W^-_\mu - M^2 W^+_\mu W^-_\mu - \\
\tfrac{1}{2}\partial_\nu Z^0_\mu \partial_\nu Z^0_\mu - \tfrac{1}{2c_w^2}M^2 Z^0_\mu Z^0_\mu - \tfrac{1}{2}\partial_\mu A_\nu \partial_\mu A_\nu - igc_w(\partial_\nu Z^0_\mu(W^+_\mu W^-_\nu - W^+_\nu W^-_\mu) - Z^0_\nu(W^+_\mu \partial_\nu W^-_\mu - \\
W^-_\mu \partial_\nu W^+_\mu) + Z^0_\mu(W^+_\nu \partial_\nu W^-_\mu - W^-_\nu \partial_\nu W^+_\mu)) - igs_w(\partial_\nu A_\mu(W^+_\mu W^-_\nu - W^+_\nu W^-_\mu) - A_\nu(W^+_\mu \partial_\nu W^-_\mu - \\
W^-_\mu \partial_\nu W^+_\mu) + A_\mu(W^+_\nu \partial_\nu W^-_\mu - W^-_\nu \partial_\nu W^+_\mu)) - \tfrac{1}{2}g^2 W^+_\mu W^-_\mu W^+_\nu W^-_\nu + \tfrac{1}{2}g^2 W^+_\mu W^-_\nu W^+_\mu W^-_\nu + \\
g^2 c_w^2(Z^0_\mu W^+_\mu Z^0_\nu W^-_\nu - Z^0_\mu Z^0_\mu W^+_\nu W^-_\nu) + g^2 s_w^2(A_\mu W^+_\mu A_\nu W^-_\nu - A_\mu A_\mu W^+_\nu W^-_\nu) + g^2 s_w c_w(A_\mu Z^0_\nu(W^+_\mu W^-_\nu - \\
W^+_\nu W^-_\mu) - 2A_\mu Z^0_\mu W^+_\nu W^-_\nu) - \tfrac{1}{2}\partial_\mu H \partial_\mu H - \tfrac{1}{2}m_h^2 H^2 - \partial_\mu \phi^+ \partial_\mu \phi^- - M^2 \phi^+ \phi^- - \tfrac{1}{2}\partial_\mu \phi^0 \partial_\mu \phi^0 - \tfrac{1}{2c_w^2}M^2 \phi^0 \phi^0 - \\
\beta_h \left(\tfrac{2M^2}{g^2} + \tfrac{2M}{g}H + \tfrac{1}{2}(H^2 + \phi^0\phi^0 + 2\phi^+\phi^-)\right) + \tfrac{2M^4}{g^2}\alpha_h - g\alpha_h M\left(H^3 + H\phi^0\phi^0 + 2H\phi^+\phi^-\right) - \\
\tfrac{1}{8}g^2\alpha_h\left(H^4 + (\phi^0)^4 + 4(\phi^+\phi^-)^2 + 4(\phi^0)^2\phi^+\phi^- + 4H^2\phi^+\phi^- + 2(\phi^0)^2H^2\right) - gMW^+_\mu W^-_\mu H - \\
\tfrac{1}{2}g\tfrac{M}{c_w^2}Z^0_\mu Z^0_\mu H - \tfrac{1}{2}ig\left(W^+_\mu(\phi^0\partial_\mu\phi^- - \phi^-\partial_\mu\phi^0) - W^-_\mu(\phi^0\partial_\mu\phi^+ - \phi^+\partial_\mu\phi^0)\right) + \\
\tfrac{1}{2}g\left(W^+_\mu(H\partial_\mu\phi^- - \phi^-\partial_\mu H) + W^-_\mu(H\partial_\mu\phi^+ - \phi^+\partial_\mu H)\right) + \tfrac{1}{2}g\tfrac{1}{c_w}(Z^0_\mu(H\partial_\mu\phi^0 - \phi^0\partial_\mu H) - \\
ig\tfrac{s_w^2}{c_w}MZ^0_\mu(W^+_\mu\phi^- - W^-_\mu\phi^+) + igs_wMA_\mu(W^+_\mu\phi^- - W^-_\mu\phi^+) - ig\tfrac{1-2c_w^2}{2c_w}Z^0_\mu(\phi^+\partial_\mu\phi^- - \phi^-\partial_\mu\phi^+) + \\
igs_wA_\mu(\phi^+\partial_\mu\phi^- - \phi^-\partial_\mu\phi^+) - \tfrac{1}{4}g^2W^+_\mu W^-_\mu\left(H^2 + (\phi^0)^2 + 2\phi^+\phi^-\right) - \\
\tfrac{1}{8}g^2\tfrac{1}{c_w^2}Z^0_\mu Z^0_\mu\left(H^2 + (\phi^0)^2 + 2(2s_w^2 - 1)^2\phi^+\phi^-\right) - \tfrac{1}{2}g^2\tfrac{s_w^2}{c_w}Z^0_\mu\phi^0(W^+_\mu\phi^- + W^-_\mu\phi^+) - \\
\tfrac{1}{2}ig^2\tfrac{s_w^2}{c_w}Z^0_\mu H(W^+_\mu\phi^- - W^-_\mu\phi^+) + \tfrac{1}{2}g^2s_wA_\mu\phi^0(W^+_\mu\phi^- + W^-_\mu\phi^+) + \tfrac{1}{2}ig^2s_wA_\mu H(W^+_\mu\phi^- - W^-_\mu\phi^+) - \\
g^2\tfrac{s_w}{c_w}(2c_w^2 - 1)Z^0_\mu A_\mu\phi^+\phi^- - g^2s_w^2A_\mu A_\mu\phi^+\phi^- + \tfrac{1}{2}ig_s\lambda^a_{ij}(\bar{q}^\sigma_i\gamma^\mu q^\sigma_j)g^a_\mu - \bar{e}^\lambda(\gamma\partial + m_e^\lambda)e^\lambda - \bar{\nu}^\lambda\gamma\partial\nu^\lambda - \\
\bar{u}^\lambda_j(\gamma\partial + m_u^\lambda)u^\lambda_j - \bar{d}^\lambda_j(\gamma\partial + m_d^\lambda)d^\lambda_j + igs_wA_\mu\left(-(\bar{e}^\lambda\gamma^\mu e^\lambda) + \tfrac{2}{3}(\bar{u}^\lambda_j\gamma^\mu u^\lambda_j) - \tfrac{1}{3}(\bar{d}^\lambda_j\gamma^\mu d^\lambda_j)\right) + \\
\tfrac{ig}{4c_w}Z^0_\mu\{(\bar{\nu}^\lambda\gamma^\mu(1+\gamma^5)\nu^\lambda) + (\bar{e}^\lambda\gamma^\mu(4s_w^2 - 1 - \gamma^5)e^\lambda) + (\bar{d}^\lambda_j\gamma^\mu(\tfrac{4}{3}s_w^2 - 1 - \gamma^5)d^\lambda_j) + (\bar{u}^\lambda_j\gamma^\mu(1 - \tfrac{8}{3}s_w^2 + \\
\gamma^5)u^\lambda_j)\} + \tfrac{ig}{2\sqrt{2}}W^+_\mu\left((\bar{\nu}^\lambda\gamma^\mu(1+\gamma^5)e^\lambda) + (\bar{u}^\lambda_j\gamma^\mu(1+\gamma^5)C_{\lambda\kappa}d^\kappa_j)\right) + \\
\tfrac{ig}{2\sqrt{2}}W^-_\mu\left((\bar{e}^\lambda\gamma^\mu(1+\gamma^5)\nu^\lambda) + (\bar{d}^\kappa_j C^\dagger_{\kappa\lambda}\gamma^\mu(1+\gamma^5)u^\lambda_j)\right) + \\
\tfrac{ig}{2\sqrt{2}}\tfrac{m_e^\lambda}{M}\left(-\phi^+(\bar{\nu}^\lambda(1-\gamma^5)e^\lambda) + \phi^-(\bar{e}^\lambda(1+\gamma^5)\nu^\lambda)\right) - \tfrac{g}{2}\tfrac{m_e^\lambda}{M}\left(H(\bar{e}^\lambda e^\lambda) + i\phi^0(\bar{e}^\lambda\gamma^5e^\lambda)\right) + \\
\tfrac{ig}{2M\sqrt{2}}\phi^+\left(-m_d^\kappa(\bar{u}^\lambda_j C_{\lambda\kappa}(1-\gamma^5)d^\kappa_j) + m_u^\lambda(\bar{u}^\lambda_j C_{\lambda\kappa}(1+\gamma^5)d^\kappa_j\right) + \\
\tfrac{ig}{2M\sqrt{2}}\phi^-\left(m_d^\lambda(\bar{d}^\lambda_j C^\dagger_{\lambda\kappa}(1+\gamma^5)u^\kappa_j) - m_u^\kappa(\bar{d}^\lambda_j C^\dagger_{\lambda\kappa}(1-\gamma^5)u^\kappa_j\right) - \tfrac{g}{2}\tfrac{m_u^\lambda}{M}H(\bar{u}^\lambda_j u^\lambda_j) - \tfrac{g}{2}\tfrac{m_d^\lambda}{M}H(\bar{d}^\lambda_j d^\lambda_j) + \\
\tfrac{ig}{2}\tfrac{m_u^\lambda}{M}\phi^0(\bar{u}^\lambda_j\gamma^5u^\lambda_j) - \tfrac{ig}{2}\tfrac{m_d^\lambda}{M}\phi^0(\bar{d}^\lambda_j\gamma^5d^\lambda_j) + \bar{G}^a\partial^2G^a + g_sf^{abc}\partial_\mu\bar{G}^aG^bg^c_\mu + \bar{X}^+(\partial^2 - M^2)X^+ + \bar{X}^-(\partial^2 - \\
M^2)X^- + \bar{X}^0(\partial^2 - \tfrac{M^2}{c_w^2})X^0 + \bar{Y}\partial^2Y + igc_wW^+_\mu(\partial_\mu\bar{X}^0X^- - \partial_\mu\bar{X}^+X^0) + igs_wW^+_\mu(\partial_\mu\bar{Y}X^- - \\
\partial_\mu\bar{X}^+Y) + igc_wW^-_\mu(\partial_\mu\bar{X}^-X^0 - \partial_\mu\bar{X}^0X^+) + igs_wW^-_\mu(\partial_\mu\bar{X}^-Y - \partial_\mu\bar{Y}X^+) + igc_wZ^0_\mu(\partial_\mu\bar{X}^+X^+ - \\
\partial_\mu\bar{X}^-X^-) + igs_wA_\mu(\partial_\mu\bar{X}^+X^+ - \partial_\mu\bar{X}^-X^-) - \tfrac{1}{2}gM\left(\bar{X}^+X^+H + \bar{X}^-X^-H + \tfrac{1}{c_w^2}\bar{X}^0X^0H\right) + \\
\tfrac{1-2c_w^2}{2c_w}igM\left(\bar{X}^+X^0\phi^+ - \bar{X}^-X^0\phi^-\right) + \tfrac{1}{2c_w}igM\left(\bar{X}^0X^-\phi^+ - \bar{X}^0X^+\phi^-\right) + \\
igMs_w\left(\bar{X}^0X^-\phi^+ - \bar{X}^0X^+\phi^-\right) + \tfrac{1}{2}igM\left(\bar{X}^+X^+\phi^0 - \bar{X}^-X^-\phi^0\right) .
\end{gathered}
$$

Figure 1: The GWS Lagrangian

algebra, vector potentials that are $\mathfrak{g}$-valed 1-forms $A = \sum_\mu A_\mu dx^\mu \in T^*M \otimes \mathfrak{g}$, and fields that are $\mathfrak{g}$-valed 2-forms $F = dA + A \wedge A$.

If γ is a gauge transformation, the potental A and the field F transform as

$$\begin{aligned} A' &= \gamma A \gamma^{-1} + \gamma d\gamma^{-1} \\ F' &= \gamma F \gamma^{-1} \end{aligned}$$

The NC solution is a principled one for overcoming the difficulty by linking the "inner" quantum non commutativity *with the new "outer" geometrical non commutativity of the external space*.

Indeed, as we already emphasized it, when $\mathcal{A}$ is no longer commutative, there exists in $\mathrm{Aut}(\mathcal{A})$ the normal subgroup $\mathrm{Inn}(\mathcal{A})$ of inner automorphisms acting by conjugation $a \to uau^{-1}$. So, the gauge group appears now as a normal subgroup of the group of NC diffeomorphisms $\mathrm{Aut}(\mathcal{A})$. As claimed by Alain Connes [6]:

> "Amazingly, in this description the group of gauge transformation of the matter fields arises spontaneously as a normal subgroup of the generalized diffeomorphism group $\mathrm{Aut}(\mathcal{A})$. It is the *non commutativity* of the algebra $\mathcal{A}$ which gives [it] for free."

In the NC framework, vector potentials A (gauge connections) are self-adjoint operator representing a 1-form

$$A = \sum_j a_j [D, b_j] \ .$$

The fields are the curvature 2-forms

$$F = dA + A^2.$$

The unitary group $\mathcal{U}(\mathcal{A}) \subset \mathrm{Inn}(\mathcal{A})$ of unitary elements $u^* = u^{-1}$ (acting by $\alpha_u(a) = uau^*$) acts upon A and F by gauge transformations

$$\begin{aligned} A &\to uAu^* + udu^* = uAu^* + u[D, u^*] \\ F &\to uFu^*. \end{aligned}$$

In the Connes-Lott interpretation of the GWS standard model (see e.g. Kastler-Schücker's [16]), the NC C^*-algebra is

$$\mathcal{A} = \mathcal{C}^\infty(M) \otimes (\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}))$$

where the "internal" algebra $\mathcal{A}_F = \mathbb{C} \oplus \mathbb{H} \oplus M^3(\mathbb{C})$ has for unitary group the symmetry group

$$U(1) \times SU(2) \times SU(3).$$

The model is the product of this internal model of the fermionic sector with a classical gauge model for the bosonic sector:

$$\begin{cases} \mathcal{A} = C^{\infty}(M) \otimes \mathcal{A}_F \\ \mathcal{H} = L^2(M,S) \otimes \mathcal{H}_F = L^2(M, S \otimes \mathcal{H}_F) \\ D = (D_M \otimes 1) \oplus (\gamma_5 \otimes D_F). \end{cases}$$

where $\mathcal{H}_F$ is the Hilbert space generated by fermions (quarks and leptons).

Computations are extremely complex but give exactly the Standard Model. One has to compute vector potentials $A = \sum_i a_i [D, a'_i]$, $a_i, a'_i \in \mathcal{A}$. As D is a sum of two terms, it is also the case for A. Its discrete part comes from $\gamma_5 \otimes D_F$ and generates the Higgs bosons. Its second part comes from $D_M \otimes 1$ and generates the gauge bosons.

6.3 "External" metric and "internal" gauge transformations

The key remark is that NC gauge connections $A = \sum a_j [D, b_j]$ can be interpreted as *internal perturbations of the metric* that is as *internal fluctuations of the spectral geometry* induced by the degrees of freedom of gauge transformations.

This coupling between the "external" metric defined by the Dirac operator and gauge transformations is a principled way for coupling gravity with QFT. It disappears in the commutative case and is a purely NC effect.

7 Quantum gravity, NCG, and spectral action (Connes - Chamseddine)

7.1 The coupling GR-QFT

In NCG, it becomes possible to *couple* the gravitational Hilbert-Einstein action with the quantum fields. As Connes [6] strongly emphasized:

> "One should consider the internal gauge symmetries as part of the diffeomorphism group of the non commutative geometry, and the gauge bosons as the internal fluctuations of the metric. It follows then that the action functional should be of a purely gravitational nature. We state the principle of spectral invariance, stronger than the invariance under diffeomorphisms, which requires that the action functional only depends on the spectral properties of $D = ds^{-1}$ in $\mathcal{H}$."

For coupling a Yang-Mills gauge theory with the Hilbert-Einstein action, Chamseddine-Connes' idea [5] is to find a C^*-algebra $\mathcal{A}$ s.t. $\mathrm{Inn}(\mathcal{A})$

is the gauge group and $\mathrm{Out}(\mathcal{A})$ plays the role of $\mathrm{Diff}(M)$. In our *Introduction* we have explained that, in the classical case, one uses principal bundles $P \to M$ with a structural group G acting upon the fibers and an exact sequence

$$Id \to \mathcal{G} \to \mathrm{Aut}\,(P) \to \mathrm{Diff}(M) \to Id$$

where $\mathcal{G} = C^{\infty}(M, G)$ is the gauge group. The symmetry group of the theory is then $\mathrm{Aut}\,(P)$, namely the semidirect product $\mathfrak{G}$ of $\mathrm{Diff}(M)$ and $\mathcal{G}$.

But in such a formalism, one must separate the space-time M and the fibers P_x of P, and make the symmetries preserve this separation. To completely geometrize the theory, one would have to find a space X s.t. $\mathrm{Aut}\,(X) = \mathfrak{G}$.

> "If such a space would exist, then we would have some chance to actually geometrize completely the theory, namely to be able to say that it's pure gravity on the space X." (Connes [7])

We have seen that such a geometrization is impossible if X is a manifold due to the simplicity theorem, but it is possible with a NC spectral triple $(\mathcal{A}, \mathcal{H}, D)$ using the exact sequence

$$Id \to \mathrm{Inn}\,(\mathcal{A}) \to \mathrm{Aut}\,(\mathcal{A}) \to \mathrm{Out}(\mathcal{A}) \to Id$$

Then, as explained Iochum, Kastler, and Schücker [13]:

> "the metric 'fluctuates', that is, it picks up additional degrees of freedom from the internal space, the Yang-Mills connection and the Higgs scalar."

Martin [20] says:

> "The strength of Connes' conception is that gauge theories are thereby deeply connected to the underlying geometry, on the same footing as gravity. The distinction between gravitational and gauge theories boils down to the difference between outer and inner automorphisms."

7.2 The spectral action

It is extraordinary that all the terms of the GWS Lagrangian can be computed from the *spectral action*, which counts the number $N(\Lambda)$ of eigenvalues of D in the interval $[-\Lambda, \Lambda]$. The formula is

$$\mathrm{Trace}\left(f\left(\frac{D^2}{\Lambda^2}\right)\right)$$

where Λ is a cut-off parameter (inverse of the Planck length) and f a smooth approximation of the characteristic function of $[0, 1]$.

As Landi and Rovelli [19] explained, the key idea is

> "to consider the eigenvalues of the Dirac operator as dynamical variables for general relativity".

It is a very deep idea. Dynamical variables are no longer values of fields (which are not Diff(M) invariant) but the *eigenvalues of the Dirac operator defining the metric*, which are spectral invariants automatically Diff(M)-invariant, and "are available in localized form anywhere." ([9], p. 184)

> "Thus the general idea is to describe spacetime geometry by giving the eigen-frequencies of the spinors that can live on that spacetime." (Landi-Rovelli [19])

As Connes says:

> "The Dirac operator D encodes the full information about the spacetime geometry in a way usable for describing gravitational dynamics."

This crucial point has also been well explained by Steven Carlip [4] (p. 47). In GR points of space-time loose any physical meaning so that GR obervables must be radically non-local. This is the case with the eingenvalues of D which provide

> "a nice set of non local, diffeomorphism-invariant obervables"

and

> "the first good candidates for a (nearly) complete set of diffeomorphism-invariant observables".

7.3 Towards number theory and zeta function

Let us emphasize a last point. The (highly complicated) computations show that the Einstein-Hilbert action can be retrieved as a term of the spectral action. Now, the counting function $N(\Lambda)$ is a step function which can be written for $\Lambda \to \infty$ as a sum of a mean value and a fluctuation term: $N(\Lambda) = \langle N(\Lambda) \rangle + N_{fluc}(\Lambda)$. Sophisticated computations of the asymptotic expansion show that the Λ^4 coefficient gives a cosmological term, the Λ^2 coefficient gives the Einstein-Hilbert action, and the Λ^0

cefficient gives the Yang-Mills action for the gauge fields corresponding to the internal degrees of freedom of the spectral metric.

Now, $N(\Lambda)$ shares many properties with arithmetic functions such as Riemann ζ function, which, as was conjectured long ago by Hilbert and Polya, could be the counting function of the eigenvalues of an operator.

In a recent and astonishing new book, *Noncommutative Geometry, Quantum Fields and Motives* [9], Alain Connes and Matilde Marcolli unify the two theories. They show that "two fundamental problems", namely "the construction of a theory of quantum gravity and the Riemann hypothesis" are linked:

> "Quite surprisingly, we shall discover that there are deep analogies between these two problems which, if properly exploited, are likely to enhance our grasp of both of them."

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