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Topological Foundations of Cognitive Science

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SHEAF MEREOLOGY AND SPACE COGNITION

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INTRODUCTION

Contemporary research concerning the cognitive links between perception, language, and action — see for instance Talmy's works — have revolutionized the dominating traditions of formal semantics. They have led to what Husserl already called in *Erfahrung und Urteil* a *genealogy of logic*, a perceptive genealogy of cognitive symbolic structures. Such a Gestalt-like perspective on symbolic structures raises a lot of new problems. It makes traditional simple — and even trivial — problems appear as complex, non trivial, ones.

It is one of these problems I want to address here, namely that of the most elementary and primitive forms of *predication in perceptive judgement*: "the snow is white", "the sky is blue", "the ball is red", etc.

For classical elementary logic these atomic formulae "S is p" stay at the lowest level of complexity. Both their syntax and their semantic are trivial. But it is no longer the case if one takes into account the perceptive situations to which these statements refer. Indeed, these situations compel to introduce in the formalization of symbolic logical structures, *topological and morphological* structures of a completely different kind. If one wants to do semantics that way, one is therefore committed to elaborate a *mathematically integrated* theory of topology and logic.

Let us see briefly why the problem of taking into account the perceptive roots of predication is a non trivial one. On the perceptive side we find *a geometrical descriptive eidetics* of the morphological (gestaltic) organization of perception. I use here the term "morphology" in the sense of Thom: a morphology is a set of qualitative discontinuities on a substrate space. But on the logical side we find a logical eidetics of judgement: a formal syntax and a formal semantics (e.g. a vericonditional theory of denotation). But between these two eidetics there exists a dramatic gap, what Thom (1988) called an "insuperable break" (*Esquisse de Sémiophysique*, p. 248). Our challenge here is to fill this gap using some sort of synthesis between logic and topology.

The gap is linked with deep traditional philosophical questions, for instance that of the relations existing between *analytic and synthetic* rules. Morphological eidetics depends on synthetic laws of perception. On the contrary, logical eidetics depends on analytic laws.

I. THE MORPHODYNAMICAL SCHEMATIZATION OF PREDICATION

Let us consider how René Thom has morphologically schematized a perceptual attributive judgement "S is p" such as "the sky is blue". The truth conditions of such a statement depend on the way, as Husserl would say, the "intentions of signification" are *intuitively filled* (in the sense of an *Erfüllung*) at the *ante-predicative and prejudicative* level. Without an adequate description of this intuitive filling-in, one can't elaborate a correct theory of predication.

Let W be the spatial extension of S. W is filled by a quality p belonging to a genus G (e.g. colours). The filling-in is then described by a map

$$f : W \to G,\ w \to f(w).$$

Husserl has given a deep analysis — but lacking of mathematical modeling — of this fact in his third Logical Investigation. I have shown that to model adequately *the unilateral dependence relation* linking the extension W with its dependent part — its *moment* — p, one has to interpret the map f as a *section* of the (trivial) *fibration*

$$\pi : E = W \times G \to W.$$

It is exactly the point of view mathematized by Thom.

Let me recall briefly what are fibrations and sections of fibrations.

Mathematically, a fibration is a differentiable manifold E endowed with a *canonical projection* (a differentiable map) $\pi : E \to M$ over another manifold M. M is called the *base* of the fibration, and E its *total space*. The inverse images $E_x = \pi^{-1}(x)$ of the points $x \in W$ by π are called the *fibers* of the fibration. They are the subspaces of E which are projected to points.

In general a fibration is required to be *locally trivial*, that is to satisfy the two following axioms:

(F_1) All the fibers E_x are diffeomorphic with a typical fiber F.

(F_2) $\forall x \in M, \exists U$ a neighborhood of x such that the inverse image $E_U = \pi^{-1}(U)$ of U is diffeomorphic with the direct product $U \times F$ endowed with the canonical projection $U \times F \rightarrow U$, $(x,q) \rightarrow x$. (See figures 1, 2).

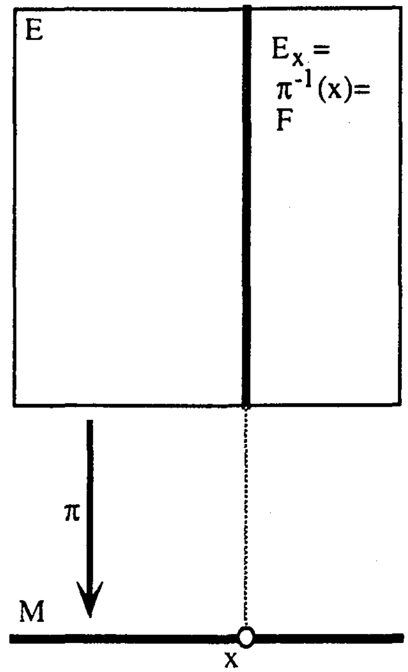

Figure 1

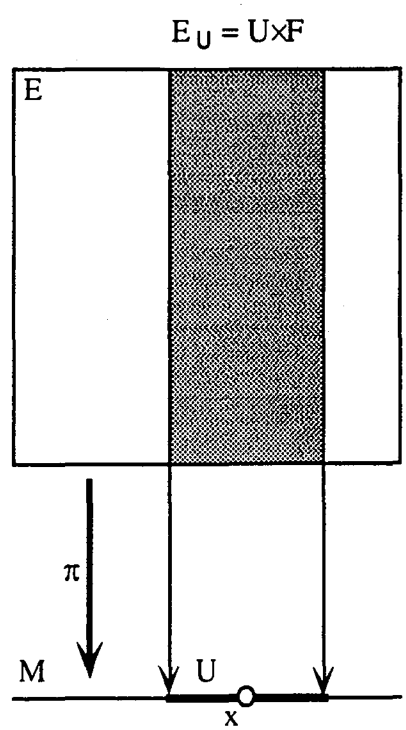

Figure 2

In our case, we have $M = W$ and $F = G$.

Let $\pi : E \rightarrow M$ be a fibration and let $U \subset M$ be an open subset of M. A *section s* of π over U is a *lift* of U to E which is compatible with π. More precisely, it is a map $s : U \rightarrow E, x \in U \rightarrow s(x) \in E_x$, i.e. such that $\pi \circ s = \mathrm{Id}_U$. In general s is supposed to be continuous, differentiable, analytic. It can present discontinuities along a singular locus. (See figures 3, 4).

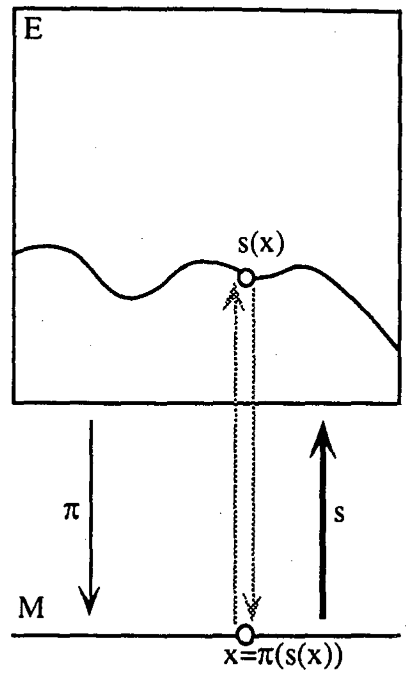

Figure 3

It is conventional to write $\Gamma(U)$ for the set of sections of π over U. If there exists a local trivialization of π over U (i.e. $E_U \rightarrow U \approx U \times F \rightarrow U$), it transforms every section $s : U \rightarrow E$ in a map $x \rightarrow s(x) = (x, f(x))$, that is in a map $f : U \rightarrow F$. Therefore, the concept of section generalizes the classical concept of map.

The main reason why it is of interest to conceive of the qualitative filling-in of spatial regions as sections of fibrations is that one can easily retreive that way the dependence relations (i.e. the ontological hierarchy) between "essences". The fact that spatial extensions are ontologically prior to qualities such as colours is modeled by the fact that the extensions W are the *base spaces* of the fibrations, and can therefore exist in an *independent* way, while the qualitative genera G are the *fibers* of the fibrations, and can therefore only exist in a *dependent* way.

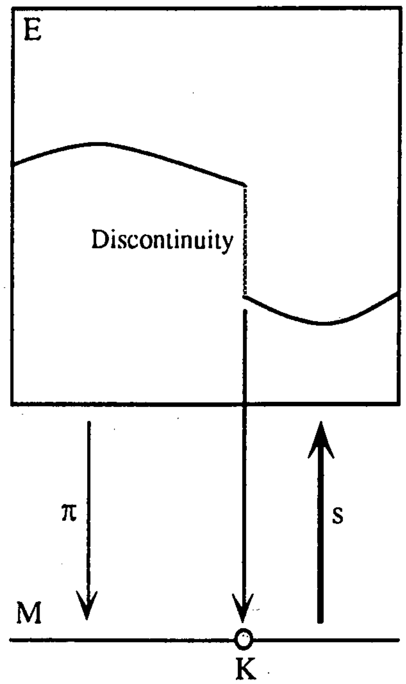

Figure 4

In general the genus G will be *categorized* and decomposed in sorts. Let p be such a category (e.g. "blue") and let ∂p be its boundary in G. The perceptive statement "S is p" is morphologically interpreted *by the following geometrical fact*:

the image $f(W)$ of the section $f: W \to W \times G$ is encapsulated in the cylinder $W \times \partial p$.

These morphological descriptions can be generalized. If for instance there exists an order relation $<$ on G (e.g. p is lighter than p'), the statement $S(W) < S'(W')$ means that $\rho(f(w)) < \rho(f'(w'))$ where ρ is the second projection $\rho : W \times G \to G$ (that is $\rho(f(w))$ is the codomain of values of f).

The problem I want to address is very simple. In the framework of a semantic theory, what can be the link between two definitions of truth (for atomic formulae):

(i) the tautological Tarskian one : "S is p" is true iff S is p;

(ii) the morphological Thomian one : "S is p" is true iff $f(w)$ is encapsulated in the cylinder $W \times \partial p$?

II. THE INDEXICAL AND MODAL STATUS OF SPATIAL EXTENSION

In the categorial syntactic structure of a statement such as "S is p", the *specificity* of the spatial extension W of S is in some way *bracketed*, "implicitated". Of course, one can focus on the extension W itself and consider its regions, its points, etc. But in doing so one changes the level of description. At the level at which S is processed as an individual entity sharing some properties — that is at the level at which the symbolic description "S is p" is relevant — the specificity of W is bracketed. This strange sort of "epoche" concerns also the *relations* between qualities, as when we say "*S* is lighter than *R*".

This problem has been well pointed out by Wittgenstein in his *Remarks on Colour* (see Mulligan 1992).

> "A language-game: Report whether a certain body is lighter or darker than another. But now there's a related one: State the relationship between the lightnesses of certain shades of colour. The form of the propositions in both language games is the same "X is lighter than Y". But in the first it is an external relation and the proposition is temporal. In the second it is an internal relation and the proposition is timeless".

The tautological aspect of Tarski's definition of truth is a symptom of this "epoche". It is what remains when syntax and semantics uncouple themselves from their perceptive rooting.

As a matter of fact, one can say that *every* perceptive statement is in some sense analog *to an indexical one*. It refers to its perceptive situation in much the same way as an indexical statement refers to its pragmatic context. The intuitive *Erfüllung* of the intentions of signification operates in a *counterfactual* fashion and one needs therefore somme sort of Kripke's semantics to take account of this fact.

If one wants to understand how classical logical structures can have a non classical *space-semantics*, one has to understand how classical variables can semantically denote *sections of fibrations* in such a way that extensions *can condition truth.*

III. COMPUTATIONAL VISION AND RECEPTIVE FIELD JET CALCULUS

Before tackling this point, I want to emphasize the *neuro-physiological* relevance of the geometrical concept of fibration.

The most influential specialists applying differential geometry to computational vision have hypothesized that cortical columns implement a *multi-scale jet calculus* and that it is essentially that way the brain does geometry. Let us consider for instance Jan Koenderink's perspective.

David Marr already showed at the end of the seventies that the retinal ganglion cells perform what is now called a *wavelet analysis* of the signal $I(x,y)$, that is a *multi-scale and spatially localized* analysis. They act as *filters*, that is they convolute the signal by the receptive profile of their receptive fields. Now, these receptive profiles are approximatively Laplacians of Gaussians ΔG. According to the fact that

$$\Delta G \ast I = \Delta(G \ast I)$$

one can conclude that such cells smooth locally the signal at a certain scale and compute its second partial derivatives.

Using such smoothed partial derivatives one can for instance detect qualitative salient *discontinuities*: it is Marr's celebrated *zero-crossing* criterium.

Now, zero-crossings are differential geometrical invariants which belongs to what is called the *2-jet* of the smoothed signal. Jets constitute an *intrinsic* version of the old idea of Taylor expansions. Let us take for instance the Taylor expansion of an R valued differentiable map $I : R^2 \to R$ up to order 2 at a point (x_0, y_0).

$$T_2I(x,y) = I(x_0,y_0) + (x-x_0)\frac{\partial I}{\partial x}(x_0,y_0) + (y-y_0)\frac{\partial I}{\partial y}(x_0,y_0) +$$

$$\frac{(x-x_0)^2}{2}\frac{\partial^2 I}{\partial x^2}(x_0,y_0) + (x-x_0)(y-y_0)\frac{\partial^2 I}{\partial x \partial y}(x_0,y_0) + \frac{(y-y_0)^2}{2}\frac{\partial^2 I}{\partial y^2}(x_0,y_0) .$$

It is constructed via simple algebraic operations from the 8-uple

$$\left\{(x,y);\ I(x,y);\ (\partial_x I, \partial_y I);\ (\partial^2_{x^2} I, \partial^2_{xy} I, \partial^2_{y^2} I)\right\}$$

Now we introduce a space $J^2 = R^8$ with coordinates

$$\{(x,y);\ z;\ (\alpha,\beta);\ (\lambda,\mu,\gamma)\}$$

and we define the *2-jet* of I as the map j^2I: $R^2 \to J^2$ given by

$$j^2I(x,y) = \left\{(x,y);\ I(x,y);\ (\partial_x I, \partial_y I);\ (\partial^2_{x^2} I, \partial^2_{xy} I, \partial^2_{y^2} I)\right\}.$$

This map can be defined in a coordinate-free (geometrically intrinsic) manner. Actually J^2 is a *fibration* of base $R^2_{(x,y)}$ and fiber $R^6_{(z;(\alpha,\beta);(\lambda,\mu,\gamma))}$ and the 2-jet j^2I is a *section* of J^2.

Jets are therefore spatial fields of differential data, and using them one can compute *geometrically relevant differential invariants and morphological features* (fold points, cusps, end points, curvature, etc.) using only *arrays of point-processors*.

If one introduces the hypothesis that the receptive fields of certain classes of cortical cells approximate partial derivatives of Gaussians of order > 2 (up to order 3 or 4) then one can give a *multi-scale* version of this jet calculus. Koenderink has strongly advocated the thesis that such a receptive field jet calculus is implemented in retinotopic arrays of cortical columns.

Let us recall briefly the *columnar structure* of the primary visual cortex (area 17 or area V1) extensively investigated since the pioneering works of Hubel, Wiesel and Mountcastle.

The basic functional module is a "cube" organized in a *retinotopic, columnar and layered* manner.

(i) The *retinotopic structure* preserves the topographic connexions of the retinian ganglion cells (the signal processing of which is transmitted and amplified by the lateral geniculate body). We get that way *a retinotopic fibration*, the base-space of which is constituted *by glued local receptive fields*.

(ii) The *columnar structure* (which is orthogonal to the cortex surface) is essentially constituted by *orientation columns*, columns of *ocular dominance*, and colour plugs. Orientation columns (the diameter of which is about 50µ) are set out orthogonally to the columns of ocular dominance. Their preferential orientation varies from 0° to 180° by 10° steps. They constitute *hypercolumns* the size of which is about 800µ-1mm.

(iii) The *layered structure* is composed of six layers. Its depth is about 1,8mm. The geniculate fibers project on layer IV. Layer VI is a feedback one. Layer V projects on the colliculus. Layers II and III receive the axons of layer IV and project their efferent fibers in other cortical regions where different attributes (shape, colour, movement, stereopsis, etc.) are further processed in a *non* retinotopic way.

The orientation columns yield a beautiful exemple of a ***neurally implemented fibration***. Actually, they implement the fibration $\pi: E \rightarrow W$ the base of which is the visual field and the fiber of which is the *projective line* $F = P^1$ of plane directions. William Hoffman (1989) has explained how the cortex implements what is called the *contact structure* of E.

Let M be a n-manifold. A *contact element* $c = (x, H)$ of M is constituted by a point $x \in M$ and by a $(n-1)$-hyperplane H of the tangent space T_xM of M at x. (See figure 5).

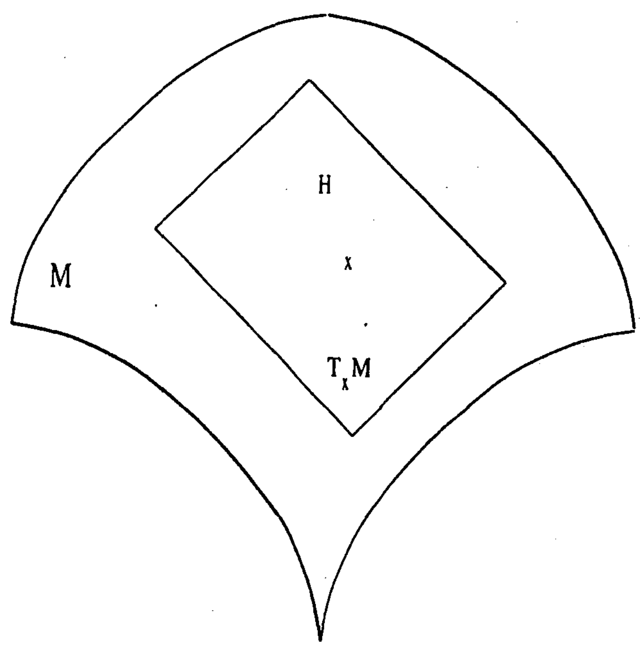

Figure 5

The *contact bundle* CM of M is the $(2n-1)$-manifold of these contact elements. CM is a fibration — a *fiber bundle* — $\pi : CM \rightarrow M$, the fiber over x being the projective space $C_xM = P(T_xM)$ of hyperplanes of the vector space T_xM. Here we have $n = 2$, the Hs are the directions and therefore $C_xM \cong P^1$. (See figure 6: M is represented as one-dimensional and in the second figure the directions are represented as points of P^1).

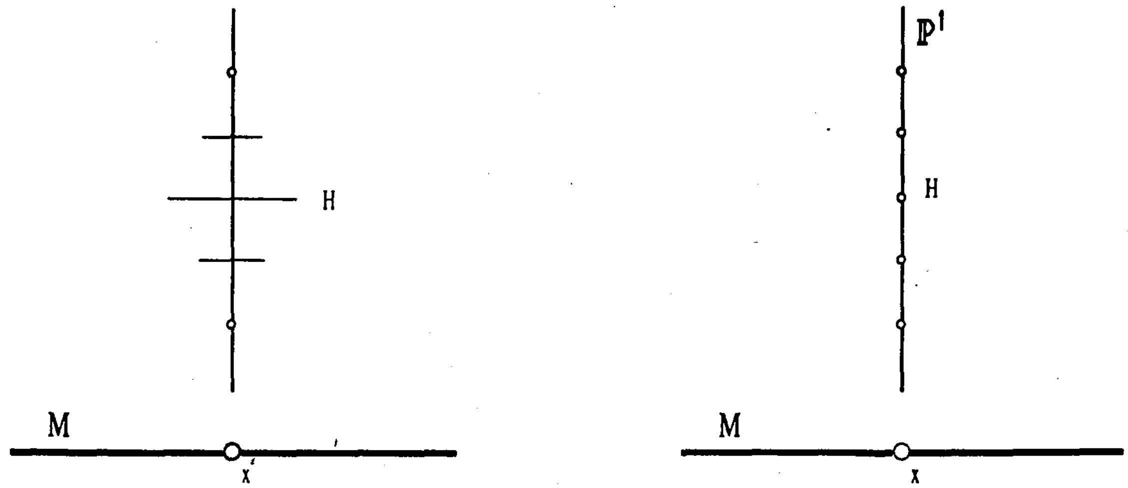

Figure 6

The contact bundle CM is isomorphic to the *projectivized* bundle of the *cotangent bundle* T^*M. Indeed a contact element (x, H) can be defined by a 1-form (a linear form) α_x on T_xM satisfying $\mathrm{Ker}(\alpha_x) = H$. But $\lambda\alpha_x$ $(\lambda \neq 0)$ defines the same H and therefore $CM \cong \mathrm{P}(T^*M)$.

There exists on the $(2n\text{-}1)$-contact bundle CM what is called *a canonical contact structure*. It reflects the *canonical symplectic structure* of T^*M. In general a *contact structure* on an N-manifold m is defined by a *non degenerate* field of $(N\text{-}1)$-hyperplanes h_c which are called *contact hyperplanes* (non degeneracy forces N to be odd). Let $c = (x, H) \in \mathrm{m} = CM$. A tangent vector

$$(\xi, \Theta) \in T_c\mathrm{m} \ = T_{(x,H)}CM$$

belongs to the contact $(2n\text{-}2)$-hyperplane h_c iff its projection ξ on T_xM belongs to H (i.e. $c = (x, H)$ moves in such a way that the x-velocity ξ belongs to H). (See figure 7).

Let Γ be a sub-manifold of M and let $C_M\Gamma$ be the manifold of the $c \in CM$ *tangent* to Γ. $C_M\Gamma$ is always of dimension n-1 and is tangent to the field h_c. $C_M\Gamma$ is therefore *an (n-1)-integral manifold* of the canonical contact structure of CM. In the visual case $n = 2$. $\mathrm{m} = CM$ is of dim = 3 (coordinates x, y, φ = direction of H). If $c = (x, H) \in CM$ and $(\xi, \Theta) \in T_{(x,\,H)}CM$, we have:

$$(\xi, \Theta) \in \mathrm{h}_c \Leftrightarrow \xi \in H.$$

If Γ is *a curve* in M and if we consider the contact elements $c = (x, H)$ which are *tangent* to Γ (i.e. $T_x\Gamma = H$) we get a *curve* $C\Gamma$ (a $(n$-1)-submanifold) in CM which is *an integral curve* of the canonical contact structure. Indeed, when x *moves along* Γ, (ξ, Θ) is such that $\xi \in T_x\Gamma$ and $T_x\Gamma = H$. Therefore $\xi \in H$ and therefore $(\xi, \Theta) \in \mathrm{h}_c$. This shows that a boundary curve (a contour curve) is a section of a fibration and that it can therefore be processed by arrays of points processors.

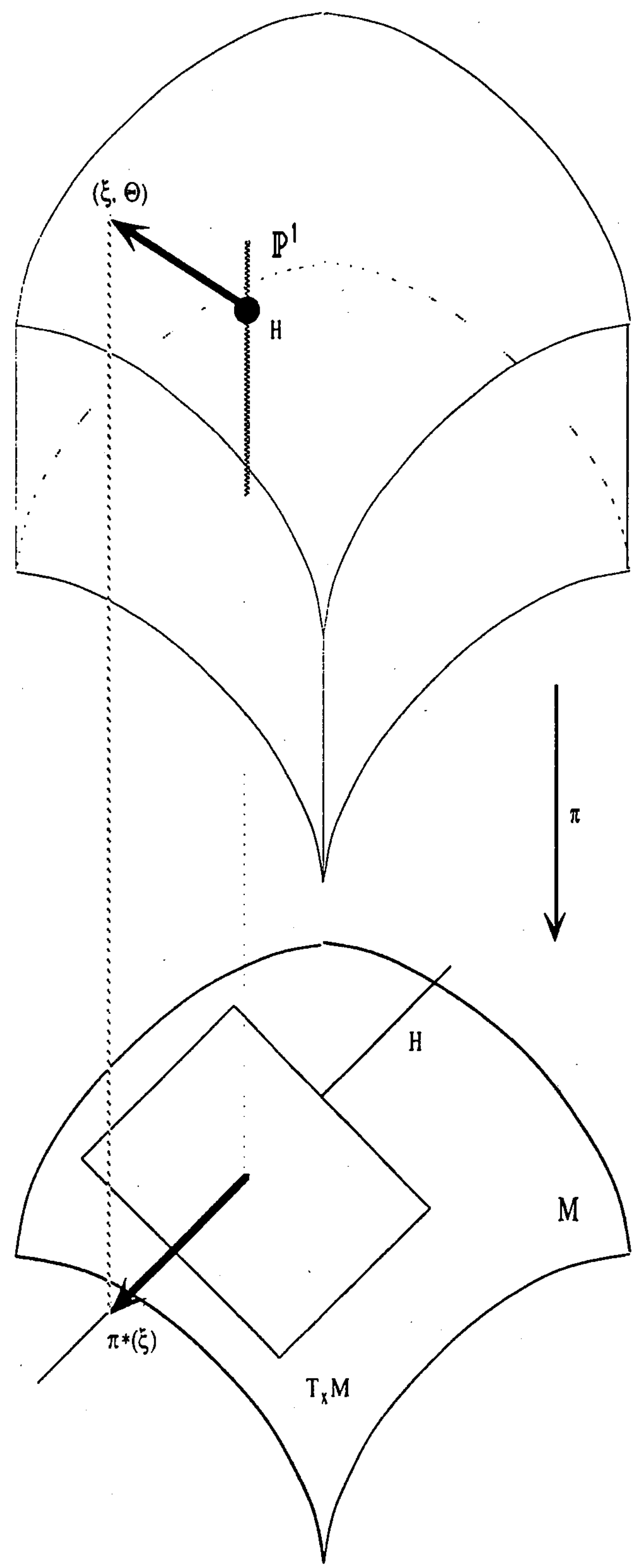

Figure 7

W. Hoffman explains very well how:

(i) the concept of receptive field corresponds to the concept of local chart in differential geometry;

(ii) the neurally implemented canonical contact structure explains the processing of visual contours as *integral curves*;

(iii) the perceptual constancies are invariant structures describable by means of actions of symmetry Lie groups on the contact structure of the projective bundle of directions.

He concludes claiming that

> "fibrations (...) are certainly present and operative in the posterior perceptual system if one takes account of the presence of "orientation" micro-response fields and the columnar arrangement of cortex"(p. 645).

For colours, the situation is a little bit more complex. There exist parvocellular layers in the lateral geniculate body constituted of ganglion cells sharing a *spectral antagonism* : (R+/G-), (G+/R-), (Y+/B-), (B+.Y-); there exist cortical plugs specialized in colour processing; their exist colour *blobs* in the area V1; and, according to Semir Zeki, the area V4 is specialized in colour processing. But it seems nevertheless plausible to hypothesize that colour is also processed through fibrations (with a dynamical antagonist spectral structure in the fibers).

This shed some light on the *morphological aspects* of perceptive statements.

Quid, now, of the logical ones ? How the morphological eidetics can be unified with a logic of judgement.

IV. HUSSERL'S *ERFAHRUNG UND URTEIL*

In *Erfahrung und Urteil* , *Untersuchungen zur Genealogie der Logik*, Husserl tries to clarify phenomenologically the *origin* of predication. He elaborates a theory of predicative statements such as "*S* is *p*", in the framework of formal apophantics (a syntactic theory) and formal ontology (a semantic theory such as set theory). But his perspective is "genealogic". According to him, classical logic *conceals* the fundamental problem of *evidence* in the logical concept of truth. By "evidence", Husserl means here *perceptive presentation as immediate acquaintance* with the world. Predicative statements such as "*S* is *p*" are rooted in ante-predicative and pre-judicative experience and it is the nature of such a founding relation which constitutes for Husserl the main problem. His "basic theme" is "categorical judgement founded in perception" (p. 70). Now, according to him, formal logic neglects perceptual evidence.

> "The formal character of logical Analytics consists in the fact that it does not consider the material quality of what is given prior to statements, and that it looks to substrates only in what concerns the categorial form they take in the statement" (p. 18).

He wants therefore to show that logic is originally based on the experience of the world, that the foundation of logic in perception has a universal scope and concerns its essence.

In *Erfahrung und Urteil* Husserl summarizes his analysis of perception previously done in the third *Logical Investigation*, *Ding und Raum* and *Ideen I*.

(i) The morphological analysis of the filling-in of spatial regions by qualities and the constitutive role of *qualitative discontinuities* in segmenting visual scenes (see the opposition between *Verschmelzung* and *Sonderung*).

(ii) The essentially *adumbrative* status of perception (*Abschattungslehre*). The object is a noematic pole unifying in a coherent way a continuous flux of adumbrations. Its noematic content expresses the *synthetic* relations of foundation between spatial extensions and their dependent moments such as contours or colours. It rule governs perceptive *anticipations* which, being filled-in by a "fluent variability" of adumbrations, are "indeterminated and general", but which are nevertheless also *prescriptively ruled by typed relations*. For instance, there exists a "typical genericity" of the filling-in relations between extensions and qualities.

He analyzes then very precisely the origin of the "primitive logical categories" from the ante-predicative synthesis. The main problem is to understand the "categorial genesis" of the *substrate/determination categories* which transforms the *perceptive synthesis*, where a substrate S possesses a dependent moment p, in the *predication* "S is p". Husserl's main thesis is that this categorial transformation arises from a reflective thematization which enables consciousness to grasp the dependent moments and typifies the substrates in "subjects" and the dependent moments in "predicates". According to him:

> "We find there the origin of the first "logical" categories" (p. 127).

Husserl points out therefore a fundamental conversion from the *synthetic* perceptive unity unifying an extension S with its dependent moments p to the *analytic* syntactic unity of the statement "S is p".

> "In the most simple predicative judgement a *double information* is processed" (p. 247).

Indeed, underlying the syntactic "subject / predicate" information concerning the "functional forms" of the terms of the proposition, there exists another information concerning the "kernel forms" "substrate=independence" and "adjectivity=dependence". This underlying information is presupposed by the syntactic one. Predication is a process based on

> "the covering of the kernel forms as syntactic material by the functional forms" (p. 248).

It is this logical typification of the synthetic dependance relations in syntactic categories I want now to model. As far as the dependence relations between extensions and filling-in qualities can be schematized by sections of fibrations, we see that, if we want to model Husserl's "genealogical" analysis of logic, we need to understand *how variables which denote sections of fibrations can be syntactically typified in such a way that the resulting semantics coud be that sort of Kripkean semantics we need to explain the "indexical" status of perceptive statements*. Moreover this must be done in the framework of formal apophantics and formal ontology.

Now, we will see that it is possible to achieve this goal using the tools of the *geometrical logic* yielded by *Topos theory*.

V. SHEAVES, TOPOÏ AND LOGIC

Some remarks to begin with.

1. Topos theory has been invented by Grothendieck for solving very sophisticated problems of algebraic geometry (he wanted to construct generalized cohomological theories). It can therefore seem strange to use it for solving elementary and non mathematical problems. But actually, when they are treated as *neuro-cognitive* ones, these "naïve" problems are not at all elementary and require sophisticated tools to be modeled adequately.

2. Category theory is certainly the best formal ontology we have now at disposal.

3. Some specialists of topos theory have already applied these tools to clarify some basic semantic problems. For instance Gonzalo Reyes (who works, with Eduardo Dubuc, Anders Koch, Ieke Moerdijk and Marta Bunge on the applications of topos theory to *Synthetic Differential Geometry*) has recently used these techniques to formalize the Kripkean theory of proper names as rigid designators.

4. In general, topos theory is used in formal logic as a tool for *typed λ-calculus* (see e.g. the works of John Mitchell, Philip Scott, Giuseppe Longo, etc.). As we will see, when variables are typed by objects in a topos, the categorical properties of the topos (to be a cartesian closed category, to possess a subobject classifier, etc.) leads to an intuitionistic "internal logic" which is a typed λ-calculus. As far as the Curry-Howard correspondance shows that (in intuitionistic logic) formulae can be treated as types, proofs as λ-terms, and the reduction of a proof by cut-elimination as the reduction of a λ-term to its normal form, topos theory has become fundamental for understanding the semantic of formal — and in particular of programming — languages.

In these applications, the geometrical origin of the sheaf and topos concepts is concealed. Here, we aim at a completely different sort of application. We want to use this geometrical origin to explain the symbolic ascent of logical forms from morphological ones.

As we already saw, in the "receptive field" perspective one needs to conceive of sections of fibrations as resulting from a glueing of local sections. This leads naturally to the concept of *sheaf* which, even if it can seem a very abstract one, is actually very deeply rooted in spatial intuition.

5.1. The concept of sheaf.

For the concept of sheaf, the primitive topological concept is no longer that of a point but that of an open set (what Peter Johnstone calls "pointless topology").

At an abstract level, a fibration is characterized by the sets of its sections $\Gamma(U)$ over the open sets $U \subset M$. If $s \in \Gamma(U)$ is a section over U and if $V \subset U$, we can consider the *restriction* $s|_V$ of s to V. The restriction is a map $\Gamma(U) \to \Gamma(V)$. It is clear that if $V=U$, then $s|_V = s$ and that if $W \subset V \subset U$ and $s \in \Gamma(U)$, then $(s|_V)|_W = s|_W$ (transitivity of the restriction). We get therefore what is called a *contravariant functor* $\Gamma : \mathbf{o}^*(M) \to \mathbf{Sets}$ from the category $\mathbf{o}(M)$ of the open sets of M in the category **Sets** of sets. (The objects of $\mathbf{o}(M)$ are the open sets of M, and its morphisms are the inclusions of open sets.)

Conversely, let Γ be such a functor — what is called a *presheaf* on M. To have a chance of being the functor of the sections of a fibration, Γ must clearly satisfy the two following axioms.

(S_1) Two sections which are locally equal must be globally equal. Let $u = (U_i)_{i \in I}$ be an open covering of M. Let $s, s' \in \Gamma(M)$. If $s|_{U_i} = s'|_{U_i}$ $\forall i \in I$, then $s = s'$.

(S_2) Compatible local sections can be collated in a global one. Let $s_i \in \Gamma(U_i)$ be a family over $u = (U_i)_{i \in I}$. If the s_i are compatible, that is if $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ when $U_i \cap U_j \neq \emptyset$, then they can be glued together : $\exists s \in \Gamma(M)$ such that $s|_{U_i} = s_i$ $\forall i \in I$.

(S_1) and (S_2) can be expressed in a purely categorical manner. For instance (S_2) says that the arrow

$$e : s \to \{s|_{U_i}\}_{i \in I}$$

is the *equalizer*

$$\Gamma(U) \xrightarrow{e} \prod_i \Gamma(U_i) \overset{p}{\underset{q}{\rightrightarrows}} \prod_{i,j} \Gamma(U_i \cap U_j)$$

of the two projections p, q corresponding to the inclusions $U_i \cap U_j \subset U_i$ and $U_i \cap U_j \subset U_j$.

In fact these axioms characterize a more general structure — and even more pervasive in contemporary mathematics — than the structure of fibration, namely the structure of *sheaf*. It can be shown that if the axioms (S_1) and (S_2) are satisfied, then one can represent the functor Γ by a general fibered structure $\pi : E \to M$ (called un "étale" space and which is not necessarily a locally trivial fibration) in such a way that $\Gamma(U)$ becomes the set of sections of π over U. In a nutshell, the fiber E_x — called in that case the *stalk* of the sheaf Γ at x — is the inductive limit (the colimit) :

$$E_x = \lim_{V \subset U \in u_x} \{(\Gamma(U), \Gamma(V \subset U))\}$$

(where u_x is the filter of the open neighborhoods of x). The stalk E_x is the set of *germs* s_x of sections at x. E is the sum of the E_x. If $s \in \Gamma(U)$, it can be interpreted as the map

$x \in U \to s(x) \in E_x$. The topology of E is then defined as the finest one making all these sections continuous.

5.2. Generalizations.

There are many generalizations of this basic situation. The best known is that of Grothendieck's topologies : open coverings are defined in terms of "sieves" and the concept of sheaf is generalized to this structure of "site".

Another generalization is that of *frames* and *locales*. One consider frames, that is lattices which share the properties of the lattices of open sets $\mathbf{o}(X)$. They are complete distributive lattices with finite meets and general joins. Locales are the objects of the dual category (in the case of topological spaces the correspondance is

$$f : X \to Y \text{ continuous } \Rightarrow f^* : \mathbf{o}(Y) \to \mathbf{o}(X)).$$

Points are then defined as morphisms $p : A \to 2 = \mathrm{o}(1)$ (in the case of topological spaces, if $A = \mathbf{o}(X)$ points correspond to true points $p : 1 \to X$). Let $Pt(A)$ be the set of points of A. A topology is defined on $Pt(A)$ by the subsets $\varphi(a) = \{p \in Pt(A) \mid p(a) = 1\}$. $\mathrm{o}(Pt(A))$ is the best approximation of A by a "spatial" locale.

5.3. The concept of topos.

Now, the main point, is that the category of sheaves on a space M shares the categorical properties of a topos and that (intuitionistic) logical languages can therefore be *interpreted* in it. According to Lawvere's perspective, such a topos t can therefore be conceived of as a universe of discourse *the objects of which are variable entities depending on the elements U of o(M)*. $\mathbf{o}(M)$ operates therefore as a set of "possible worlds". The "possible worlds" are here *spatial extensions*. This dependence on spatial extensions shares the two main properties we need:

(i) it is constitutive of the concept of logical truth and therefore of semantic relevance;

(ii) but it is not directly "visible" in the syntax of the internal logic of t and it is therefore syntactically irrelevant.

Actually, as long as the formal constructions in such a universe are constructive (in the intuitionistic sense) one can completely bracket and forget the spatial dependence at the syntactic level. We get therefore a very elegant explanation of the puzzling fact that spatial extension operates in perceptive statements in a rather "indexical" and "pragmatic" manner.

Ieke Moerdijk and Gonzalo Reyes (1991) have proposed an interesting philosophical commentary to Lawvere's main ideas.

> "Topos theory has brought to light and given the means to exploit a complementarity (or duality) principle between *logic* and *structure*" (p. 10).

In the framework of classical model theory, a theory T is of the general form $T = S + I$ where S is the axiomatics of a type of structures and I a set-theoretical interpretation satisfying the "trivial" and "tautological" Tarskian semantics. But one can also interpret S

in a topos t = **Sh**(c) by means of a *sheaf semantics*. One can then work in t as one would work in a standard universe of sets provided one does it with a non classical (intuitionistic) logic reflecting the new interpretation.

Let us now recall breafly the structure of a topos. If A is a sheaf on M, $A(U)$ will denote the set $\Gamma_A(U)$ of sections of A over U. It is easy to show that the sheaves on a base space M constitute a category **Sh**(M).

5.3.1. Exponentials

Sh(M) is a *cartesian closed* category. This means that it has products and fibered products or pullbacks, a terminal object — classically denoted by 1 — and *exponentials* B^A. An exponential object is an object which "internalizes" in the objects of **Sh**(M) the morphisms $f: A \rightarrow B$. Such "internalization" of functorial structures are called *representable* functors. Technically, the functor $(\bullet)^A$ is the *right adjoint* of the functor $A \times (\bullet)$. This means that we have for every object C of **Sh**(M) a functorial isomorphism $\mathrm{Hom}(C,B^A) \cong \mathrm{Hom}(A \times C,B)$. E.g. for $C=1$, we get $\mathrm{Hom}(1,B^A) \cong \mathrm{Hom}\ (A,B)$. But an arrow $f: 1 \rightarrow B^A$ is like an "element" of B^A. In fact, if A is a sheaf, an arrow $s : 1 \rightarrow A$ is a *global section* of A, that is an element $s \in A(M)$.

If we take $C=B^A$ and Id_{B^A}, the right adjunction defines what is called a *counit* $\varepsilon : A \times B^A \rightarrow B$ such that for every $f: C \rightarrow B^A$ the associated $h : A \times C \rightarrow B$ is given by $h = \varepsilon \circ (1 \times f)$.The counit generalizes the map $(x,f) \rightarrow f(x)$ in set theory and is therefore called the *evaluation map*.

The sheaf B^A is defined using the evident restrictions $A|_U$ and $B|_U$ to open sets: $B^A(U) = \mathrm{Hom}(A|_U, B|_U)$. It is called the "internal Hom" or the sheaf of germs of morphisms from A to B.

5.3.2. Subobject classifier

Sh(M) possesses also what is called a *subobject classifier* Ω, that is an object which "internalizes" the sets of subobjects, making the subobject functor representable. A subobject $m : S$ 0. A is a monomorphism (an injective map in the case of the category of sets). This means that if f and g are two morphisms from an object R to S, then $m \circ f = m \circ g$ implies $f=g$. It is equivalent to say that the fibered product $S \times_A S$ defined by m is isomorphic with S. A subobject classifier is a monomorphism True : $1 \rightarrow \Omega$ such that every subobject $m : S$ 0. A can be retrieved from True by a pull-back :

$$\begin{array}{ccc} S & \rightarrow & 1 \\ \downarrow & & \downarrow \mathrm{True} \\ A & \underset{\varphi_S}{\rightarrow} & \Omega \end{array}$$

We get therefore a functorial isomorphism $\mathrm{Sub}(A) \cong \mathrm{Hom}(A,\Omega)$. φ_S is called the *characteristic map* of the subobject S.

In the category **Sets** of sets, $\Omega = \{0,1\}$ is the classical set of boolean truth-values. Here — and it is perhaps the main difference between a topos like **Sh**(M) and the classical topos **Sets** —, $\Omega(U)$ *depends essentially on the topological structure. It expresses the localization of truth in a sheaf topos.*

By definition $\Omega(U) := \{W \subset U\}$. It is trivial to verify that Ω is a sheaf. The $\underline{\text{True}}$ map $\underline{\text{True}} : 1 \to \Omega$ is defined by $\underline{\text{True}}(U) : 1 \to U \in \Omega(U)$ that is by the *maximal* element of $\Omega(U)$: to be true over U is to be true "everywhere" over U. The global section $T \in \Omega(M)$ selected by $\underline{\text{True}}$ is nothing else than the whole base space M itself and $\underline{\text{True}}(U)$ is its localization to U. If S is a subsheaf of the sheaf A, its characteristic map $\varphi_S : A \to \Omega$ is given by the maps $\varphi_S(U) : A(U) \to \Omega(U)$ which map $s \in A(U)$ to the largest $W \subset U$ s.t. $s|_W \in S$. It is easy to verify that the monic map $S\ 0.\ A$ is effectively the pull-back of $\underline{\text{True}}$ by φ_S. As Mac Lane says: φ_S gives "the shortest path to truth" $W \subset U$ for $S\ 0.\ A$.

5.3.3. Elements, properties and parts

In a topos, the morphisms $a : B \to A$ are called *generalized elements* of A, or elements *defined on B* (this denomination comes from algebraic geometric and, more precisely, from Grothendieck's theory of schemes). Among the elements, the most important are those *defined on open sets U*, that is precisely the sections $s \in A(U)$. If $U \in \mathbf{o}(M)$, we can consider the Yoneda sheaf $\mathbf{y}(U) = \{V \subset U\}$ and verify that $A(U) - \text{Hom}(\mathbf{y}(U),A)$. The elements defined on the terminal object 1 are "global". We will see that an arrow $\theta : A \to \Omega$ is a "predicate" for A, that is a "property" of its generalized elements. Among all predicates, there is the predicate $\underline{\text{True}}_A : A \to 1 \to \Omega$. It is easy to verify that an element $a : B \to A$ factorizes through a subobject $S0.A$ iff $\text{char}S(a) = \varphi_S \circ a = \underline{\text{True}}_B.\varphi_S$ is therefore the predicate of A which is true exactly for those elements of A which are in S. The *unicity* of φ_S expresses the *extensionality principle*.

Using the exponentials and the subobject classifier we can define the *parts* of an object A as another object $P(A) = \Omega^A$. We get the functorial isomorphisms:

$$\text{Sub}(A) - \text{Hom}(A,\Omega) - \text{Hom}(A\times 1,\Omega) - \text{Hom}(1,\Omega^A) = \text{Hom}(1,P(A)).$$

We have therefore $\Omega = P(1)$.

This shows that there are 3 equivalent descriptions of a subobject $m : S0.A$.

(i) its "extension" S: we will see that it can be symbolized as in **Sets** by $\{a | \varphi_S(a)\}$;

(ii) its characteristic map $\varphi_S : A \to \Omega$ which is a "predicate" of A;

(iii) the global section $s : 1 \to P(A)$ which is its "*name*".

The evaluation map $\varepsilon_A : A \times \Omega^A = A \times P(A) \to \Omega$ is a "membership" predicate: if $a : B \to A$ is an element of A, and if $s : 1 \to P(A)$ is (the name of) a subobject of A, then $\varepsilon_A \circ (a \times s) = \underline{\text{True}}_{B\times 1}$ iff $\varphi_S \circ A = \underline{\text{True}}_B$, that is iff a is an element of S.

5.3.4. Towards logic

The existence of an intuitionistic "internal logic" in a topos depends essentially on the fact that, being the set of the open sets W of the topological space U, $\Omega(U)$ is (fonctorially) an *Heyting algebra*. Ω is therefore a sheaf of Heyting algebras (an Heyting algebra object in $\mathbf{Sh}(M)$). The consequence is that the "external" set of subobjects $\text{Sub}(A)$ and the "internal" one $P(A)$ are also Heyting algebras, the canonical isomorphism $\text{Sub}(A) - \text{Hom}(1, P(A))$ being an isomorphism of Heyting algebras.

5.4. Topoi and logic

Now, the central fact is that a topos is exactly the categorical structure which is needed for doing logic. *But this logic is spatially localized.* For details see e.g. Saunders Mac Lane, Ieke Moerdijk 1992.

5.4.1. Types and localization

We can associate with each topos $\mathbf{Sh}(M)$ a formal language l_M called its Mitchell-Bénabou language, and a forcing semantics called its Kripke-Joyal semantics. The crucial point is that a sheaf X can be considered as a *type* for variables x which are interpreted as sections $s \in X(U)$ of X. *We get therefore at the same time a typification and a localization of the variables*. This achievement fits perfectly well with Husserl's description and explains Wittgenstein's remark. It provides them with a correct mathematical status.

(i) Sections are tokens denoted by variables belonging to types (species, essences). They are "concretely" particularized by the specification of their localization U and by their specific values. But as an element of type X, s particularizes an abstract unilateral relation of dependence, the relation "quality→extension" which is constitutive of X.

(ii) The relations between particular sections $s \in X(U)$, $t \in Y(V)$ are *external*. The relations between X and Y are *internal*. Nevertheless the linguistic expressions which express them are formulas in the formal language l_M associated to $\mathbf{Sh}(M)$, and are *the same*.

5.4.2. Syntax

How are the terms and the formulas of l_M syntactically constructed? Here is a summary of their inductive construction.

A term σ of type X constructed using variables y, z of respective types Y, Z has a *source* $Y \times Z$ and is interpreted by a morphism $\sigma : Y \times Z \to X$ which expresses its structure.

(i) To each $X \in \mathbf{Sh}(M)$ considered as a type are associated variables $x, x', \ldots$ They are interpreted by the identity map $1_X : X \to X$.

(ii) Terms $\sigma : U \to X$, $\tau : V \to Y$ of respective types X and Y yield a term $\langle\sigma,\tau\rangle$ interpreted by $\langle\sigma,\tau\rangle : W = U \times V \to X \times Y$ of type $X \times Y$.

(iii) Terms $\sigma : U \to X$, $\tau : V \to X$ of the same type X yield the term $\sigma = \tau$ of type Ω interpreted by

$$(\sigma = \tau) \ : \ W = U \times V \to X \times X \xrightarrow{\delta_X} \Omega$$

where δ_X is the characteristic function of the diagonal subobject $\Delta : X \to X \times X$.

(iv) A term $\sigma : U \to X$ of type X and a morphism $f : X \to Y$ yield by composition a term $f \circ \sigma$ of type Y.

(v) Terms $\theta : V \to Y^X$ and $\sigma : U \to X$ of respective types Y^X and X yield a term $\theta(\sigma)$ of type Y interpreted by

$$\theta(\sigma) \; : \; W = V \times U \to Y^X \times X \xrightarrow{e} Y$$

where e is the evaluation map.
(vi) In particular, terms $\sigma : U \to X$ and $\tau : V \to \Omega^X$ yield a term $\sigma \in \tau$ of type Ω interpreted by

$$\sigma \in \tau \; : \; W = V \times U \to X \times \Omega^X \xrightarrow{e} \Omega \, .$$

(vii) A variable x of type X and a term $\sigma : X \times U \to Z$ of type Z and of source $X \times U$ yield a term of type Z^X interpreted by $\lambda x \sigma : U \to Z^X$.
(viii) Ω is the type of the *formulae* of l_M. As it is an Heyting algebra, we get the logical operations of propositional calculus : $\varphi \wedge \psi$, $\varphi \vee \psi$, $\varphi \Rightarrow \psi$, $\neg \varphi$. It is easy to verify that if $\varphi(x,y) : X \times Y \to \Omega$ is a formula, we can write the subobject of $X \times Y$ classified by its interpretation in the "set theoretic" manner : $\{(x,y) \in X \times Y | \varphi(x,y)\}$.
(ix) One of the most remarkable facts of topoi theory is that it is possible to define *quantification* in a purely categorical manner. Let $f : A \to B$ be a morphism of **Sh**(M) and consider the "inverse image" functor $f^* : \mathrm{Sub}(B) \to \mathrm{Sub}(A)$ defined by composition with f. Its internal version is the morphism $P(f) : P(B) \to P(A)$. The fact is that $P(f)$ has *two adjoint functors* : a left adjoint one $\exists_f : P(A) \to P(B)$ and a right adjoint one $\forall_f : P(A) \to P(B)$. They generalize the two adjunctions in **Sets**. If $f : A \to B$, $S \subset A$ and $T \subset B$,

$$\exists_f(S) \;=\; \{b \in B \,|\, \exists a \in S \; (f(a) = b)\} \;=\; f(S),$$

$$\forall_f(S) \;=\; \{b \in B \,|\, \forall a \in A \; (f(a) = b \;\Rightarrow\; a \in S)\}$$

$$\left\{ b \in B \,|\, f^{-1}(b) \subset S \right\},$$

$$\text{and} \begin{cases} f^*(T) \subset S \;\Leftrightarrow\; T \subset \forall_f(S) \\ S \subset f^*(T) \;\Leftrightarrow\; \exists_f(S) \subset T \end{cases} .$$

We have:

$$S \in \exists_f(S)(U) \text{ iff } \{V \,|\, \exists t \in S(V) f(t) = s_{|V}\} \text{ covers } U,$$

$$S \in \forall_f(S)(U) \text{ iff } \forall V \subseteq U f^{-1}(s_{|V}) \subseteq S(V).$$

Now, let $\varphi(x,y) : X \times Y \to \Omega$ be a formula of two variables in **Sh**(M). Let $p : X \to 1$ the canonical projection and $P(p) : P(1) \to P(X)$. We get the adjunctions:

$$\Omega^X \;=\; P(X) \underset{\xrightarrow{\exists_p}}{\overset{\xrightarrow{\forall_p}}{\xleftarrow{P(p)}}} P(1) = \Omega \, .$$

We consider $\lambda x \varphi(x,y) : Y \to \Omega^X = P(X)$ and we get the formula of source Y:

$$\forall x \varphi(x,y) \;=\; \forall_p \circ \lambda x \varphi(x,y) \;:\; Y \to \Omega \, .$$

These categorical constructs show that the formal language l_M is, at the "linguistic" level, exactly of the same nature as the classical formal language of sets theory. The main difference is that we have introduced a subtle dialectics *between the type of the variables and their localization.*

5.4.3. Semantics

The Kripke-Joyal semantics of topoi is a *forcing* semantics generalizing Cohen's one. A variable x of type X denotes a *section* $s \in X(U)$, that is a morphism $s : U \to X$ where U is now the sheaf defined by U. The semantic rules are rules $U \Vdash \varphi(s)$ (U forces $\varphi(s)$). Let $s : U \to X$ and let $\mathrm{Im}(s) \in \mathrm{Sub}(X)$ be the image of s. One defines

$$U \Vdash \varphi(s) := \mathrm{Im}(s) \subseteq \{x \mid \varphi(x)\},$$

that is $U \Vdash \varphi(s)$ iff $U \xrightarrow{s} X \xrightarrow{\varphi} \Omega$ factorizes through $\{x / \varphi(x)\}$:

$$U \Rightarrow \{x \mid \varphi(x)\} \to 1 \xrightarrow{\text{True}} \Omega .$$

The semantic rules are:

(i) $U \Vdash \varphi(s) \wedge \psi(s)$ iff $U \Vdash \varphi(s)$ and $U \Vdash \psi(s)$.

(ii) $U \Vdash \varphi(s) \vee \psi(s)$ iff there exists an open covering $(U_i)_{i \in I}$ of U s.t. for every i, $U_i \Vdash \varphi(s|_{U_i})$ or $U_i \Vdash \psi(s|_{U_i})$ (intuitionistic rule for disjonction)

(iii) $U \Vdash \varphi(s) \Rightarrow \psi(s)$ iff, for all $V \subseteq U$, $V \Vdash \varphi(s|_V)$ implies $V \Vdash \psi(s|_V)$.

(iv) $U \Vdash \neg \varphi(s)$ iff there does not exist $V \subseteq U$, $V \neq \varnothing$ s.t. $V \Vdash \varphi(s|_V)$ (the negation is intuitionistic because Ω is a Heyting algebra and not a Boolean one).

(v) $U \Vdash \exists y \varphi(s,y)$ (y being of type Y) iff there exist an open covering $(U_i)_{i \in I}$ of U and sections $\beta_i \in Y(U_i)$ s.t. for every $i \in I$ $U_i \Vdash \varphi(s|_{U_i}, \beta_i)$.

(vi) $U \Vdash \forall y \varphi(s,y)$ iff for every $V \subseteq U$ and $\beta \in Y(V)$ we have $V \Vdash \varphi(s|_V, \beta)$.

The subtilities of this form of semantics result from the following problem. In a topos, one can interpret set-like expressions such as

$$\left\{ (x_i)_{i \in I} \in \prod_{i \in I} F_i \;\middle|\; \varphi(x_i) \right\}$$

defined by $(a_i) \in \{(x_i) \mid \varphi(x_i)\}(U)$ iff $U \Vdash \varphi(a_i)$. But one has to be sure that such expressions define *subsheaves*. For this, one has to "sheafify" the sub-functors appearing in the set-like constructions.

5.5. Sheaf mereology and boundaries.

Two last points to conclude.

1. There is a natural *mereology* applying to sections of fibrations and sheaves. It satisfies all the axioms of mereo-topology (Smith 1993) except the one asserting the *unicity* of the universe of objects under consideration. Actually, the union of sections is a merging, a fusion, a glueing of compatible sections. The union of the sections s satisfying a predicate φ is therefore in general not a single section but only a set of *maximal* ones. (See Petitot 1994)

2. In perceptive statements, *boundaries* play a fundamental role. And one knows since Brentano that boundaries are somehow paradoxical entities. To tackle this point, we can use Lawvere's idea of *co-Heyting* algebras. In a Heyting algebra of open sets, the negation $\neg U$ of U is the *interior* of its complementary set, that is the largest open set V such that $U \cap V = \varnothing$. In a co-Heyting algebra of closed sets, the negation $\neg F$ of F is dually the smallest element H such that $F \cup H = 1$ (that is the closure of its complementary open set). One has

$$\neg(F \cap H) = \neg F \cup \neg H, \text{ but only } \neg(F \cup H) \subseteq \neg F \cap \neg G.$$

One defines then the *boundary* ∂F of F as the intersection $F \cap \neg F = \partial F$. ∂F is therefore defined by logical "contradiction". The boundary operator satisfies the Leibniz rule :

$$\partial(F \cap H) = (\partial F \cap H) \cup (F \cap \partial H).$$

Boundaries are characterized by $\partial B = B$ that is by $\neg B = 1$ or $\neg\neg B = 0$. In general, the double negation $\neg\neg F \subset F$ is the "regular core" of F (the closure of its interior). One has of course $F = (\neg\neg F) \cup \partial F$.

CONCLUSION

1. There exists a topological geometrical eidetics of the morphological structures of perception. This eidetics is neurologically relevant.

2. There exists an indexical status of spatial extensions in perceptive statements. This fact requires some sort of Kripkean modal semantics.

3. It is relevant to try to model Husserl's ante-predicative "genealogy" of logic.

4. The geometric logic yielded by topos theory can do the job. It has therefore a deep cognitive meaning.

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