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Sheaf mereology and Husserl's morphological ontology

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This paper begins with Husserl's phenomenological distinction between formal ontology (analytic theory of general objects) and "material" regional ontologies (types of "essences" of objects which prescribe "synthetic *a priori*" rules). It then shows that, as far as its "ontological design" is concerned, transcendental phenomenology can be seen as an "object-oriented" epistemology (opposed to the classical "procedural" epistemology). The paper also analyses the morphological example, which constitutes the core of Husserl's third Logical Investigation, of the unilateral relation of foundation between sense qualities and spatio-temporal extension. It gives a geometrical model using the geometrical concepts of fibration, sheaf and topos.

1. Formal ontology vs. regional ontologies

In Husserlian phenomenology, the concept of *formal* ontology cannot be dissociated from that of "*material*" *regional* ontology. Formal ontology concerns a formal analytic theory of general objects. It axiomatizes formal categories and concepts like those of: object, quality, relation, identity, equality, plurality, number, magnitude, whole, part, genus, species, state of affairs, etc. On the contrary, a "material" regional ontology—as for instance the ontology of perception—concerns the objective laws governing a phenomenal domain of a specific type. For Husserl, such an objective type has a *normative* status. It is a noematic essence, a set of constitutive eidetic rules, which *a priori* prescribes and determines, prior to empirical data, what properties belong *typically and generically* to the objectivity of the empirical phenomena under consideration.

According to Husserl, this possibility of coherent rule-governed *anticipations* is the main characteristic of objectivity. For instance, in his theory of perceptive adumbrations (*Abschattungslehre*), he explains that it is the possibility of anticipating the geometrical deformations of the apparent contours of an object which founds perceptive intentionality, that is the directionality of consciousness towards external real objects.

This fundamental thesis is corroborated by contemporary research. For instance, Jan Koenderink (1976: p. 59), one of the leading geometers in computational vision, claims:

> If an observer has been permitted to visually explore a certain body by means of changing his vantage point voluntarily, he can gather enough information to predict future changes of the visual image [anticipation]. He is able (...) to interpret such changes as proprioceptive, and may consider the object as an unchanging entity, despite the changing visual input [unity and objective identity]. Our geometrical theory

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[singularity theory] enables us to understand the structure of the observer's internal models of external bodies.

The main Husserlian idea is that the temporal flux of the manifold of mental states (*Bewußtseinsmannigfaltigkeit*) is governed by rules which prescribe (*vorzeichnen*) its connexions and series (*Bewußtseinszusammenhänge*). The fundamental problems of phenomenology, which Husserl calls the "functional" ones (1913; §86 "*Die funktionellen Probleme*"), concern

how, according to absolutely invariant eidetic laws, an existing object can be a correlate for linked series of states of consciousness of a perfectly determined eidetic status, and conversely how the being inherent to such linked series is equivalent with an existing object. (p. 177).†

As Husserl [1913: §16 on *Region and Category in the material sphere. Synthetic a priori knowledge* (Husserl's emphasis)]:

The region is nothing else than the generic total and highest unity (...) *which belongs to a concretum.* (p. 30).‡

Every regional essence determines *eidetic 'synthetic' truths, that is truths which have their foundation in it, in so far as it is such a generic essence,* and which *are not simply particular forms of truths belonging to formal ontology.* (p. 31).§

Regional categories

do not only express, as do concepts in general, particular forms of purely logical categories, (...) they *express in terms of eidetic generality what must a priori and 'synthetically' occur to an individual object of the region.* (p. 31).||

On the contrary, as is explained in §10 on *Region and Category. The analytic region and its categories,* the "formal region" is not an authentic one. It is "the empty form of a region in general".

Die sog. 'formale Region' (...) *ist eigentlich nicht Region, sondern leere Form von Region überhaupt.* (p. 22).

It prescribes only a common general formal law-governedness to the material regional ontologies.

† "Wie, nach absolut festen Wesensgesetzen seiender Gegenstand Korrelat ist für Bewußtseinszusammenhänge ganz bestimmten Wesengehaltes, sowie umgekehrt das Sein so gearteter Zusammenhänge gleichwertig ist mit seisndem Gegenstand".

‡ "Region ist nichts anderes als die *gesamte zu einem Konkretum gehörige oberste Gattungseinheit.*"

§ "Jedes regionale Wesen bestimmt *"synthetische" Wesenswahrheiten, d.h. solche, welche in ihm als diesem Gattungswesen gründen, nicht aber bloße Besonderungen formal-ontologischer Wahrheiten sind.*"

|| "Diese Begriffe [die regionalen Kategorien] *in eidetischer Allgemeinheit ausdrücken, was einem individuellen Gegenstand der Region 'a priori' und 'synthetisch' zukommen muß.*"

We can therefore set down the following list of oppositions:

Formal ontology	Regional ontologies
Formal	Material
General	Specific
Analytic	Synthetic *a priori*
Logical categories	Essence: typicality, genericity
General legislation	Anticipation

2. Transcendental phenomenology as "object-oriented" epistemology

Since Dreyfus' (1982) *Husserl, Intentionality and Cognitive Science,* many philosophers have emphasized the close links between Husserlian phenomenology, contemporary research in artificial intelligence and computational cognitivism (see e.g. McIntyre & Woodruff Smith 1982; McIntyre 1986). It is true that the *noetico-noematic correlation* shares many common features with cognitive *functionalism* and that the *epoche* and *transcendental reduction* yield remarkable cases of methodological solipsism. The main difference with, for instance, a theory of mental representations of a Fodorian type is that, according to Husserl, intentionality is constituted on the *solipsistic* basis of "*narrow*" cognitive contents—which are "narrow" in the sense they are no longer denotative. Intentionality is therefore no longer a semantic property linking the cognitive contents with the external world via some causal theory of reference. According to Husserl, it is defined *after* the bracketing of the external world, as a *primitive* property of mental contents.

If we try to interpret the opposition between formal and regional ontologies in the framework of AI and computational cognitivism, we are led to the following analogy (it is of course only an analogy) with *object-oriented programming* (OOP). The idea is that transcendental phenomenology (that is a systematic theory of regional ontologies) is a sort of "object-oriented" epistemology.

In OOP, generic objects are pre-defined in a modular way. They are types, generic classes characterized by specific attributes and methods which are informationally encapsulated and which prescribe specific responses to external general messages.† More precisely, an object is a data structure characterized by attributes and associated with routines, procedures, actions—called methods—which operate *specifically* on it. This specificity is called *encapsulation.* The objects belong to classes which describes the object's data, the methods and their implementation, and also the messages by which they are acted on. Classes are hierarchically organized and sub-classes inherit the attributes and the methods of their super ordered classes. Finally, *polymorphism* is the possibility of sending the same message to objects of different types and to get nevertheless different specific responses (because different methods have been activated by the message).

In much the same manner, in transcendental phenomenology generic objects are pre-defined in an *a priori synthetic way.* They are types, classes, characterized by

† See e.g. Booch (1991).

specific eidetico-constitutive rules which are *a priori* synthetic and which prescribe specific responses to scientific investigation.

The parallel can be summarized in the following way:

Transcendental phenomenology	*Object oriented programming*
Domain: constitution of knowledge.	Domain: programming.
General categories of formal ontology *which are applicable to any types of objects.*	External messages and non-specific universal methods *which are applicable to any types of objects.*
Regional ontologies (Essences). Typical objects. Eidetico-constitutive rules.	Classes. Typical objects. Specific data structures and routines (attributes and methods).
Anticipation and specification of what must a priori and synthetically occur to an individual object of the region.	Anticipation and specification of the characteristic behavior of an individual object of the class.
Object of a region (token) = instance of its generic regional essence (type).	Particular object (token) = instance of its class (type).
Uncoupling between analytic a priori (generic formal ontology) and synthetic a priori (specific "material" regional ontology).	*Encapsulation* of the particular attributes and methods which are specific of a class.
Application of the categories of formal ontology to the regional ontologies.	Communication with the different classes of objects by means of messages and universal methods.
Transcendental schematism: the specific interpretation of the general formal categories in a regional ontology.	*Polymorphism*: the specific response of an object type to the external messages via the selection of appropriate internal methods.

I think that the main interest of this analogy is to show that the celebrated (and much maligned) "synthetic a priori" is essentially an epistemological and ontological strategy of *modularization and encapsulation* of the objects. It is a strategy for ontological design and the constitution of objective knowledge.

The main error of logical positivism in its obsessional attempt to refute the existence of synthetic *a priori* judements is to have searched for absolute criteria for characterizing them. Indeed, it is of course impossible. To ask if a judgement is "an sich" an analytic or an *a priori* synthetic one is as vain as asking if a procedure is "an sich" an encapsulated method or a universal one (that is a non-encapsulated one which can be applied to every class of objects). It is only a question of "ontological design".

In my perspective, synthetic *a priori* is therefore no more—but also no less—than a strategic move in ontological design. It yields a powerful polymorphic method of modular encapsulation for ontologies. This transcendental strategy is to its positivist counterpart what OOP is to procedural programming.

3. The unilateral foundation of sense qualities in spatio-temporal extension

I now want to give an explicit example of Husserl's "dialectic" between formal ontology and regional ontologies. It belongs to the regional ontology of perception and concerns the *morphological* Gestaltist description given by Husserl in the third *Logische Untersuchung* (*Zur Lehre von den Ganzen und Teilen*) concerning the Whole/Part theory. This text has been extensively (and deeply) analysed by Smith and Mulligan (1982). My purpose is to give here a *geometric schematization* of Husserl's pure eidetic description and to correlate it with the mereological axiomatics proposed in Smith (1993).

I will first recall some elements of Husserl's description.

3.1. HUSSERL'S PURE EIDETIC DESCRIPTION

Husserl begins with the difference between abstract and concrete contents, which he identifies with the (Stumpfian) opposition between dependent (*unselbstständigen*) and independent (*selbstständigen*) contents. According to him, such a difference is fundamental for the "pure (a priori) theory of the objects as such" which concerns the "formal objective categories" and the essential truths (*Wesenswahreiten*) of formal ontology.

But it is only in the second chapter, *Gedanken zu einer Theorie der reinen Formen von Ganzen und Teilen*, that Husserl develops an axiomatics of whole/parts relations. In the first chapter, *Der Unterschied der selbstständigen und unselbständigen Gegenstände*, he develops, in fact, a "material" analysis of empirical morphologies.

As is well known, the central problem analysed by Husserl is that of the *unilateral* dependence between qualitative moments (e.g. colour) and spatial extension (*Ausdehnung*). According to him, qualities are abstract essences (species) and constitute categorized manifolds, that is "*Mannigfaltigkeiten*" decomposed in different categories. He thought of the "quality$\rightarrow$extension" dependence as an eidetic law binding generic abstracta or types.

> The dependence [*Abhängigkeit*] of the immediate moments [*der unmittelbaren Momente*] concerns a certain relation conform to a law existing between them, a relation which is determined only by the immediately super-ordered abstracta of these moments. (p. 233).

There exists a *functional* dependence (*funktionelle Abhängigkeit*) connecting the *immediate* moments of quality and extension. The same qualitative *Abschattung* can be spread (*ausgedehnt, ausgebreitet*) over every extension, and conversely the same extension can be covered (*bedeckt*) by every quality. But this functional dependence—which associates to every point x of the extension W the value $q(x)$ of the quality q at this point—is objectively legalized by a pure law (*objektive-ideale Notwendigkeit, reine und objektive Gesetzlichkeit*) which acts only at the level of pure essences (*reine Wesen*). This "ideal a priori necessity grounded in the material essences" (*in den sachlichen Wesen gründenden idealen oder apriorischen Notwendigkeit*) is, according to Husserl, a typical example of the *synthetic a priori*.

Husserl (1931: §8–9) analyses the difference between the contents which set themselves into relief intuitively against a background (*anschaulich sich abhebenden Inhalten*) and the contents which are intuitively merged and fused together (*verschmolzenen*). Perceptual grasping presupposes a global unity of the intuitive moments and a "*phänomenal Abhebung*", that is, a saliency in Thom's or Gibson's sense. It is such a saliency which is expressed by the difference between, on the one hand, contents intuitively separated (*gesonderten, sich abhebenden, sich abscheidenden*) from the neighbouring ones and, on the other hand, contents merged with the neighbouring ones (*verschmolzenen, überfließenden, ohne Scheidung*).

3.2. THE CONCEPT OF *VERSCHMELZUNG*

The concept of fusion or merging—of *Verschmelzung*—is a key one. It expresses the spreading of qualities, that is the topological transition from the local level to the global one. It is *Verschmelzung* which generates the moment of global unity, of totality, of an object. Its complementary concept is that of separation, of disjunction, of cleavage—of *Sonderung*. *Sonderung* is an obstacle to *Verschmelzung*. It generates boundaries delimiting parts. At the intuitive synthetic *a priori* level, the "whole/part" difference grounds itself on the "Verschmelzung/Sonderung" one.

Here also, Husserl's pure description fits very well with contemporary research. For instance Stephen Grossberg, one of the leading scientists in the field of vision, concludes from his numerous works that there are two fundamental systems in visual perception:
(i) The *Boundary Contour System* (BCS) which controls the segmentation of the visual scenes. It detects, sharpens, enhances and completes edges, especially boundaries, by means of a "spatially long-range cooperative process". The boundaries organize the image geometrically (morphologically).
(ii) The *Featural Contour System* (FCS) which performs featural filling-in, that is spreading of qualities. It stabilizes qualities such as colour or brightness. These diffusion processes are triggered and limited by the boundaries provided by the BCS. Therefore, according to Grossberg (1988: p. 35),

> "Boundary Contours activate a boundary completion process that synthesizes the boundaries that define perceptual domains. Feature Contours activate a diffusion filling-in process that spreads featural qualities, such as brightness or color, across these perceptual domains" (p. 35).

In fact, the concept of *Verschmelzung* does not come from Stumpf but from the German psychologist Johann Friedrich Herbart (1776–1841), who developed a *continuous* theory of mental representations and contents. Essentially in the same vein as Peirce, Herbart was convinced that mental contents are *vague* and can vary continuously. For him, a "serial form" (*Reihenform*) was a class of mental representations which undergo a graded fusion (*abgestufte Verschmelzung*) gluing them together via continuous transitions. He coined the neologism of *synechology* for his conception (Peirce's neologism of *synechism* is clearly equivalent).

It is not sufficiently known that Herbart's point of view was one of the main interests of Bernhard Riemann when he was elaborating his key concept of Riemannian manifold. Even if Riemann did not agree with Herbart's metaphysics, he strongly claimed that he was "a Herbartian in psychology and epistemology". Scholtz (1992) has shown that in Riemann's celebrated *Über die Hypothesen, welche der Geometrie zu Grunde liegen* (1867) the role of the differentiable manifold underlying a Riemannian manifold "is taken in a vague sense by a Herbarian-type of 'serial form', backed by mathematical intuition".

3.3. *SONDERUNG* AND QUALITATIVE DISCONTINUITIES

In the third *Logical Investigation,* Husserl (1900–1901: §8) claims that "*Sonderung beruht* (...) *auf Diskontinuität*". *Verschmelzung* corresponds to a continuous (*stetig*) spreading of qualities in an undifferentiated unity (*unterschiedslose Einheit*) (*p.* 244) *and Sonderung* corresponds to qualitative discontinuities in the way in which extension is covered (*Deckungszusammenhang*) by qualities.

These qualitative discontinuities are salient only if

(i) they are contiguously unfolded (*sie angrenzend ausgebreitet sind*) against the background of a moment which varies continuously (*ein kontinuierlich variierendes Moment*), namely the spatial and temporal moment, or
(ii) they present a sufficient gap (threshold of discrimination).

Husserl's (p. 246) morphological description is precise and remarkable.

> It is from a spatial or temporal limit [*einer Raum- oder Zeitgrenze*] that one springs from a visual quality to another. In the continuous transition [*kontinuierlichen Übergang*] from a spatial part to another, one does not progress also continuously in the covering quality [*in der überdeckenden Qualität*]: in some place of the space, the adjcent neighboring qualities [*die angrenzenden Qualitäten*] present a finite (and not too small) gap [*Abstand*].

This Husserlian pure eidetic description of the unilateral dependence "quality → extension" yields therefore the following correspondences.

Totality	Parts
Verschmelzung	Sonderung
Spreading activation (featural filling-in)	Boundaries
Continuity	Discontinuity

3.4. THE LINKS WITH COMPUTATIONAL VISION

This Gestaltist morphological description is very akin to contemporary theories of computational vision. For instance, Marr (1982) introduced the hypothesis that the main function of the ganglionary cells of the retina is to extract the qualitative discontinuities (zero-crossings) which are encoded in the signal, and that the higher levels of visual processing are grounded in this primal morphological organization of the image (primal sketch). In fact, it has been shown that the convolution of the signal by the receptive profiles of the ganglionary cells (which are essentially Laplacians of Gaussians), is a *wavelet analysis*, that is a *spatially localized* and *multiscale* Fourier analysis. Now, wavelet analysis is actually the best known device for extracting discontinuities. Let us explain very briefly the main idea in the one-dimensional case.

Consider the Hilbert space $L^2(\mathbb{R})$ of square integrable functions on $\mathbb{R}$. Fourier analysis provides an orthogonal decomposition of every $f \in L^2(\mathbb{R})$ relative to the orthonormal basis of trigonometric functions $e^{-i\omega x}$. That is, the Fourier transform (FT) $\hat{f}(\omega)$ of $f(x)$ is given by the formula:

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-i\omega x} dx.$$

It can be shown that $\hat{\hat{f}}(\omega) = f(x)$ and that the norms $\|f(x)\|$ and $\|\hat{f}(\omega)\|$ are equal (that is, the FT is an isometry).

The problem is that the information provided by $\hat{f}$ is *delocalized* (because the plane waves $e^{-i\omega x}$ are). In order to localize it, Gabor introduced the idea of the *Window Fourier transform* (WFT):

$$Gf(\omega, u) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\omega x} g(x-u) f(x) dx.$$

$Gf(\omega, u)$ is localized by a spatial "window" $g(x)$ translated along the x-axis. The WFT depends not only on the frequency ω but also on the position u. It generalizes the FT. The inverse transform is given by:

$$f(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} Gf(\omega, u) e^{i\omega x} g(u-x) d\omega dx.$$

It is an isometry between $L^2(\mathbb{R})$ and $L^2(\mathbb{R}^2)$, that is $\|f\| = \|Gf\|$. It is in general highly redundant and it is therefore possible to sample u and ω.

The problem with Gabor's WFT is that it operates at only *one* level of resolution. If the signal is a multiscaled one (e.g. fractal), this is a drastic limitation.

The idea of the *wavelet transform* (WT) is to find decompositions of $L^2(\mathbb{R})$ starting from *a single* function $\psi(x)$ (the "mother" of the wavelets), and using its translated transforms $\psi(x-u)$ and its rescaled transforms:

$$\psi_s(x) = \sqrt{s}\, \psi(sx) \qquad \left(\text{or } \psi_s(x) = \frac{1}{s} \psi\left(\frac{x}{s}\right)\right).$$

One then gets the following WT:

$$Wf(s, u) = \int_{\mathbb{R}} f(x) \psi_s(x-u) dx = f * \bar{\psi}_s(u)$$

with $\bar{\psi}(x) = \psi(-x)$. It is well defined if an *admissibility condition* C_ψ on the FT $\hat{\psi}$ is satisfied, to the effect that $\hat{\psi}(0) = 0$ and that $\hat{\psi}$ is sufficiently flat near 0:

$$(C_\psi) \qquad \int_0^\infty \frac{|\psi(\omega)|^2}{\omega} d\omega < \infty$$

The main result of the theory is that appropriate ψ *exist*. A typical example is Marr's wavelet. The *amplitude* $|Wf(s, u)|$ of the WT is an indicator of the *singularities* encoded in the signal. More precisely, the Lipschitzian order of f at x can be deduced from the asymptotic decreasing of $|Wf(x, us|$ in the neighbourhood of x when the scale tends towards 0. As was emphasized by Mallat (1989):

> the ability of the WT to characterize the type of local singularities is a major motivation for its application to detect the signal's sharper variations.†

As the WT is generally highly redundant,‡ it is possible to discretize it by sampling the u and ω variables. With such devices it becomes possible to *compress* an image in an *intrinsic* way, that is according to its specific structure.§ The main fact we want to stress here is that the compression of information—which is an information processing constraint—appears as identical with a morphological analysis—which relates to a geometrical objective fact.

4. The problem of a morphological geometry

Husserl's morphological description is precise and remarkable. But nevertheless it raises a fundamental problem. Qualitative discontinuities, he says,

> concern the minimal specific differences [*die niedersten spezifischen Differenzen*] in an immediately super-ordered [*übergeordnet*] pure genus [*Gattung*]. (p. 246).

They are discontinuities of the functional dependence "quality → extension".

The consequence is that, according to Husserl, it is impossible to formalize them. Formalization can only operate at a *higher* level of abstraction, the level of the general eidetic law of dependence.

We meet here a *formalist* thesis which subordinates the regional material ontologies to formal ontology, and therefore the synthetic a priori (*synthetisch-a priorische*) laws to the analytic (*analytisch-a priorische*) ones. As regards its material content, the eidetic law of dependence "quality → extension", belongs to the sphere of vague intuitions (*vage Anschaulichkeiten*). But, according to Husserl, these vague—*inexact*—morphological essences cannot be *geometrically* modelled. As he claims (§9, p. 245):

> Kontinuität und Diskontinuität sind natürlich nicht in mathematischer Exaktheit zu nehmen.

It is not possible to clarify here this fundamental point. But nevertheless I want to emphasize the fact that one of the main limitations of phenomenology is its divorce

† For an introduction to the use of wavelet analysis in computational vision, see also Mallat & Zhong (1989). For a discussion of the link with morphological phenomenology, see e.g. Petitot (1989*b*, 1990, 1993*b*, *c*).

‡ When the redundancy is zero one speaks of orthogonal wavelets.

§ Wavelet analysis can be refined—in particular for the applications to data compression problems—by means of wavelet *packet* algorithms and methods. Many wavelets are used in parallel so as to adapt in the best way the choice of the decomposition basis to the particular structure of the signal. The fit criterion is the minimizing of the information entropy (e.g. Wickerhauser, 1991).

of any "material descriptive eidetic" of *Erlebniss* from any form of geometry. Husserl has always rejected the possibility of a morphological geometry. In some outstanding sections of the *Ideen I* (§§71–75), he explains the fundamental difference between, on the one hand, the vague inexact descriptive concepts correlated with morphological essences, and, on the other hand, the exact ideal mathematical concepts. According to him, *ideation,* which leads exact essences to ideality, is drastically different from *abstraction,* which leads inexact essences to genericity (categorization and typicality). This opposition is a key one for Husserl. Nevertheless, it can be shown (Petitot, 1985, 1989*a,* 1992*a,* 1993*a*) that it is no longer acceptable.

5. The synthetic a priori law of dependence "quality → extension" and the geometrical concept of fibration

5.1. FIBRE BUNDLES AND SECTIONS

Indeed, *there is* a fundamental geometric structure which fits perfectly well with Husserl's eidetic pure description of the spreading [*Ausbreitung*] of a quality in an extension or, equivalently, of the covering [*Überdeckung, Deckungszusammenhang*] of an extension by a quality. It is the key geometrical concept of *fibration.*

According to Herbart, Riemann and Stumpf, a spatial substrate (*Ausdehnung*) can be geometrically modelled by a differentiable manifold W. let Q be the qualitative genus under consideration (e.g. the space of colours). Q can be modelled by a manifold endowed with a categorization, that is with a decomposition in domains (categories) centred around central values (prototypes).

The functional dependence expressing a spreading–covering relation between W and Q can be naively defined as a map $q: W \to Q$ which, to any given point $x \in W$, associates the value $q(x) \in Q$ of the quality at this point. This models Husserl's functional dependence. *Verschmelzung* is then expressed by the differentiability of q and *Sonderung* by the discontinuities of q. These discontinuities constitute a closed subset K of W which expresses geometrically the salient morphology profiled in W. (See Petitot & Smith, 1991.)

But this naive model is too naive. Indeed, we need to have *all* the space Q at hand at *every* point $x \in W$. This requisite is imposed by Hussel's pure description (§III.1.), but also by theories of perception. Actually, since Hubel's and Wiesel's pioneering works, it has been shown by many neurophysiological experiments that the covering of extensions by qualities such as colours or by local geometrical elements such as directions are neurally implemented by (hyper)columns, that is by retinotopic structures where, "over" each retinal position, there exists a "column" implementing the same set of possibilities.

This leads to the fundamental and pervasive concept of *fibration* introduced by Whitney, Hopf and Stiefel and which concerns, in modern geometry and mathematical physics, all the situations where fields of non spatio-temporal entities functionally depend on space–time positions.

Mathematically, a fibration is a differentiable manifold E endowed with a

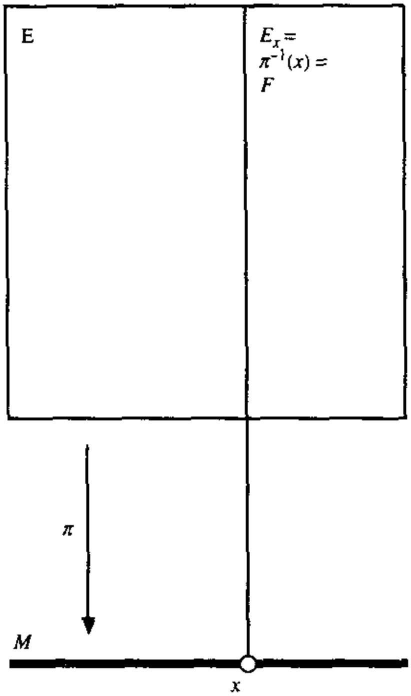

FIGURE 1. The general structure of a fibration. E is the total space, M the base space, π the canonical projection and $E_x = \pi^{-1}(x)$ the fibre over x. All the fibres are isomorphic to a typical fibre F.

canonical projection (a differentiable map) π: $E \to M$ over another manifold M. M is called the *base* of the fibration, and E its *total space*. The inverse images $E_x = \pi^{-1}(x)$ by π of the points $x \in W$ are called the *fibres* of the fibration.

The two axioms for π are then:

(F_1) All the fibres E_x are diffeomorphic with a typical fibre F.

(F_2) The fibration is *locally trivial*, that is $\forall x \in M$, $\exists U$ a neighbourhood of x such that the inverse image $E_U = \pi^{-1}(U)$ of U is diffeomorphic with the direct product $U \times F$ endowed with the canonical projection $U \times F \to U$, $(x, q) \to x$. (See Figures 1 and 2.)

In our case, we have $M = W$ and $F = Q$. The main point is that the geometric structure of a fibration geometrizes the Husserlian eidetic law of unilateral dependence "quality $\to$ extension". Therefore, this eidetic law possesses a *geometric* eidetic content.

How can we interpret the concept of functional dependence in this new context? It corresponds to the key concept of a *section* of a fibration. Let $\pi: E \to M$ be a fibration and let $U \subset M$ be an open subset of M. A section s of π over u is a *lift* of U in E which is compatible with π. More precisely, it is a map $s: U \to E$, $x \in U \to s(x) \in E_x$, i.e. such that $\pi \circ s = \mathrm{Id}_U$. In general s is supposed to be continuous, or differentiable, or analytic. Sometimes it can present discontinuities along a singular locus. (See Figures 3, 4 and 5.)

It is conventional to write $\Gamma(U)$ for the set of sections of π over U. A local

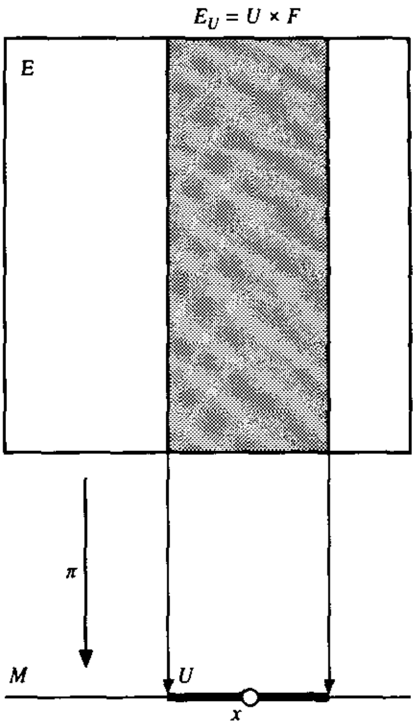

FIGURE 2. The local triviality of a fibration. For every point x of the base space M, there exists a neighbourhood U of x such that the inverse image $E_U = \pi^{-1}(U)$ of U by the canonical projection π is trivial, that is diffeomorphic with the direct product $U \times F$.

trivialization of π over U (i.e. $E_U \rightarrow \approx U \times F \rightarrow U$) transforms every section $s: U \rightarrow E$ in a map $x \rightarrow s(x) = (x, f(x))$, that is in a map $f: U \rightarrow F$. Therefore, the concept of section generalizes the classical concept of map, that is of functional dependence.

5.2. FROM HUSSERL'S PURE EIDETIC DESCRIPTION TO THOM'S TOPOLOGICAL ONE

With the concepts of fibration and section at hand, we can identify the Husserlian pure eidetic description with the topological one proposed by Thom (1972, 1980). This link between morphological phenomenology and geometrical eidetics is in our eyes an essential one, and we have up to now devoted to it a lot of work (see Petitot, 1979, 1985, 1989, 1992*b*, 1993*a*,*c*). It unifies qualitative physics, Gestalt theory, phenomenology and structuralism in a general mathematical theory of forms and structures.†

Let S be a material substrate. The problem is to explain its observable morphology. *We suppose that an internal dynamical mechanism X defines the internal states* of S. More precisely:

† For the link between morphodynamics and qualitative physics see Petitot and Smith (1991). For the link between phenomenology and Gestalt theory, see Smith and Mulligan (1982) and Smith (1988). For the link with structuralism see Holenstein (1992) and Petitot (1985, 1986, 1989*a*,*b*, 1993*c*).

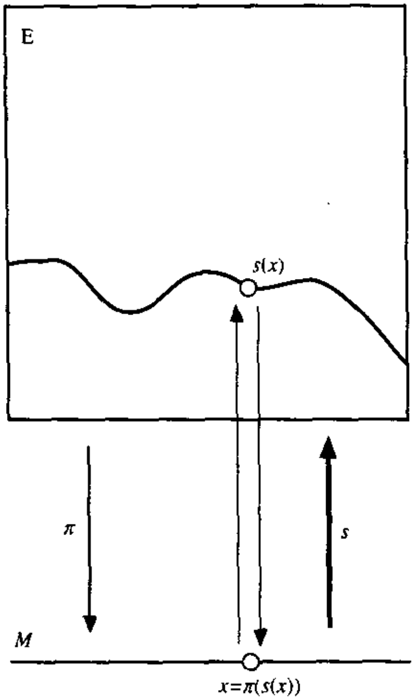

FIGURE 3. A section of a fibration is a lifting of π, that is a map $s:M\to E$ such that $\pi(s(x))=x$.

(i) There exists a configuration space (or a phase space) M of S which is a differentiable manifold and whose points x represent the instantaneous transient states of S. M is called the *internal space* of S.
(ii) X is a flow on M, that is, a system of ordinary differential equations $\dot{x}=X(x)$ which shares three properties: it is first complete (its trajectories are integrable from $t=-\infty$ to $t=+\infty$); second deterministic; and third smooth relatively to the initial conditions. The smooth vector field X is called the *internal dynamics* of S.

As a flow, X is identifiable with the one parameter sub-group of diffeomorphisms of M, Γ_t, where Γ_t is the diffeomorphism of M which associates to every point $x\in M$ the point x_t which is the point at time t on the trajectory of X leaving x at time $t=0$. Clearly, $\Gamma_{t'}\circ\Gamma_t=\Gamma_{t+t'}$ and $\Gamma_{(-t)}=(\Gamma_t)^{-1}$. $\Gamma:\mathbb{R}\to\mathrm{Diff}(M)$ is therefore a morphism of groups from the additive group $\mathbb{R}$ to the group of diffeomorphisms of M. Γ is the integral version of the vector field X. The internal states of S are then the (asymptotically stable) *attractors* of X.

It is very difficult to define rigorously the notion of an attractor for a general dynamical system. The usual definition is the following. Let $\omega^+(a)$ be the positive limit set of $a\in M$, that is the topological closure "at infinity" of the positive trajectory of a. Let $A\subset M$ be a subset of the internal space M. A is an attractor of the flow X if it is topologically *closed*, X-*invariant* (i.e. if $a\in A$ then $\Gamma_t(a)\in A$ $\forall t\in\mathbb{R}$), *minimal* for these properties (i.e. $A=\omega^+(a)$ $\forall a\in A$), and if it attracts asymptotically every point x belonging to one of its neighbourhoods U (i.e. $\exists U$ s.t.

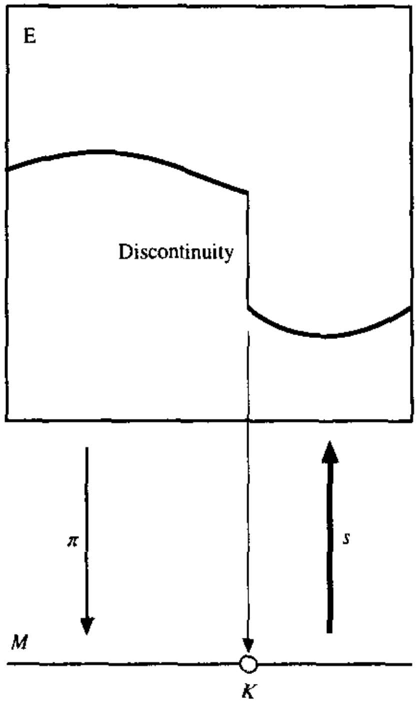

FIGURE 4. A discontinuous section.

$A = \omega^+(x)$ $\forall x \in U$). A is asymptotically stable if in addition it *confines* the trajectories of its sufficiently neighbouring points. If A is an attractor of X, its *basin* $B(A)$ is the set of points $x \in M$ which are attracted by A (i.e. s.t. $\omega^+(x) = A$). If U is an attracted neighbourhood of A, we have of course $B(A) = \bigcup_{t<0} \Gamma_t(U)$.

Now, as only one internal state A can be the *actual* state of S, there exists necessarily some criterion I (for instance a physical principle of minimization of energy) which selects A from among the possible internal states of S. The system S is also *controlled* by control parameters varying in the extension W of S. W is called the *external space* of S. The internal dynamics X is therefore a dynamics X_w which is parameterized by the external points $w \in W$ and varies smoothly relative to them.

Phenomenologically, the system S manifests itself through observable and measurable qualities $q^1(w), \ldots, q^n(w)$ which are characteristic of its actual internal state A_w and are sections of fibrations having as typical fibres the quality types $Q^1, \ldots, Q^n$. When the control w varies smoothly in W, X_w and A_w vary smoothly. If A_w subsists as the actual state, then the q^i also vary smoothly. But if the actual state A_w bifurcates towards another actual state B_w when w crosses some critical value, then some of the q^i must present a discontinuity. Thom has called *regular* the points $w \in W$ where locally all the qualities q^i vary smoothly and *singular* the points $w \in W$ where locally some of the q^i present a qualitative discontinuity. The set R_w of regular points is by definition an open set of W and its complementary set K_W, the set of singular points, is therefore a closed set. By definition, K_W is the morphology yielded by the dynamical behaviour of the system S.

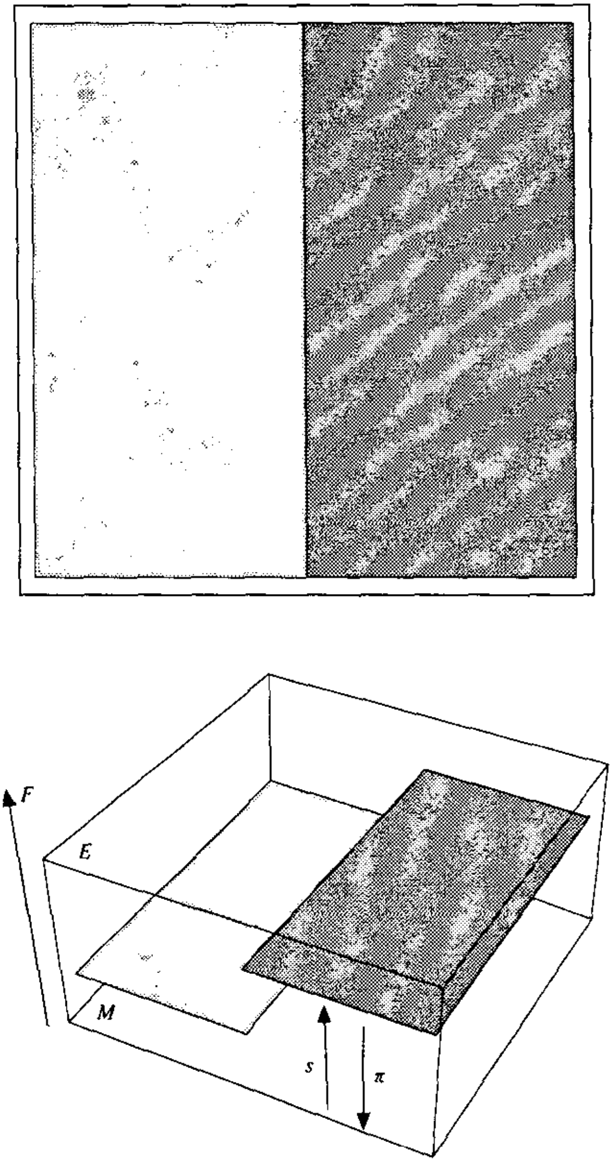

FIGURE 5. (a) A covering (*Überdeckung*) of a space by a quality (colour or grey level). (b) The corresponding section.

The singular points $w \in K_W$ are *critical values* of the control parameters and, in the physical cases, the system S presents for them *a critical behaviour*. Thom was the first scientist to stress the point that qualitative discontinuities are phenomenologically dominant, that every qualitative discontinuity is a sort of critical phenomenon and that a general mathematical theory of morphologies presented by general systems had to be an enlarged theory of critical phenomena.

Now, it is clear that Thom's description is a topological version of Husserl's one.

Regular points corresponds exactly to *Verschmelzung,* and singular ones to *Sonderung.*

5.3. GEOMETRY AND PHENOMENOLOGICAL EIDETICS

(1) The functional dependencies determined at the level of minimal specific differences correspond to *particular sections* $\sigma: W \rightarrow E$ of *particular* fibrations $\pi: E \rightarrow W$ with fibre Q.
(2) The qualitative salient discontinuities are discontinuities of sections $\sigma \in \Gamma(U)$.
(3) The eidetic law "concretely determined by its material contents" corresponds to a particular fibration $\pi: E \rightarrow W$ of fibre Q *but without any particular given section.* Such a fibration models an *abstract* relation between the genus W and Q (first level of abstraction). It implicitly contains an infinite universe of potential functional dependences, namely, all the sets of sections $\Gamma(U)$ for $U \subset W$.
(4) The synthetic *a priori* law of dependence "quality$\rightarrow$extension" corresponds to *the general mathematical structure* of fibration. It concerns the most abstract genus—the essences—of space and quality (second level of abstraction).
(5) Last but not least, the "analytic axiomatization" of this synthetic law in the framework of formal ontology corresponds to the *axiomatics* of fibrations.

6. The axiomatics of fibrations and the concept of sheaf

6.1. GLUING AND COHOMOLOGY

The axiomatics of the concept of fibration rests essentially on the concept of gluing—of fusion, of merging, of collating—of sections.†

One constructs global sections by gluing local ones in the following way. Let $\pi: E \rightarrow M$ be a fibration of fibre F and let $\mathcal{U} = (U_i)_{i \in I}$ be an open covering of U which locally trivializes π. This means that there are diffeomorphisms $\varphi_i: U_i \times F \rightarrow \pi^{-1}(U_i) = E_{U_i}$ over the open sets U_i which induce fibre diffeomorphisms $\varphi_{i,x}: F \rightarrow E_x = \pi^{-1}(x)$.

The condition for gluing to be possible is that $\forall i,j \in I$ such that $U_i \cap U_j \neq \emptyset$, then $\forall x \in U_i \cap U_j$, the automorphism of F $\theta_{i,j}(x) = (\varphi_{i,x})^{-1} \circ \varphi_{j,x}: F \rightarrow F$ belongs to a certain Lie group G of diffeomorphisms of F. This group is called the *structural group* of the fibration. For instance, in the case of a linear fibre bundle of which the fibre F is a vector space, the structural group G will be the linear group $GL(F)$. It is trivial to verify that if $U_i \cap U_j \cap U_k \neq \emptyset$, then $\theta_{i,j} \circ \theta_{k,i} = 1$.

In fact we meet here the concept of *simplicial structure,* which lies at the basis of what is called the *cohomology* of fibrations. Let $\mathcal{U} = (U_i)_{i \in I}$ be an open covering of the base M. Let F and G be as above. The *skeleton* K of $\mathcal{U}$ is the simplicial structure over I defined in the following manner:

- the 0-simplexes are the indices $i \in I$;
- the 1-simplexes are the pairs $(i, j) \in I \times I$ such that $U_i \cap U_j \neq \emptyset$;
- the 2-simplexes are the triples $(i, j, k) \in I \times I \times I$ such that $U_i \cap U_j \cap U_k \neq \emptyset$; etc.

For any open set U of M, let $\Gamma(U) = \{\theta: U \rightarrow G,\ \theta$ a differentiable map$\}$. If s is a

† For an introduction to local/global structures, fibrations, sheaves, topoi, etc. see Petitot (1979, 1982), and, for more details, Mac Lane and Moerdijk (1992).

p-simplex of K, a *p-cochain* is an element $\sigma \in \Gamma_s = \Gamma(\bigcap_{i \in s} U_i)$. The p-cochains form a group C^p. It is then easy to define a *coboundary* operator and therefore a cohomology theory. For instance, let $\varphi = (\varphi_i)$ be a 0-cochain. Its boundary is the 1-cochain $\partial\varphi = (\theta_{ij} = \varphi_i^{-1} \cdot \varphi_j)$. If $\theta = (\theta_{ij})$ is a 1-cochain, its coboundary is the 2-cochain $\partial\theta = (\psi_{ijk} = \theta_{ij} \cdot \theta_{jk} \cdot \theta_{ki})$, etc.†

It is trivial to verify that $\partial^2 = 1$. One can therefore consider the group of *p-cocycles* Z^p, that is of p-cochains σ without boundary ($\partial\sigma = 1$) and of *p-coboundaries* B^p, that is of p-cochains of the form $\sigma = \partial\tau$ with $\tau \in C^{p-1}$. As $\partial^2 = 1$, $B^p \subset Z^p$ and one can therefore consider the quotient groups $H^p = Z^p/B^p$. They are called the *cohomology groups* of the fibration. One can then prove the following theorem.

Theorem. A fibration is characterized by a 1-cocycle $\theta = (\theta_{ij})$ ($\partial\theta = 1$ is the gluing condition). It is *globally* trivial if θ is a 1-coboundary ($\theta_{ij} = \varphi_i^{-1} \cdot \varphi_j$).

6.2. SECTIONS AND SHEAVES

At a more abstract level, a fibration is characterized by the sets of its sections $\Gamma(U)$ over the open sets $U \subset M$. If $s \in \Gamma(U)$ and if $V \subset U$, we can consider its *restriction* $s|_V$ to V. The restriction is a map $\Gamma(U) \to \Gamma(V)$. It is clear that if $V = U$, then $s|_V = s$ and that if $W \subset V \subset U$ and $s \in \Gamma(U)$, then $(s|_V)|_W = s|_W$ (transitivity of the restriction). We get therefore a *contravariant functor* $\Gamma: \mathcal{O}^*(M) \to \mathcal{S}$ from the category $\mathcal{O}(M)$ of the open sets of M‡ in the category $\mathcal{S}$ of sets.

Conversely, let Γ be such a functor. To have a chance of being the functor of the sections of a fibration, Γ must clearly satisfy the two following axioms.

(S_1) Let $\mathcal{U} = (U_i)_{i \in I}$ be an open covering of M. Let $s, s' \in \Gamma(M)$. If $s|_{U_i} = s'|_{U_i}\ \forall i \in I$, then $s = s'$. Two sections which are locally equal must be globally equal.
(S_2) Let $s_i \in \Gamma(U_i)$ be a family of sections over $\mathcal{U} = (U_i)_{i \in I}$. If the s_i are compatible, that is, if $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ when $U_i \cap U_j \neq \emptyset$, then they can be glued together: $\exists s \in \Gamma(M)$ such that $s|_{U_i} = s_i\ \forall i \in I$. Compatible local sections can be collated in a global one.

In fact these axioms characterize a more general structure—and one that is even more pervasive in contemporary mathematics—than the structure of fibration, namely the structure of *sheaf*. It can be shown that if the axioms (S_1) and (S_2) are satisfied, then one can represent the functor Γ by a general fibred structure $\pi: E \to M$ (called an "étale" space and which is not necessarily locally trivial as a fibration must be) in such a way that $\Gamma(U)$ becomes the set of sections of π over U. In a nutshell, the fibre E_x—called in that case the *stalk* of the sheaf Γ at x—is the inductive limit:

$$E_x = \lim_{V \subset U \in \mathcal{U}_x} \{(\Gamma(U), \Gamma(V \subset U)\}$$

(where $\mathcal{U}_x$ is the filter of the open neighbourhoods of x). E is the sum of the E_x. If $s \in \Gamma(U)$, then s can be interpreted as the map $x \in U \to s(x) \in E_x$. The topology of E is then defined as the finest topology making all these sections continuous.

† In these formula, products and inverses are those defined by G.

‡ The objects of $\mathcal{O}(M)$ are the open sets of M, and its morphisms are the inclusions of open sets.

7. Axiomatics of fibrations, sheaf mereology and the formal ontology of wholes and parts

The concept of section is a key mathematical concept which permits us to build up models for a large class of *dependent* parts. It therefore provides a good model (in the sense of model theory) for axiomatic mereology,† what we will call *a sheaf model.* This is why, to conclude this reflexion, I want to stress the relation between the axiomatics of fibrations and the piece of formal ontology proposed by Smith (1993).‡

7.1. SMITH'S MEREOLOGICAL AND TOPOLOGICAL SYSTEM.

Smith (1993) proposed an axiomatic for two primitives:

(i) the mereological primitive $x\mathbf{C}y$: "x is a *constituent* of y";
(ii) the topological primitive $x\mathbf{P}y$: x is an interior part of y".

From $x\mathbf{C}y$ some other concepts can be immediately derived:

DC1. "x overlaps y" $x\mathbf{O}y := \exists z(z\mathbf{C}x \wedge z\mathbf{C}y)$;
DC2. "x is discrete from y"; $x\mathbf{D}y := \neg x\mathbf{O}y$.

The first two axioms characterizing the $\mathbf{C}$ relation are:

AC1. $x\mathbf{C}y \Leftrightarrow \forall y(z\mathbf{O}x \Rightarrow z\mathbf{O}y)$;
AC2. $x\mathbf{C}y \wedge y\mathbf{C}x \Rightarrow x = y$.

This mereological system is extensional and $\mathbf{C}$ is an order relation.

A fundamental axiom is that, for every predicate φ of x, one can define the *sum*—the fusion, the merging—of all the x which satisfy φ. This yields the definition:

DC4. $[x:\varphi(x)] := \iota y(\forall w(w\mathbf{O}y \Leftrightarrow \exists v(\varphi(v) \wedge w\mathbf{O}v)))$, for any satisfied predicate.

Of course, we need axioms to guarantee that the matrix of **DC4** is a definite description to which Russell's operator ι can be applied. **AC1** implies unicity. For existence, we need:

AC3. $\exists x\varphi(x) \Rightarrow \exists y(y = [x:\varphi(x)])$.

One can then prove easily the following theorems.

TC3. $y = [x:\varphi(x)] \Rightarrow \forall x(\varphi(x) \Rightarrow x\mathbf{C}y)$;
TC4. $\exists x\, \forall y(y\mathbf{C}x)$ (the universe exists);
TC5. $y\mathbf{C}[x:\varphi(x)] \Leftrightarrow \forall w(w\mathbf{C}y \Rightarrow \exists v(\varphi(v) \wedge w\mathbf{O}v))$.

One can also define the following fundamental concepts.

$1 := [x:x = x]$ (the universe); $x\mathbf{O}y := [z:z\mathbf{C}y \vee z\mathbf{C}y]$ (union);
$x \cap y := [z:z\mathbf{C}y \wedge z\mathbf{C}y]$ (intersection);
$x' := [z:z\mathbf{D}x]$ (complement).

As concerns the topological primitive $\mathbf{P}$, the following six axioms are clearly needed.

† *For mereology in the framework of formal ontology, see* Poli (1992).

‡ In the Padova Workshop on Formal Ontology, I met Dr. Graham White who has also used sheaf cohomology in relation to Smith's mereology. This unexpected convergence enhances the relevance of sheaf mereology (see White, 1993).

AP1. $x\mathbf{P}y \Rightarrow x\mathbf{C}y$:
AP2a. $x\mathbf{P}y \wedge y\mathbf{C}z \Rightarrow x\mathbf{P}z$ (left monotonicity);
AP2b. $x\mathbf{C}y \wedge y\mathbf{P}z \Rightarrow x\mathbf{P}z$ (right monotonicity);
AP3. $x\mathbf{P}y \wedge x\mathbf{P}z \Rightarrow x\mathbf{P}(y \cap z)$ (condition on finite intersections);
AP4. $\forall x(\varphi(x) \Rightarrow x\mathbf{P}y) \Rightarrow [x:\varphi(x)]\mathbf{P}y$;
AP5. $\exists y(x\mathbf{P}y)$;
AP6. $x\mathbf{P}y \Rightarrow x\mathbf{P}[t:t\mathbf{P}y]$.

From these axioms it is possible to derive all the traditional topological concepts. For instance the *interior* of an object x is defined by:

DP6. $\mathbf{int}\,(x) := [y:y\mathbf{P}x]$.

For defining the *closure* of x one can define first the *boundary* relation $x\mathbf{B}y$ in the following manner: x *crosses* y if x overlaps both y and its *complement* $1-y := [z:z\mathbf{D}y]$. x *straddles* y if when $x\mathbf{P}z$ then z crosses y. Finally $x\mathbf{B}y$ if $z\mathbf{C}x$ implies that z straddles y. One can then define the closure of x by:

DP4. $\mathbf{cl}\,(x) := x \cup [y:y\mathbf{B}x]$.

Closure thus defined can easily be shown to satisfy the usual Kuratowski axioms for topological space.

7.2. SHEAF MEREOLOGY

Let us now apply this general mereological axiomatics—which belongs to formal ontology—to the sheaf-theoretic modelling of Husserl's pure eidetic description of the *Überdeckung* of extension by dependent qualities. The basic elements of our universe are sections of a sheaf defined by a contravariant functor $\Gamma:\mathcal{O}^*(M)\to\mathcal{S}$. Given such an object $s \in \Gamma(U)$ we must carefully distinguish between

(i) the domain on which it is defined, $\mathrm{Dom}\,(s) = U \subset M$, which is a detachable part of a geometrical extensive whole (the base manifold M), and
(ii) its values $s(x)$, which belong to an intensive space of qualities (the fibre F).

It must be emphasized again that the key concept of section supports a *local/global* dialectic: restriction from global to local and gluing from local to global. The domains of sections correspond to the purely extensional (topological) part of the axiomatics. Their values correspond on the other hand to the truly mereological part.

As the base space M is a manifold, all the concepts of open set, interior part, boundary, closure, etc. are ipso facto well defined.† But the concepts of fibration and sheaf deepen what it is for a section s to be a *constituent* of another section t. There are in fact (at least) *two* meanings of constituency, a weak one and a strong one.

(1) *Weak sense.*
$s \in \Gamma(U)\mathbf{C}t \in \Gamma(V) := U \subset V$ (that is the domains are included one in the other).
(2) *Strong sense.*
$s \in \Gamma(U)\mathbf{C}t \in \Gamma(V) := (U \subset V) \wedge (t|_U = s)$ (that is the sections agree on the included domain).

† In some cases one can generalize the concept of section and define it for non-open subsets of M. But in general "good" sections must share some properties of continuity, differentiability, analyticity, etc. These are all local properties which are well defined only on open sets.

Of course, it is the strong sense which is the most interesting. With it, *overlaps* become *gluing conditions*. More precisely, the gluing condition $s|_{U\cap V} = t|_{U\cap V}$ is a condition of *maximal* overlap (that is of overlapping on $\mathrm{Dom}(s) \cap \mathrm{Dom}(t)$).

The strong sense of constituency is imposed by the **AC2** axiom:

AC2. $x\mathbf{C}y \wedge y\mathbf{C}x \Rightarrow x = y$.

Indeed, if we retain only the weak sense, we must introduce an equivalence relation $x \equiv y := \mathrm{Dom}(x) = \mathrm{Dom}(y)$ and take the modified axiom:

AC2*. $x\mathbf{C}y \wedge y\mathbf{C}x \Rightarrow x \equiv y$.

This shows that **AC2** is by no means evident for the case of dependent parts.

The strong sense of constituency is also imposed by the axioms **AP1** and **AP2a, b** linking the primitives **C** and **P**. Indeed, if $s \in \Gamma(U)$, the unique plausible meaning for $s\mathbf{P}t$, $t \in \Gamma(V)$ is that $U \subset \mathbf{int}(V)$ and $t|_U = s$.† But then **AP1** and **AP2a, b** imply immediately $s\mathbf{C}t$ in the strong sense.

Of course, we can also introduce a third meaning of constituency, a *mixed* one. The objects are now *sets* S of sections and we define:

$$S\mathbf{C}T := \forall s \in S\ \exists t \in T,\ s\mathbf{C}t \text{ in the strong sense.}$$

It is relevant to use the mixed sense if for instance we want to get a good definition of the complement $x' := [z : z\mathbf{D}x]$ of a section s. Indeed, a section t can be discrete from s for two completely different reasons:

(i) $(U = \mathrm{Dom}(s)) \cap (V = \mathrm{Dom}(t)) = \varnothing$;
(ii) $(U = \mathrm{Dom}(s)) \cap (V = \mathrm{Dom}(t)) \neq \varnothing$ but $t(x) \neq s(x)\ \forall x \in U \cap V$.

The complement s' of s is therefore a *set* of sections.

We can even introduce a fourth meaning of constituency if we take as objects *multivalued* sections. In that case a $s \in \Gamma(U)$ is no longer a map $s: U \to E$ lifting the projection π. Instead, it is a map which associates to every point $x \in U$ a *subset* $s(x)$ of the fibre E_x over x. But, even if they can be interesting by themselves, such extensions of the concept of section are somewhat artificial. For a section, the natural meaning of being a constituent is to be a restriction of some larger section.

Now, the main point is that the mereological concept of *sum* (union and fusion), *splits* into two different concepts.

(1) The *union* of any two sections $s \in \Gamma(U)$ and $t \in \Gamma(V)$ can be defined as the section $s \cup t \in \Gamma(U \cup V)$ such that $s \cup t(x) = \{s(x), t(x)\}$ (we take $s(x)$, $t(x) = \varnothing$ if $x \notin U, V$). If $s(x) \neq t(x)$ for $x \in U \cap V$ then $s \cup t$ is a multivalued section.
(2) The *fusion* of two sections $s \in \Gamma(U)$ and $t \in \Gamma(V)$ is more restrictive. It requires the gluing condition $s|_{U\cap V} = t|_{U\cap V}$. The fundamental consequence is that, if $\varphi(s)$ is a predicate of sections, the sum $[s : \varphi(s)]$ is no longer a *single* element. It is the set of all *maximal* sections satisfying φ.

In some cases, we can also suppose that there exists some algebraic "superposition" structure $u * v$ in the fibre F (think of the superposition of colours). We can then define the union of any two sections $s \in \Gamma(U)$ and $t \in \Gamma(U)$ as the section

† Of course **int** means here the topological interior. In general, V will be open and therefore $\mathbf{int}(V) = V$.

$r \in \Gamma(U \cap V)$ s.t. for every $x \in U \cup V$, $r(x) = s(x)*t(x)$. But this is no longer a true union. It is rather a "sum" in an algebraic sense.

The concept of sum has to be redefined according to the fact that in sheaf theory the concept of union (of gluing) depends on the concept of *prolongation* of a section. Let φ be a predicate for sections. Let us use the term φ-section for a section satisfying φ. We look at the possibility of *extending* a φ-section $s \in \Gamma(U)$ to a larger open set $U \subset V$. We look therefore at sections $t \in \Gamma(V)$ s.t. $U \subset V \wedge s\mathbf{C}t \wedge \varphi(t)$. t is a maximal φ-section if:

$$\forall r(\varphi(r) \wedge t\mathbf{C}r \Rightarrow r = t).$$

A maximal φ-section t satisfies the matrix of **DC4**:

$$\forall w(w\mathbf{O}t \Leftrightarrow \exists v(\varphi(v) \wedge w\mathbf{O}v)),$$

but this is no longer a definite description. Let $[x:\varphi(x)] = \Gamma_\varphi(M)$ be the *set* of maximal φ-sections. The **TC3** theorem becomes:

$$\forall x(\varphi(x) \Rightarrow \exists z \in [x:\varphi(x)](x\mathbf{C}z)).$$

The universe is no longer a single element. It is the set $\Gamma_m(M)$ of maximal sections. We have of course $\Gamma(M) \subset \Gamma_m(M)$: global sections are maximal ones.

As concerns the topological part of Smith's system, too, the sheaf model permits also to solve some difficulties. The **AP1**, **AP2a, b** and **AP3** axioms are trivially satisfied. The **AP4** axiom $\forall x(\varphi(x) \Rightarrow x\mathbf{P}y) \Rightarrow [x:\varphi(x)]\mathbf{P}y$ is also evident. Indeed, if all the φ-sections x are interior parts of *one* single section y, then the set $[x:\varphi(x)]$ of maximal φ-sections is reduced to one element x_m and $x_m\mathbf{P}y$. But, on the other hand, the "very strong" **AP5** axiom $\exists y(x\mathbf{P}y)$ will not be satisfied in general, not because M cannot be an interior part of itself (it is an axiom for any topology that the global space is always clopen, that is close and open) but because there exist in general *many* global sections $t \in \Gamma(M)$, and it is therefore possible that different sections can be prolongated to *different* maximal sections.

The definition **DP6** of the interior of an object, $\mathbf{int}(x) := [y:y\mathbf{P}x]$, does not raise any problems. Let $t \in \Gamma(V)$. The sum $[s:s\mathbf{P}t]$ of the s such that $s\mathbf{P}t$ is a singleton $\{\mathbf{int}(t)\}$ with $\mathbf{int}(t) := t|_{\mathbf{int}(V)}$.† But the situation is not as straightforward for the definition **DP4** of the closure of an object, $\mathbf{cl}(x) := x \cup [y:y\mathbf{B}x]$.

In fact, we meet here a very delicate point. As we have already stressed, all the classical topological concepts (interior, closure, boundary, etc.) are at hand in the sheaf model because the base space M is a manifold. But, as we have seen, the topological basis of a sheaf of sections constitute only one half of the structure. The other half is constituted by the values of the sections. And there is a worrying problem concerning the extension of topological concepts to this last level. Indeed, if we impose the constraints of continuity, or differentiability, or analyticity, etc. on sections, then it is a well known fact that the problem of extending sections to the boundary of their domain is an extremely difficult one, and that, in general, it is even without solution.

Let us evoke briefly only one example concerning the theory of holomorphic dynamical systems, and in particular the (filled connected) Julia sets and the

† In general the domain V of t will be open, and therefore $\mathbf{int}(t) = t$.

Mandelbrot set which have become so popular as typical "beautiful" fractals.† They give examples of what can be an infinitely complex compact, connected, closed set in $\mathbb{C} = \mathbb{R}^2$, the complement of which is connected and simply connected in the Riemann sphere $\mathbb{C} \cup \{\infty\}$, and the interior of which is constituted by infinitely many open discs of different scales. Let K be such a closed set. A well known theorem of Riemann says that there exists a conformal map ψ from $\mathbb{C} - K$ to $\mathbb{C} - \Delta$ (where Δ is the closed unit disc). On the other hand, a deep theorem due to Caratheodory says that the inverse ψ^{-1} of ψ can be prolongated *continuously*—but *not* holomorphically in general—to the boundary $\partial\Delta = S^1$ of Δ. But in general ψ cannot be prolongated, even continuously, to the boundary ∂K of K. In short, it is in general difficult, even impossible, to extend maps defined on open sets to the boundary of their domain.

The definition **DP4**, $\mathbf{cl}\,(x) := x \cup [y : y\mathbf{B}x]$, is therefore problematic for sections in the sheaf model. In fact, it is meaningful only if the sheaf model is endowed with more restrictive structures (differentiable, holomorphic, etc.) than the simple topological one.

We can therefore conclude that the mereology of sections—in the sheaf model which axiomatizes Husserl's pure eidetic description—shows that some mereological axioms are "evident" only for purely *extensional* mereology, and are by no means "evident" for the more sophisticated sort of (non-extensional) mereological axiomatics.

8. Conclusive remark

It is a well known fact that the category of sheaves defined on a base space constitutes what is called a *topos* and that every topos possesses an internal (intuitionist) logical language.‡ This internal logic provides a natural formal language for the phenomenon of *Überdeckung*. This point will be further developed in another work (Petitot, in press).

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† For a brief introduction to these technical topics, see Petitot (1992*a*).

‡ For details concerning the deep links relating the theories of sheaves and topoi to logic—links discovered by W. Lawvere and M. Tierney in the 1960s—see Mac Lane and Moerdijk (1992) and Petitot (1979, 1982).

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