“Weak” objectivity and relativity in Kant’s Phoronomy. A groupoid and functorial approach

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1 Introduction

In philosophy of physics, the difficulties of a realist ontology are particularly acute in quantum mechanics (QM). Even if realism is the spontaneous philosophy of most physicists, many philosophers consider that quantum objectivity is a "weak" objectivity restricted to measured observables and cannot be interpreted in terms of the ontology of a *mind-independent* – i.e. *transcendent* – substantial reality: physical theories imply themselves that physical objectivity is restricted to informations retrieved from measured observables and cannot be "ontologized". The literature on this subject is extensive.

However, it is widely accepted that these issues are specific to QM, have no equivalent in classical mechanics (CM) and that there are therefore no obstructions to realist interpretations of CM. As Brigitte Falkenburg explains in [17], this is mainly due to the fact that in CM, since there is no incompatibility between observables, a *complete determination* of objects is possible, while it is not the case in QM. But complete determination of objects does not justify in any way an ontological realism.

In this investigation, we would argue that, contrary to this common wisdom, in CM, objectivity is already a "weak" objectivity and that many problems of a "non ontologizable" objectivity are already present.

The point is that every time a physical theory has what is called a "background structure" with an associated relativity group G, it can concern only a "weak" form of objectivity since only G-invariant entities can have a well-defined physical content. Non G-invariant entities are *contextual* with respect to G and cannot refer to an independent reality. So if some of them are constitutive of the objects, then the objectivity is necessarily "weak". There is also a large litterature on symmetries in physics. See e.g. the book [9] edited in 2003 by Katherine Brading and Elena Castellani.

It is possible to accommodate relativity and symmetries in physics with some sort of realism if we shift from ontological realism to *structural* realism from Poincaré, Eddington, Weyl to John Worrall [67] or Steven French and James Ladyman [18], [39] (see e.g. Holger Lyre's [42]).

Our purpose here is to come back to the first philosophical thematization of this key issue.

In the 1970s, when I was a young researcher, I have been strongly impressed by Jules Vuillemin's *Physique et métaphysique kantiennes* [64] claiming that in physics "transcendental" means "relativity and symmetry groups". Vuillemin analyzed Kant's *Metaphysische Anfangsgründe der Naturwissenschaft* [32] (*Metaphysical Foundations of Natural Science*: *MFNS*) which fits very well with Newtonian mechanics. It was an illumination. Of course, I already knew that Kant was the first philosopher, long before his first *Critique*, to draw the metaphysical dramatic conclusions from the existence of *non conceptual* symmetries, congruences, and equivalence relations in the definition of objects of knowledge. But quite all philosophers focused on the *Critique of Pure Reason* while the key book was the *MFNS*. Since then, thanks to Vuillemin, the *MFNS* became my bible for philosophy of physics. I worked a lot on them and reached the conclusion that, in spite of the devastating critiques raised against it, the transcendental approach remains, largely beyond Kant, highly relevant in philosophy of physics. I developed this hypothesis in many papers. A synthetic summary can be found (in French) in *La Philosophie transcendantale et le problème de l'Objectivité* [50] (1991) and (in English) in "Actuality of Transcendental Aesthetics for Modern Physics" [51] (1992). Further developements can be found in other papers cited in the bibliography, in particular in my tribute to Bernard d'Espagnat "Objectivité faible et philosophie transcendentale" [54] (1997).[1]

Transcendental epistemology for physics can be considered as a precursor of structural realism. Its interest is that, at the time of Kant's *MFNS*, as the concept of a mathematical structure did not exist, the foundational problems of mathematical physics had to be metaphysically formulated.

Before getting into the substance of these issues, we must strongly emphasize the fact that for us transcendentalism is, as are empiricism or realism, a type of epistemological perspective and is in no way reducible to Kant, as empiricism is not reducible to Hume or realism to Plato. For us, transcendentalism can be generalized and historicized and applied to any domain of reality. Of course, Kantism must be updated, but, as emphasized by Bernard d'Espagnat:

> "It turns out that that quantum physics as well as the outcomes of recent physical experiments yield strong support to two of [Kantism's] most essential features, the ideality of space (or space-time as now we would preferably say) and the (correlated) fact that, far from being independently existing out there, phenomena are essentially representations in our mind." ([16], 481).

2 Generalizing and "historicizing" transcendentalism

2.1 Conceptual analysis ans computational synthesis

The difference between "weak" (i.e. transcendental) objectivity and "strong" (i.e. transcendent) ontology is quite easy to understand. Since physical theories are construed on the basis of (i) categorial concepts such as "system",

[1] Since I met him for the first time in 1969, Bernard d'Espagnat has been one of my main masters in philosophy of QM. In 2009 he founded the *Collège de Physique et Philosophie*. He asked Michel Bitbol, Hervé Zwirn and myself to be his cofounders and we organized with him seminars at the Academy (Institut de France). See the site http://www.cphi2.org.

"state", "observable", "conservation", "causality", "interaction", etc., and (ii) geometric and dynamical intuitions such as space, time or motion, the categorial concepts have to be *mathematically interpreted* in a *specific* defined way using the geometro-dynamical intuitions in order to constitute a well-behaved physical objectivity. Transcendental philosophy has essentially to do with such interpretations. Categorial concepts can change and evolve, and also intuitions. So transcendentalism can be applied to many different domains of reality, far beyond CM as Kant did in the *MFNS*. Now, if we look at the history of modern physics, we find that the categorial concepts remained fairly stable while their mathematical interpretation changed dramatically. But this evolution is by no means an argument against transcendentalism.

In fact, a generalized and "historicized" transcendental perspective on modern physics can be based on very general principles.

1. Physics deals only with phenomena, and phenomena are *relational* entities that are inseparable from their conditions of observation. Therefore, access conditions (observation, measurement, retrieving information, etc.) are constitutive of the very concept of a physical object. It is for this very reason that physical objectivity cannot be the ontology of a mind-independent, transcendent and substantial, reality.

2. But even if they lose any "strong" ontological content and are contextually dependent upon their access conditions, observable phenomena must be converted into objective entities, that is, as Kant said, into determined "objects of experience". For that, categorial concepts must be applied to them. Indeed, in order to be transformed into true scientific objects, phenomena have to be "constituted", that is lawfully "legalized". In the *MFNS*, Kant explained for the first time how categories and principles (he organized in four groups) specialize in Newtonian Mechanics into Phoronomy (Kinematics), Dynamics, Mechanics, and Phenomenology.

3. *Constitution* is the core of a transcendental approach to objectivity. As was strongly emphasized by many philosophers, e.g. Peter Mittelstaedt, to be "carriers of observable properties", objects have to be constituted as objects, that is be made independent from their subjective part as phenomena. For empiricism objects are "chimera", for ontological realism objects are *given* as mind-independent, for transcendentalism, objects are constituted as observer-independent. (See Mittelstaedt [46])

4. Let us be more precise. The essential feature of physics is the existence of mathematical interpretations which transform the categorial concepts into *algorithms* for a mathematical *reconstruction* of observed phenomena. This is a critical point. Until recently, physics has been the only science able to solve the *inverse problem* of the abstraction problem and to supplement conceptual analysis with a *computational synthesis* of phenomena. In Kant, computational synthesis is first based on the *schematization* and then on what he called the "*construction*" (*Konstruktion*) of categories. It is here that *Transcendental Aesthetics* (T.Ae.) comes on stage.

So, the critical difficulty for generalizing transcendentalism is to generalize classical T.Ae., and for that, it is necessary to understand its general function. Let us come back once again to Kant. T.Ae. presents two aspects corresponding to what Kant called two "expositions" (*Erörterung* = "clear representation of what belongs to a concept") in the *Kritik der reinen Vernunft* (*Critique of Pure Reason*: *CPR*): the metaphysical one and the transcendental one. First, phenomena are observable and therefore must appear to an observer. They appear in a specific medium of manifestation (space and time for sensible phenomena) which provides "forms of intuition". Second, these "forms" can be mathematically determined and converted into what Kant called "formal intuitions" (see the celebrated footnote to §26 of the Transcendental Deduction in the *CPR*). To determine phenomena as objects of experience, we need therefore a link between mathematically determined forms of observability (what is "gegeben") and categorial forms of lawfulness (what is "gedacht"). In Kant this link is worked out at two levels. At the level of the *CPR* it is provided, as we already seen, by transcendental schematism which converts the categories into principles (*Grundsätze*). At the level of the *MFNS*, it is provided by the construction of categories. Construction (*Konstruktion*) is a mode of presentation (*Darstellung*). It means that it is possible to interpret mathematically the schematized categorial contents by using mathematics stemming from the transcendental exposition of T.Ae. I think that it is in this very special sort of "*mathematical hermeneutics*" – not only for the intuitive forms of manifestation but also for the categorial forms of lawfulness themselves – that the *synthetic a priori* finds its true function and its deep transcendental meaning. Indeed, the relevant question is not to know if there can exist synthetic a priori *judgments*, but to know if there exists a *function* of synthetic a priori in modern physics. We think that the response is positive: background structures, relativity groups and symmetries are synthetic a priori components of physical theories.

2.2 The *MFNS*

Let us remind very briefly (in modern terms) the four chapters of CM according to the *MFNS*.[2]

1. **Phoronomy (Kinematics).** "Mathematical" categories of quantity and "Axioms of Intuition" ("*Axiomen der Anschauung*") governing "extensive" magnitudes. The Euclidean metric of space is a background (a priori) geometrical and arithmetic structure enabling numerical measures of physical motions. These motions must comply with the Galilean relativity of inertial frames,[3] which manifests that phenomena are not, as such, objects of experience. Moreover, trajectories of material bodies have to

[2] See Petitot [51]. For a recent extremely detailed analysis of the *MFNS*, see Michael Friedman's monumental opus *Kant's Construction of Nature* [22]. For another commentaries, see e.g. Kerszberg [35].

[3] We will see later in sections 3.7 and 4.1 that Galilean relativity is not completely achieved in Phoronomy.

be described by differential entities (velocities, accelerations, etc.) varying covariantly (link with relativity). Physics must be a kind of differential geometry and not a "logic" in the traditional sense. Motions can be "*composed*" and their composition can be computed using an *algebraic* structure (a vector space structure, in modern terms). Instantaneous states of motions are described by position and velocity which are *both relative*. So the basic concept of a state of motion is contextual.

2. **Dynamics.** "Mathematical" categories of quality and "Anticipations of Perception" ("*Anticipationen der Wahrnehmung*") governing "intensive" magnitudes. Matter is distributed in space and occupies it with different densities. There exists a universal force of attraction (gravitation) but, to balance gravitational collapse, there must also exist repulsive forces inside matter. Attraction was already quite well understood at Kant's time, but internal forces of repulsion remained a mystery (Kant speculated deeply on them in the *Opus Postumum* [33]). And they remained a mystery until the advent of QM and quantum principles such as Pauli exclusion principle for fermions. For Mechanics, matter can be reduced to *mass* (a scalar number) and inertial mass is the same as the gravitational mass.

3. **Mechanics.** "Dynamical", i.e. physical, categories of relation (substance = *Inhärenz und Subsistenz*, causality = *Causalität und Dependenz*, community, coexistence, reciprocity and interaction = *Gemeinschaft*) and "Analogies of Experience" ("*Analogien der Erfahrung*"). The category of substance (schematized as transcendental principle of temporal permanence: *die Beharrlichkeit des Realen in der Zeit*) is reinterpreted as the transcendental principle of conservation laws; the category of causality as that of forces (Newton law); and the category of community as that of interactions driven by the principle of equality between action and reaction. Mechanics is the core of the determination of phenomenal motions into objects of experience.

4. **Phenomenology.** Categories of modality and "Postulates of empirical thought in general" ("*Postulate des empirischen Denkens überhaupt*"). Because of relativity, motion cannot be a real (*wirklich*) but only a "possible"(*möglich*) predicate of matter (it is a purely relational phenomenon). Position and velocity are not observable properties whose values could individuate objective physical systems. The sentence "The body S "has" such position and such velocity" (in the sense of "having a property") is not a physical judgment. We find here the root of the transcendental ideality of space and time. But forces (causality) are real and determine real properties of objects of experience which are governed by necessary laws. Necessity is not here a logical but a transcendental modality. It is conditional, relative to the radical contingency of possible experience.

2.3 Galoisian physics

As we already pointed out, a fundamental component of the transcendental structure of physical theories is provided by *symmetries*. In general relativity and abelian or non abelian gauge theories, the radical enlargment of the symmetry groups does not nullify but support transcendentalism.They are far-reaching manifestations of what Daniel Bennequin [6] called the "Galoisian" essence of modern physics. Exactly as the Galois group of an algebraic equation (that is the group expressing the degree of indiscernibility between solutions) is the main tool for determining the solutions themselves, symmetries that express entities which cannot be physical observables are the main tool for determining the physical observables themselves. As claimed Jean-Marie Souriau in [62]:

> "there is nothing more in physical theories than symmetry groups, except precisely the mathematical construction that allows to prove that there is nothing more."

This Galoisian essence of physics increased in the successive theories. For instance, in CM, the Euclidean metric of space-time and the Galilean group act as background structures. In General Relativity (GR), the metric is no longer a background structure and becomes a physical component of the theory. The relativity group is the group $\mathrm{Diff}(\mathcal{S})$ of diffeomorphisms of space-time $\mathcal{S}$ and the $\mathrm{Diff}(\mathcal{S})$-invariance implies that localization becomes relational so that points lack any physical content. But this so-called "background independence" is not a refutation of transcendentalism. It means that in GR the background structure is the *differentiable* structure (and no longer the metrical structure) of space-time. So T.Ae. corresponds no longer to the Euclidean level of geometry but to the differentiable level.

With gauge theories, the role of symmetries became even more constitutive. A specialist as Michio Kaku can claim:

> "The secret of this mystery [the mystery of unified theories] most likely lies in the power of gauge symmetry. (...) Nature demands symmetry. (...) Symmetry, instead of being a purely aesthetic feature of a particular model, now becomes its most important feature." ([31], p. 8).

In [51] (1992) we gave for the first time a transcendental approach of:

1. Hamiltonian (symplectic) mechanics, in particular Noether theorem and the formalism of the momentum map worked out by Bertram Kostant [38], Jean-Marie Souriau [62], Aleksandr Kirillov [36], Vladimir Arnold [2], Alan Weinstein [65], Ralph Abraham and Jerrold Marsden [1]. With this formalism it becomes possible to deduce directly first integrals from relativity groups even without specifying any Lagrangian.

2. General relativity and the a priori determination of Einstein equations proposed by John Archibald Wheeler, Charles Misner, and Kip Thorne in their *Geometrodynamics* [45] using the cohomology of differential forms.

3. Quantum field theory, non abelian gauge theories (see e.g. Quigg [59] for an introduction) and superstrings theory (see e.g. Kaku [31] for an introduction).

4. Later we gave also a transcendental approach of Alain Connes' articulation between general relativity and the Standard Model of quantum field theory using noncommutative geometry. See [57].

It turned out later that this perspective shared many theses with Michael Friedman's *Dynamics of Reason* [20] (1999).

(i) The development of modern physics does not invalidate the transcendental constitutive perspective:

> "We still need superordinate and highly mathematical first principles in physics – principles that must be injected into our experience of nature before such experience can teach us anything at all." (p. 14)

(ii) The conditions of possibility of physical theories (a priori synthetic principles of coordination) are not analytic logical judgments.

(iii) Kant's a priori principles can be generalized, relativized and historicized:

> "What we end up with (...) is thus a relativized and dynamical conception of a priori mathematical-physical principles, which change and develop along with the development of the mathematical and physical sciences themselves, but which nevertheless retain the characteristically Kantian constitutive function of making the empirical natural knowledge thereby structured and framed by such principles first possible." (p. 31);

(iv) The central role of constitutive principles:

> "What characterizes the distinguished elements of our theories is rather their special constitutive function: the function of making the precise mathematical formulation and empirical application of the theories in question first possible." (p. 40)

There are nowadays a lot of converging works of many authors on the actuality of such approaches. For a collection of papers see the book *Constituting Objectivity. Transcendental Perspectives in Modern Physics* [8] (2009) we edited with Michel Bitbol and Pierre Kerszberg. The convergence is particularly strong with philosophers of physics as Michael Friedman, Tom Ryckman, Bas van Fraassen, and those of the "Stanford School" as the late Pat Suppes or Nancy Cartwright. We share the same interest for model diversity, mathematical specificity, practical pluralism, and especially the same critique of logical generality in philosophy of science. As liked to claim Pat Suppes, "we need mathematics, not meta-mathematics".[4]

[4] From September 2013 to June 2014, I had the privilege of being invited as a fellow by the Stanford Humanities Center. I had many opportunities of attending seminars and discussing

2.4 Transcendental physics and Lagrangian formalism

The a priori structure of CM becomes evident in the Lagrangian (and Hamiltonian) formulation, that is in the framework of *symplectic geometry* (for some informations see, e.g., Abraham-Marsden [1], Arnold [2], Kostant [38], Souriau [62], Weinstein [65]). In the variational approach founded on the least action principle introduced by Joseph-Louis Lagrange in his 1788 *Mécanique analytique* [40], Newton's equation is equivalent to minimizing a Lagrangian $\mathcal{L} = T - U$ where T = kinetic energy and U = potential energy, e.g. for gravitation $\mathcal{L} = T - U = \frac{1}{2}mv^2 - \frac{k}{r}$. Motions minimize the action $\int_{t_1}^{t_2} \mathcal{L}dt$ and the background structure (Kant's "a priori") is simply what is not varied in the Lagrangian, in particular the metric structure of space. Contrariwise, in the Hilbert variational formulation of GR, the Riemannian metric of space-time is precisely what is varied.

Indeed, let $\mathcal{S}$ be space-time endowed with a locally Minkowskian metric $g_{\mu\nu}$. If e_α is a basis of the tangent space $T_x\mathcal{S}$ and if ω^α is the dual basis of $T_x^*\mathcal{S}$, the Riemann curvature tensor $\mathcal{R}$ is given by $R^\alpha{}_{\beta\gamma\delta} = \langle \omega^\alpha, [\nabla_\gamma, \nabla_\delta] e_\beta \rangle$ (where ∇ is the covariant derivative and, for $\alpha \in T_x^*\mathcal{S}$ and $v \in T_x\mathcal{S}$, $\langle \alpha, v \rangle = \alpha(v)$). For $\alpha \in T_x^*\mathcal{S}$ and $u, v, w \in T_x\mathcal{S}$, $\mathcal{R}(\alpha, u, v, w) = \langle \alpha, R(v, w)u \rangle$ with $R(v, w) = [\nabla_v, \nabla_w] - \nabla_{[v,w]}$. By contraction, we get the Ricci curvature tensor $R_{\mu\nu} = R_\mu{}^\alpha{}_{\alpha\nu}$. By a second contraction, we get the scalar curvature $R = R_\mu{}^\mu$. Einstein curvature tensor is then given by $G = \text{Ricci} - \frac{1}{2}gR$. It satisfies the contracted Bianchi identities $\nabla G \equiv 0$.

Now, as was shown by Hilbert in 1915 [27] and developed later by Richard Arnowitt, Stanley Deser, Charles Misner in 1962 [3], it is possible to derive the metric $g_{\mu\nu}$ from a variational principle using the scalar curvature R as Lagrangian density. The action is then:

$$S = \frac{1}{16\pi} \int R \sqrt{|g|} d^4x$$

where $|g|$ is the determinant of the metric tensor g and we see that the Lagrangian contains only terms derived from g. But let us again emphasize that the differentiable structure of space-time remains an a priori background structure since it is not involved in the Lagrangian.

2.5 Noether theorem

We have often explained (see e.g. [51], [52], [54]) that in Lagrangian Mechanics the transcendental relation between Relativity and Conservation laws is *internalized* into the theory. It derives no longer from a metaphysical principle but from a mathematical theorem, namely Noether theorem, which expresses that the symmetries of the Lagrangian are constitutive of the physical objectivity of the objects and that physical theories are therefore effectively "Galoisian".

with Michael Friedman, Tom Ryckman and Pat Suppes. It is a pleasure to acknowledge the SHC, its Director Caroline Winterer and all its staff.

Noether theorem says that to each Lagrangian symmetry expressing the unobservability of a certain entity is correlated a law of conservation allowing the measure of a correlated quantity. The most classical examples are collected in the following table:

Relativity (unobservable entities)	**Symmetries**	**Conservation laws (measurable quantities)**
Origin of time	Time translations	Energy
Origin of space	Space translations	Momentum
Priviledged direction	Space rotations	Angular momentum
Priviledged velocity	Galilean boosts (inertial frames)	Initial position of the center of mass

3 Phoronomy in the *MFNS*

3.1 Motions and relative spaces

Let us try to go to the root of the metaphysical problem of relativity, even if it is a difficult task since we have forgotten the foundational difficulties of classical mechanics. For that, we come back to Kant's *Phoronomy* (Kinematics), which explains why physical objectivity departs from ontology. The inconsistency of the concept of an "absolute" space (absolute space is only a "regulative Idea") implies that motions of material bodies are relative to relative "spaces". These relative spaces have two characteristics: (i) they are frames, and (ii) they are bounded domains of different sizes, which can be nested one into the other. We could say that they are "local" frames. Here, we will consider them only as frames.

Phenomenality (i.e. relativity) of motion (which is different from objectivity since "objective" means invariance w.r.t. relativity) is therefore characterized by the transformations of particular frames, namely *inertial* frames corresponding to linear uniform motions (LUM). The point is that the realist concept of "absolute" space is superseded by an infinite equivalence class of inertial frames.

Relativity was already well-known in Kant's time since it had been introduced by Galilei long before his 1638 *Discorsi* [23]. But Kant was the first philosopher to draw the ultimate metaphysical consequences. Due to relativity, neither "natural" (Aristotelian) places nor changes of places (velocities) can be *attached* to bodies.

3.2 Changes of inertial frames and Kant's "Principle" (*Grundsatz*)

Let us be a little bit more precise. According to Explication (*Erklärung*) 1,

> "*Matter* is the *movable* in space. That space which is itself movable is called material, or also *relative space*." (480/15)[5]

[5] Usually we work on the French translation of the *Anfangsgründe* by François de Gandt

In Phoronomy, only the general property of movability is taken into account,[6] and movability is an *empirical* concept "given through experience". It must be determined "in accordance with a priori principles" (482/17, Remark 2). Material bodies and relative spaces are *both* movable and the phenomena under scrutiny in Phoronomy are pure motions as changes of places. Therefore material bodies can be reduced to points (the positions of their center of mass) and no other properties than the motions of these positions are considered.

In Phoronomy, the issue is to determine the basic phenomenon of motion as "quantity in motion (speed [*Geschwindigkeit*] and direction [*Richtung*])". We must be able to associate to an oberved motion a mathematical entity which, in modern terms, is of the type $v = |v| u$ where $|v|$ is a positive scalar (a *quantitas*) and u the unit vector of a direction D on a line L. Such a measurable entity of a certain type is called by Kant a *quantum* whose measure yields a *quantitas*. It is the first step of the "*construction*" of the concept of motion: *to associate to it a type of mathematical entity*. We can say that motion as *quantum* is Kant's anticipation of what will be called later a vector.

Of course, as is emphasized in Remark 2, the key point is that *absolute* space is antinomic and cannot be introduced in physics. "Absolute space is (...) *in itself* nothing, and no object at all." (481/16). Kant heavily stressed that: "an absolute motion (...) cannot be experienced at all" (487/23), "absolute space is nothing for all possible experience" (488/23). An absolute space can be thought of as an idea but not *given*, the difference between *gedacht* and *gegeben* being crucial. It is an Idea of Reason, an ideal limit of nested relative spaces.[7] All given spaces are relative, that is moving and never absolutely at rest. At the end of the *MFNS*, in his *General Remark to Phenomenology*, Kant develops this thesis:

> "[The concept of absolute space] cannot be an object of experience (...) and yet it is a necessary concept of reason, and thus nothing more than a mere *idea*." (559/98).

Of course, far beyond Phoronomy, if we look at all the gravitational interactions between all bodies in the universe, we can choose a special *cosmological* inertial frame attached to the center of mass of the universe and acting as a surrogate of the absolute space. But this has no sense at the phoronomical level.

In Explication 2, "motion of a thing" is defined as "the *change of its outer relations* to a given space." (482/17) and, since no internal motions of bodies is taken into account, motions can be reduced to changes of places, i.e. of positions of points. The core of Phoronomy concerns *inertial* motions of constant velocity $C = S/T$ (where C means "celeritas", S "spatium" and T "tempus"), that

(Pléiade edition, Gallimard, Paris, 1985). Here we will use Friedman's English translation and refer to pages by first the page of the German edition and second the English version.

[6] "In phoronomy nothing is to be at issue except motion" (480/15, Remark 1)

[7] From Descartes to Leibniz and Newton, the metaphysics of space includes also a theological component (space as *sensorium dei*). For a discussion of this aspect, see e.g. Friedman [21].

is the boost part of the Galilean group we will introduce later in section 4.2.[8] So velocity is an intensive quantity which is in some sense extensive-like: "In phoronomy we use the word 'speed' purely in a spatial meaning $C = \frac{S}{T}$." (Remark 3, 484/20).[9] This remark is extremely important since, if we put $T = 1$, we get $C = S$,which means in modern terms that, even if the dimensions $[C]$ and $[S]$ are different, we can identify a velocity vector v with a vector connecting two points (a, b) in the affine space, by putting $b - a = v$.

So, Phoronomy concerns the a priori framework (the background structure) of motions of material points. The properties of velocities as vectors (module, direction) were already well known at Kant's time, even if the general structure of a vector space was of course not thematized as such. The composition of velocities using the parallelogram rule was common knowledge for mathematicians and physicists, as well as the multiplication of a velocity by a scalar number. But if there exist no absolute space and if positions and velocities (i.e. the variables which define the instantaneous state of a material point) are therefore relative and not objective properties of the material points, *they cannot be attached to these points*. Of course, we can attach to a body K a moving frame F_K, but w.r.t. this frame the body is by definition at rest. It has a relative velocity w.r.t. any inertial frame F and, precisely because this velocity is relative, it cannot be attached to K. So how to compose velocities? We will see that Kant had to challenge an highly non trivial problem, which is for velocities the same problem as the difference (whose subtlety is well known by secondary school students) between *affine* space and *vector* space for positions. In an affine space, points a *cannot* be composed; only classes of *congruent bipoints* (a, b) can be composed as vectors.

Kant had to understand that it was the same for velocities. To overcome this obstacle his idea was to look first at changes of inertial frames, because they are the most basic entities, more basic than the motions themselves (inertial frames are a priori space structures while motions are empirical phenomena), and then to compose velocities using changes of frames.

Composition of motions in a composite motion (*Zusammengesetzen Bewegung*) is the possibility of correlating (i) a composition of changes of positions and frames, and (ii) an algebraic composition of velocities as entities of type $v = |v| u$. This correlation constitutes the main step of the "*construction*" of the concept of motion. Explication 4 says that

> "To construct the concept of a composite motion means to present a motion a priori in intuition, insofar as it arises from two or more

[8] In Phoronomy Kant discussed also non inertial motions, such as non uniform (accelerated) rectilinear motions of circular motions. But such motions have mechanical causes and are not purely phoronomic. They are nevertheless very different because along a rectilinear motion the acceleration is due to the change of $|v|$ (*Geschwindigkeit*) and not to the geometry (straight lines are Euclidean geodesics) while along a curvilinear motion an acceleration is always due to the geometrical change of direction (*Richtung*) even if $|v|$ is constant.

[9] What is called in physics a *dimension* was already clear since a long time: if $v = |v| u$ is a velocity (celeritas) its dimension $[C]$ is $[S] [T]^{-1}$.

given motions united in the movable." (486/22).[10]

And Explication 5 says that

> "The *composition of motion* is the representation of the motion of a point as the same as two or more motions of [this point] combined together." (489/24).

Composition is a "union", a "combination". But it must be mathematical since "Phoronomy is (...) the pure theory of quantity (*mathesis*) of motions." (489/25). Phoronomy is a *Größenlehre*. So we must derive a mathematical (algebraic) structure (quite in the modern Bourbakist sense) from the core phenomenon of movability and, as there cannot exist any pre-established harmony between empirical phenomena and mathematical operations, we need an intermediate level, namely the categories. It is the application of the *categories of quantity* to motions which enables their constructive mathesis. Using modern terms, we could say that in Phoronomy the transcendental construction of motion as a magnitude concerns the possibility of converting the diversity of inertial motions into an algebraic structure:

> "Phoronomy has first to determine the construction of motions in general as *quantities* (...) with respect to both their speed and direction, and, indeed, with respect to their composition." (487/22, Remark to Explication 4).

The first step is the *Principle* asserting the relativity of the concept of motion:

> "**Principle**. Every motion, as object of possible experience, can be viewed arbitrarily as motion of the body in space at rest, or else as rest of the body, and, instead, as motion of the space in the opposite direction with the same speed." (487/23)

Kant formulates things in natural language with quite no formulas. To clarify his analysis we will use some simple symbolic notations. Let F be an inertial frame and K a body (a material point) with velocity $v = v_{K/F}$ w.r.t. F. v corresponds to K moving while F is at rest. To consider K at rest, is to consider a frame F_K attached to K. Let us denote v_{F/F_K} by $v_{F/K}$. The Principle says that

$$v_{F/K} = -v_{K/F}$$

This formula seems completely trivial since we consider today that there is no problem in considering motions as vectors. But, at a foundational level in Kant's time, it was by no means trivial. Indeed it synthesizes two completely different things which become very clear if we shift from Kant's rather complicated descriptions in natural language to symbolic notations.

[10] Explication 3 concerns the interpretation of rest when all relative spaces are movable and there is no absolute rest.

1. The notation K/F symbolizes the contextual relation of K w.r.t. the frame F. Such relations can be composed by concatenation. Let us denoted it by $*$. We get

$$(F/K) * (K/F) = F/F = Id_F \text{ (the identical change of } F\text{)}$$

2. On the other hand, there is an *algebraic* composition law on the v which is an *additive* structure.

3. Finally, there is a compatibility between the two compositions. Indeed, let us write $v_{F/K} = -v_{K/F}$ as $v_{F/K} + v_{K/F} = 0$. Then we get

$$v_{F/K} + v_{K/F} = v_{(F/K)*(K/F)} = v_{F/F} = v_{Id_F} = 0$$

So we see that the Principle describes conceptually a correlation between transformations of frames and some system of numbers quantifying them: the composition $*$ of transformations is interpreted as the additive law of composition $+$ and $F/F = Id_F$ (F is at rest w.r.t. itself) is interpreted by 0. We note that all the *different* Id_F are interpreted by the *same* number 0, which means that algebraization involves a quotient by an equivalence relation.

The non triviality of the categorico-algebraic construction of motions as magnitudes (mathesis) is particularly evident when we consider the meaning of the *equality* in $v_{F/K} = -v_{K/F}$. Kant is very clear concerning what means here "to be the same", that is what means *phoronomic identity*: the two cases "K moving and F at rest" and "K at rest and F moving" are "completely the same for all experience, and every consequence of experience." (488/23). Since there is no absolute space, the two cases are the same.

> "For any concept is entirely the same as a concept whose differences from it have no possible example at all, being only different with respect to the connection we wish to give it in the understanding." (488/23).

So *algebraic equality means physical indiscernibility*: as we say today, inertial motions cannot be detected by physical measures. And what is the case for velocities is also the case for positions: in modern terms space is an *affine* space.

> "We are also incapable, in any experience at all, of assigning a fixed point in relation to which it would be determined what motion and rest are to be absolutely." (488/23-24).

Kant go further at the end of the Remark to Explication 4. In fact velocity can be arbitrarily distributed between the body K and the frame F. At the algebraic level this corresponds to the case where $v_{K/F} = v_1 + v_2$. We introduce an intermediary inertial frame F' whose velocity $v_{K/F'}$ has the same direction as $v_{K/F}$ and we take $v_{K/F'} = v_1$ and $v_{F'/F} = v_2$.

"In phoronomy (...) it is completely undetermined and arbitrary how much speed, if any, I wish to ascribe to the one or to the other." (488/24).

So, more generally, we can say that the Principle says the following. Let F_0 be an inertial frame, and F_1 another inertial frame moving w.r.t. F_0 with constant velocity $v_{F_1/F_0} = v$. Let K be a body (a material point) with velocity $v_0 = v_{K/F_0}$ w.r.t. F_0, all these relative velocities being parallel. Then the velocity of K w.r.t. F_1 is $v_1 = v_0 - v$:

$$v_1 = v_{K/F_1} = v_{K/F_0} - v_{F_1/F_0} = v_{K/F_0} + v_{F_0/F_1} = v_0 + v_{F_0/F_1}$$

So if we put F_K (the frame attached to K) = F_2, we get[11]

$$(F_2/F_0) * (F_0/F_1) = (F_2/F_1)$$
$$(F_0/F_1) * (F_1/F_0) = Id_{F_0} \text{ (the identical change of } F_0\text{)}$$

with the fundamental relation correlating $*$ and $+$:

$$v_{F_2/F_0} + v_{F_0/F_1} = v_{(F_2/F_0)*(F_0/F_1)}$$

It is in that sense that, in this first step of Phoronomy, the composition (*Zusammensetzung*) of changes of inertial frames is coded by an algebraic additive structure with identity isomorphic to $(\mathbb{R}, =, +, 0)$. Let us strongly emphasize the fact that, in a technical modern sense, be it interpreted as a Bourbakist structure or as a model of a logical theory, a structure such as $(\mathbb{R}, =, +, 0)$ is by no means a trivial entity. Our thesis is that Kant's "construction" is a metaphysical justification (using categories, schematism and principles) of the applicability of such a structure to inertial motions. It is not only a great philosophical achievement but also a remarkable precursor of the deepest foundational problems of mathematical physics.

3.3 Relativity and cognition: *Zuschauer* and subject

Kant was well aware that Phoronomy was a consequence of the embedding of our embodied perception, action and cognition into our environment. The moving frame F_K attached to a material body K is like the frame attached to our own body and correlating our vision to our motor control, what is called today an *ego-centered* frame. And a fixed frame F w.r.t. which the motion of K is computed is a frame attached to reference objects in the external space where we navigate, what is called today an *allo-centered* frame. The dialectic between moving (ego-centered) frames and fixed (allo-centered) frames is implemented in a very complex way in our brain and our body and is an important domain

[11] Kant considered such situations, e.g. in the case of "a ball moving on the table in the cabin of a ship". F_0 is attached to the riverbank, F_1 to the ship, the cabin and the table, and F_2 to the ball.

of cognitive neuroscience (e.g. the former imply basal ganglia while the later imply hippocampus and the transformations ego/allo-centered frames is neurally implemented). See e.g. Berthoz [7], Atallah *et al.* [5] and our *Neurogéométrie de la vision* [56] (2008).

It has been a very long time since I have argued that contemporary cognitive neurosciences support many tenets of transcendental philosophy and that there are neurophysiological roots for what Kant called “pure intuitions”. Recently, some among the best neuroscientists defended similar theses. For instance in June 2014, John O’Keefe of the “Institute of Cognitive Neuroscience”, University College, London, award recipient of the 2013 Horwitz prize “for significant advancements to the field of neuroscience”, gave a public lecture titled “Immanuel Kant: Pioneer neuroscientist” (O’Keefe [48]). The abstract was:

> “In his *Critique of Pure Reason*, Kant argued that our concept of space was not derived from sensations arising from our interaction with the physical world but instead represented the a priori basis for our perception of the world in the first place. Extensive work in modern neuroscience has provided strong evidence in support of this position. We now know that there is an extensive network of brain areas in the temporal lobes dedicated to the construction of an allocentric space framework and that some parts of this network develop relatively independently of the animal’s experience. This map-like spatial representation is constructed from more primitive representations of places, directions and distances and allows the animal to know where it is in an environment and how to navigate to desired locations. In my talk, I will present the evidence for these more primitive representations and discuss how they may interact with each other to produce a Kantian map-like representation of space. In the latter part of my talk I will discuss how our understanding of these brain systems sheds light on some of the postulates of Euclidean geometry, one of the conceptual domains used by Kant to support his view of the synthetic a priori nature of our spatial representations.”

The same year (2014), John O’Keefe was awarded the Nobel Prize for his discovery of “place cells” in 1971, with May-Britt and Edvard Moser for their discovery of “grids cells” in 2005.

In 2012, another great specialist of cognitive neuropsychology, Stanislas Dehaene, Professor at the Collge de France of Paris and member of the French Académie des Sciences proposed a “Kantian” research project for the *Human Brain Project*, the most important European program in neurosciences. With Elizabeth Brannon he formulated the project in *Trends in Cognitive Sciences, Special Issue: Space, Time and Number* (Dehaene-Brannon [14]). The title was “Space, time, and number: a Kantian research program” and the abstract was the following:

> “What do the representations of space, time and number share that

> might justify their joint presence in a special issue? In his *Critique of Pure Reason*, Immanuel Kant famously argued that they provide "a priori intuitions" that precede and structure how humans experience the environment. Indeed, these concepts are so basic to any understanding of the external world that it is hard to imagine how any animal species could survive without having mechanisms for spatial navigation, temporal orienting (e.g. time-stamped memories) and elementary numerical computations (e.g. choosing the food patch with the largest expected return). In the course of their evolution, humans and many other animal species might have internalized basic codes and operations that are isomorphic to the physical and arithmetic laws that govern the interaction of objects in the external world. The articles in this special issue all support this point of view: from grid cells to number neurons, the richness and variety of mechanisms by which animals and humans, including infants, can represent the dimensions of space, time and number is bewildering and suggests evolutionary processes and neural mechanisms by which Kantian intuitions might universally arise. (...) If Immanuel Kant were born today, he would probably be a cognitive neuroscientist!"

Of course, there exists an unfathomable abyss between ego/allo-centered frames studied by cognitive neuroscience and a symbolic algebra for mathematical vector spaces. Phoronomy is deeply rooted in this cognitive ground but it retains only the *interface* between its "subjective component" and the objects of experience. It ignores the whole living interiority of perception, motricity and navigation, and retains only the general formatting of phenomena by inertial frames of space-time. So relativity is a "residual subjectivity", the subjectivity of a *Zuschauer* (observer). Its "subjective" status justifies the so well known (and so criticized) thesis of the transcendental *ideality* of space. Its "residual" status explains why the ego/allo dialectics, once reduced to a composition of transformations of inertial frames, can be mathematically converted into algebraic operations.

3.4 Measuring vector magnitudes

Kant extended the Principle $v_{F/K} = -v_{K/F}$ to the general composition of velocities and explained in a very detailed way how the categories of quantity converted by schematization into "Axioms of intuition" apply to inertial motions, and how this application converts the "intuitive" and "subjective" relativity of space and time into an algebra of numerically evaluable magnitudes. *Measure* is a key problem. Indeed, a motion is not by itself a magnitude (*Größe*) and a magnitude is not, by itself, a numerically assessable "*quantitas*", that is a *Zahlgröße*. It must be evaluated, i.e. measured, to become a numerical entity, and measure requires a specific system of numbers. This is impossible without schematism since it is the scheme of the category of quantity (*quantitatis*) which provides numerically evaluable entities endowed with algebraic operations enabling to "construct" magnitudes mathematically.

The measure problem is quite delicate for Kant because it mixes three realms: (i) intuitions (geometry, motion), (ii) arithmetics (numerical values of evaluated magnitudes), (iii) algebra (symbolic operations). According to Kant, algebra computes "symbolic constructions". It is not conceptual but "intuitive", and intuitive in a very particular, "non ostensive", sense.[12]

Since composition must compose homogeneous entities and "since nothing is homogeneous with motion except motion in turn" (489/25), Phoronomy must combine velocities but *phoronomically*, that is with a purely "geometric construction" without any reference to some mechanical causes of the motions. The equality relation (to be the same) of velocities is a physical indiscernibility and not a physical process involving mechanical forces:

> "Geometrical *construction* requires that one quantity be the *same* as another or that two quantities in composition be the *same* as a third, not that they produce the third as causes, which would be mechanical construction." (493/29, Remark 1 of the Proposition).

Equality means *congruence*:

> "Complete similarity and equality, insofar as it can be cognized only in intuition [that is without mechanical construction], is *congruence*. All geometrical construction of complete identity rests on congruence." (Idem)

Kant was very much aware of the issues concerning identity (Identität), to be the same (einerlei) and equality (Gleichheit) and the fact that in Phoronomy equality is a physical equivalence w.r.t. experience and not a logical equivalence. It is why relativity has dramatic consequences for the logic of judgments as we will see later in section 7.

3.5 Phoronomy "Proposition" (*Lehrsatz*)

We have seen how the algebraic composition (*Zusammensetzung*) of velocities of inertial motions of the same direction is deduced from the composition of transformations of inertial frames. Kant used this Principle to define the composition of velocities for material bodies. His Proposition (*Lehrsatz*) is the following:

> "The composition of two motions of one and the same point can only be thought in such a way that one of them is represented in absolute space, and, instead of the other, a motion of the relative space with the same speed occurring in the opposite direction is represented as the same as the latter." (490/26).

[12]This thesis will be later developed by Hilbert (with an emphatic reference to Kant) in *Über das Unendliche* [28] (see [58]).

Let us formulate the proposition symbolically. Here "absolute space" is not the absolute space as an Idea of Reason but a fixed frame of reference F_0. If K is the point, the first motion is K/F_0 with velocity v_{K/F_0}. The second motion would be a second motion $(K/F_0)'$ with velocity v'_{K/F_0}. We want to give a sense to the composition of K/F_0 with $(K/F_0)'$. Formulated symbolically, the key obstruction Kant pointed out becomes very clear: the composition $(K/F_0) * (K/F_0)'$ is *not defined* because composition of transformations is a *concatenation* and a $A/B * C/D$ is defined iff $B = C$. To overcome the obstruction, we can introduce a moving frame F_1 with $(K/F_0)' = (F_0/F_1)$ and velocity $v_{F_1/F_0} = -v'_{K/F_0}$. Now, the concatenation $(K/F_0) * (K/F_0)' = (K/F_0) * (F_0/F_1)$ is well defined and gives K/F_1. According to the principle,

$$v_{K/F_1} = v_{K/F_0} + v_{F_0/F_1} = v_{K/F_0} - v_{F_1/F_0} = v_{K/F_0} + v'_{K/F_0}.$$

So the velocity of K w.r.t. F_1 is the sum of the two velocities which could not be summed when considered both as velocities w.r.t. the fixed frame F_0. It is the same if we consider two initial frames with $v_{K/F_0} = v_0$ and $v_{K/F_1} = v_1$. We take a third frame F_2 with $v_{F_2/F_0} = -v_{K/F_1} = -v_1$ and we look at

$$v_{K/F_2} = v_{K/F_0} + v_{F_0/F_2} = v_0 + (-(-v_1)) = v_0 + v_1.$$

In other words, the key relation is $v_{(A/B)*(B/C)} = v_{A/B} + v_{B/C}$ for *all* velocities (and not only those with the same direction). If K is a point moving at velocity v w.r.t. a frame F_0, to add w to v means

1. to take the inertial frame F_0 where K has velocity v: $v_{K/F_0} = v$,
2. to take the inertial frame F_1 with velocity $-w$ w.r.t. F_0: $v_{F_1/F_0} = -w$,
3. and to take the velocity $v + w$ of K w.r.t. F_1 :

$$v_{K/F_1} = v_{K/F_0} + v_{F_0/F_1} = v + (-(-w)) = v + w.$$

To add a velocity w to v is not to change a property of K but to change the context (the frame) of K.

In the Remark 2 to the Proposition Kant commented the particular case $w = -v$ which means that K is at rest in F_1 because F_1 has the same velocity v as K w.r.t. F_0. While it is possible to think two opposite forces applied at the same point,

> "it is impossible to think two equal motions in the same body in opposite directions immediately, that is, in relation to precisely the same space at rest." (494/30)

This impossibility is an "*impossibility of construction*" (Ibidem, Kant's emphasis) The construction requires to combine "the motion of the body with the *motion of the space*." (Ibidem, Kant's emphasis).

Kant strongly emphasized the obstruction that it is impossible in Phoronomy (i.e. without resorting to mechanical causes) to add two motions defined w.r.t. the *same* space and that composition requires *two* spaces. Clearly, he considered that it was an important discovery.

> "The composition of two speeds in one direction cannot be represented intuitively in the same space." (490/26)
> "[The] congruence of two combined motions with a third (as with the *motus compositus* itself) can never take place if this two combined motions are represented in one and the same space." (493/29, Remark 1 of the Proposition).

The obstruction can be overcome only with good account being taken that the equality of velocities as magnitudes corresponds to the *congruence* of motions:

> "The composition of motions (...) must take place in accordance with the rules of congruence, which is only possible (...) by means of the motion of the space congruent to one of the two given motion, so that the two [together] are congruent to the composite [motion]." (494-495/31).

3.6 The algebra of motions

The phoronomic "construction" of motion requires an algebraic *vector* calculus (even if the term was not coined at that time), these particular type of magnitudes having a module, a line and a direction.

Kant described very precisely this algebraic calculus in three steps. Let v and w be two velocities $\neq 0$:

1. When v and w have the same direction D (i.e. are supported by parallel lines L and have the same orientation), they can be added as positive real numbers (their modules). In modern terms, we would say that their space is $\mathbb{V}u$ where $\mathbb{V} = (\mathbb{R}^+, +)$ is the additive monoid of positive reals and u the unit vector of L associated to D.

2. When v and w have the underlying L and any directions D, they can be added as real numbers. In modern terms, we would say that their space is $\mathbb{V}u$ where $\mathbb{V} = (\mathbb{R}, +)$ is the additive group of reals and u a unit vector of L.

3. When v and w have any of the possible directions, they can be added using the parallelogram rule. In modern terms, we would say that their space is the direct sum $\mathbb{V}u_1 \oplus \mathbb{V}u_2 \oplus \mathbb{V}u_3$ where $\mathbb{V} = (\mathbb{R}, +)$ is the additive group of reals and the $\{u_i\}$ define an orthonormal frame.

So, the algebraic system of vectors and numbers mathematizing velocities as magnitudes, is given by:

$$\begin{cases} v+w:(\mathbb{R}^+,+)\text{, same line, same direction} \\ v+w:(\mathbb{R},+)\text{, same line} \\ v+w:\left(\mathbb{R}^3,+\right)\text{, parallelogram rule} \end{cases}$$

And in the last lines of Phoronomy (495/32), Kant explained that these three steps apply the three categories of quantity to velocities: the category of unity correspond to the object "line + direction", the category of plurality to the object "line" and the category of totality to the set of all lines and directions.

3.7 Velocities and tangent space

In fact the phoronomic problem is even more subtle because Kant considered also

(i) *instantaneous* velocities which, as *differential* entities (derivatives $\frac{dx}{dt}$), are *intensive* magnitudes and therefore

(ii) linear uniform motions "tangent" to any motions.

As velocities are entities of the form $v = |v|\, u$, the scalar module $|v|$ (*Geschwindigkeit*) can be interpreted as an intensive degree. In the Remark 2 to the Proposition, Kant tackle this issue:

> "The parts of the speed are not external to one another like the parts of the space (...) then the concept of its [the speed] quantity, since this is *intensive*, must be constructed in a different way from that of the *extensive* quantity of space." (493/29/30).

I emphasized in [54] that it is quite clear that Kant worked in what are called in differential geometry the *tangent spaces* $T_a\mathcal{E}$ of the space $\mathcal{E}$ at $a \in \mathcal{E}$ and considered, on the one hand, the translations in the $T_a\mathcal{E}$, that is the relativity of velocities for the observer a and, on the other hand, the translations in $\mathcal{E}$, but without being able of articulating correctly the two.

Michael Friedman arrived independently to the same conclusion. He says:

> "For Kant, the problem of conceptualizing speed or velocity as a magnitude (...) involves the construction or exhibition of an addition operation directly on the set of *instantaneous* speeds defined at *a single given spatio-temporal point*." ([22], 61)

And he adds in a note:

> "From a contemporary point of view, the set of instantaneous velocities defined at a single given spatio-temporal point constitutes what we call the tangent space at the space-time point in question." (Ibidem, note 42)

He also converges in part with our thesis in insisting on the fact that

> "Kant's principle of the relativity of motion (...) is not a relativity principle at all, in our modern sense; for such principles, in our terms, essentially characterize the (spatio-temporally extended) *affine structure* of a given space-time, and therefore characterize spatio-temporal *curvature*. (...) Kant's principle at this [phoronomic] stage, merely characterizes the structure of the spatio-temporal tangent space at each point, and it says nothing at all about the affine structure that may then coordinate these various tangent spaces together." ([22], 81, note 69)

But perhaps Friedman is too strict when he claims that

> "Kant's structure is purely infinitesimal, (...) and, accordingly, Kant, unlike Newton, entirely fails to articulate what we would call Galilean relativity." ([22], 90, note 81).

We think that the concept of an *affine* structure of space and time is (of course without the name) explicitly present in Kant since the impossibility of an absolute position in space and moment in time is strongly asserted. The problem is that Kant delved so deeply into the metaphysical foundations of Newton's Mechanics that

(i) he arrived at a *double* structure: an "extensive" affine structure for space $\mathcal{E}$ and time $\mathcal{T}$ *and* an "intensive" Galilean structure for the tangent spaces $T_{(a,t)}(\mathcal{E} \times \mathcal{T})$, while

(ii) he did not succeed in *constructing* the articulation between the two, given by the *inertia principle*.

But the problem is so deep that it remained unsolved not only until general relativity in 1915 but until the introduction of the concept of an *affine connection* by Cartan in the 1920s (see [11]) precisely for generalizing to smooth manifolds the identification of tangent spaces through translations in affine space.

However, there is in Kant a sort of surrogate of this Himalayan difficulty. Indeed, as we have seen above in section 3.2, using the formula for constant velocities $C = \frac{S}{T}$ with $T = 1$, we can identify the *intensive* "internal" instantaneous velocity v at $a \in \mathcal{E}$ with the extensive vector $b - a$ in $\mathbb{R}^3$, that is we can *externalize* v to connect in $\mathcal{E}$ the point a with the point $b = a + v$ along the LUM of velocity v during the unit time.

3.8 Accelerations and invariance

Before we refine the links between Galilean relativity and Phoronomy, let us make a short remark on accelerated motions. The problem of the Phoronomy is that velocities are relative to inertial frames and that inertial frames have themselves a velocity. For accelerations $\gamma_{K/F}$ and γ_{F_0/F_1} it is completely different because inertial frame have no accelerations and all $\gamma_{F_0/F_1} = 0$. So to say that *physically* accelerations are invariant, or are non contextual, or can be attached to points, or have a physical content, or are properties of object of experience is equivalent to say that *algebraically* all $\gamma_{F_0/F_1} = 0$, 0 being the neutral element of the algebraic structure of accelerations as *quanta*.

4 Galilean relativity and Phoronomy

4.1 The Galilean structure

The picture we arrived at is therefore by no means trivial. With the limited concepts and mathematics of his time, Kant explained quite clearly *a part* of the Galilean structure of classical Newtonian space-time.[13] This structure says:

1. That the space $\mathcal{E}$ which serves as background structure to motions is the *affine* space $\mathbb{A}^3$ endowed with its Euclidean *metric* structure. The $\mathbb{R}$-vector space $\mathbb{R}^3$ acts naturally by *translations* on $\mathbb{A}^3$. Positions in $\mathbb{A}^3$ cannot be added but differences of positions are vectors and have therefore an algebraic structure.

2. That the choice of a frame identifies $\mathbb{A}^3$ with $\mathbb{R}^3$.

3. That time is an affine line $\mathbb{A}^1$ and the choice of an origin of time identifies it with $\mathbb{R}$.

4. That in fact, $\mathcal{E} = \mathbb{A}^3$ is the space of simultaneous positions and therefore $\mathcal{E}$ is a space $\mathcal{E}_t$ indexed by t: space-time is an affine space $\mathbb{A}^4$ fibered by time. Time is not absolute (there is no well-defined origin of time) but is universal (it is the same for all slices $\mathcal{E}_t$ of space-time).

5. That velocities as intensive magnitudes belong to the tangent vector spaces $T_a\mathcal{E}$.

6. That all the $T_a\mathcal{E}$ are also isomorphic to $\mathbb{R}^3$.

7. That, according to the inertia principle, inertial (i.e. linear uniform) motions connect the different tangent spaces $T_a\mathcal{E}$ along their trajectories (straight lines which are the Euclidean geodesics, it is the simplest example of a Cartan connection) and enable to correlate the velocities as intensive magnitudes to distances as extensive magnitudes.

8. That the group of symmetries of this "Galilean" structure is the relativity group of Mechanics.

Using a metaphysical foundational language, Kant was the conceptual precursor of all these points except (7). We already commented the points (1), (2),

[13] For an introduction to the Galilean structure of Newtonian space-time, see e.g. Arnold [2], Stein [63] or Malament [44]

(3). As for the point (4), i.e. $\mathcal{E}$ is the space of simultaneous positions, remember that for Kant simultaneity belongs to the third category of Relation. The universality of time is also clear. The points (5), (6) and (8) restricted to the tangent spaces are the core of Phoronomy. And (7) is the crux of the issue.

What is most remarkable is not so much the description of a so large part of the Galilean structure but rather the way in which it is deduced from the transcendental structure of objectivity (Transcendental Aesthetics, Analytic of Concepts, Analytic of Principles).

4.2 The Galilean group

The simplest way of understanding the Galilean group $\mathbb{G}$ is to look at the arithmetic Galilean structure $\mathbb{R}^3 \times \mathbb{R}$, that is to take an arbitrary frame F at $a \in \mathcal{E}$, an arbitrary frame T of $T_a\mathcal{E}$, and consider Galilean relativity w.r.t. to the vector spaces $\mathbb{R}^3$ associated to $\left(\mathbb{A}^3, F\right)$ and $(T_a\mathcal{E}, T)$. Let (x,t) be the points of $\mathbb{R}^3 \times \mathbb{R}$. We get the elementary Galilean transformations:

- **(i)** translations of time (those of the component $\mathbb{R}$) $(x,t) \mapsto (x, t+\tau)$, where $\tau \in \mathbb{R}$;
- **(ii)** translations of space (those of the component $\mathbb{R}^3$) $(x,t) \mapsto (x+\xi, t)$, where $\xi \in \mathbb{R}^3$;
- **(iii)** rotations of space $(x,t) \mapsto (R(x), t)$, where R is a rotation of $\mathbb{R}^3$
- **(iv)** linear uniform motions (LUMs) of velocity v: $(x,t) \mapsto (x+tv, t)$, $v \in \mathbb{R}^3$.

The later (LUMs) are well-defined, since x is a vector of $\mathbb{R}^3$ and not a position in $\mathbb{A}^3$ and v is a vector of $T_x\mathbb{R}^3$ identified with $\mathbb{R}^3$. All Galilean transformations are compositions of these elementary ones (see e.g. [2]), which correspond to the symmetries of the Lagrangian and to the conservation laws via Noether theorem (see above section 2.5).

The Galilean group is 10-dimensional: 1 dimension for the τ, 3 dimensions for the ξ, 3 parameters (Euler angles) for the R, and 3 dimensions for the v. Its group composition law is easy to define. Let $T = (R, v, \xi, \tau)$ be a Galilean transformation g. It acts on the (x,t) by $g(x,t) = (R(x) + vt + \xi, t+\tau)$. Today all students learn to present this action in a matrix form, i.e. as a linear transformation. As translations are affine but not linear (they do not preserve 0), a small trick is necessary. We identify the 4-dimensional vector (x,t) with the 5-dimensional vector $(x,t,1)$ and write:

$$\begin{pmatrix} R & v & \xi \\ 0 & 1 & \tau \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ t \\ 1 \end{pmatrix} = \begin{pmatrix} R(x) + vt + \xi \\ t+\tau \\ 1 \end{pmatrix}$$

where R is a 3×3 rotation matrix, and v and ξ vertical 3-vectors. Using this matrix representation, the composition of Galilean transformations becomes simply the product of matrices. That is if $g, g' \in \mathbb{G}$, the matrix of their composition $g' \circ g$ is the product.[14]

$$\begin{pmatrix} R' & v' & \xi' \\ 0 & 1 & \tau' \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} R & v & \xi \\ 0 & 1 & \tau \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} R'R & R'v + v' & R'\xi + \tau v' + \xi' \\ 0 & 1 & \tau' + \tau \\ 0 & 0 & 1 \end{pmatrix}$$

(i) translations of time correspond to matrices $\begin{pmatrix} I_3 & 0 & 0 \\ 0 & 1 & \tau \\ 0 & 0 & 1 \end{pmatrix}$ (where I_3 is the identity of $\mathbb{R}^3$);

(ii) translations of space correspond to matrices $\begin{pmatrix} I_3 & 0 & \xi \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$;

(iii) rotations of space correspond to matrices $\begin{pmatrix} R & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(iv) linear uniform motions (LUMs or boosts) of velocity v correspond to matrices $\begin{pmatrix} I_3 & v & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Kant's Phoronomy concerns in particular the subgroup $\mathbb{G}_U$ of $\mathbb{G}$ of boosts.

5 Phenomenology, relativity and "alternative" judgments

Another remarkable result of Kant in the *MFNS* was to understand that Phoronomy and relativity ran into a deep *logical* problem when motion has to be defined as a *predicate* determining an object. As he explained, motion is given as a phenomenon w.r.t an observer (a subject) and to become an object of experience, he has to be *determined* since, as explained in the *CPR*, a phenomenon is *undetermined* as object.

> "The movable (...) becomes an object of experience, when a certain *object* (here a material thing) is thought as *determined* with respect to the *predicate* of motion." (554/93).

The correlation between, on the one hand, predicative judgments and, on the other hand, objects which are not given as objects but *constituted* as objects according to a transcendental process of constitution is by no means evident because the predicate must at the same time

[14] Remember that composition is conventionally written from right to left because the action $f(x)$ of a function f on an argument x is written from left to right and so $g(f(x))$ is naturally written $(g \circ f)(x)$.

(i) determine the phenomenon as object and enable the constituted object to become a carrier of observable properties, and

(ii) have possibly truth values.

It is here that *transcendental logic* comes to the stage as different from general logic, which is concerned only by point (ii).

Phoronomy presents an a priori quasi-Galilean[15] background structure for motions considered as a *specific* class of phenomena to be determined as objects of experience. We will say that it defines an "object framework". But, as a knowledge, physics is also a set of *judgments* which must have logical truth values and be true or false. So physical knowledge must *articulate an object framework with a logic of judgments.*

If there were an "absolute" space and no relativity, then a realist interpretation of motions would be justified, general logic could be used, and we could say that *either* "K is moving" *or* "K is at rest" is a true judgment. But as it is not the case and since neither places nor changes of places can be *attached* as properties to bodies, we must *subordinate the logic of judgments to the object framework*: Kant tackles the problem that, since the *choice* of an inertial frame is completely *free*, these judgments are *neither* true *nor* false. It is here that transcendental logic departs from general logic. To fix this issue, we have to come back to the pivotal role of categories in the transcendental approach. On the one hand, categories are associated to logical forms of judgments. On the other hand, they are associated to principles of mathematical constructions of concepts and algorithmic reconstructions (models) of phenomena. They act as a bridge between a logic of judgments and the mathematical background structures of objects of experience.

The departure from general logic is tackled by Kant in the *Phenomenology*, which develops the categories of Modality and the "Postulates of the empirical thought in general". Due to relativity, a judgment such as "the body K has velocity v_K" has *no physical content.* It concerns a phenomenal motion and not a well-determined object of experience. v_K is relative to an arbitrary freely chosen frame F and cannot be a predicate, a property, of the body K. If judgments on velocities v_K would be objective, either $v_K = v$ or $v_K \neq v$ would be true. But it is not the case, since two predicates objectively contradictory can be ascribed: $v_K = v$ and $v_K \neq v$ can be both true at the same time.

Yet general logic concerns the relation of knowledge to well-determined (constituted) objects: its judgments are, in Kant's terminology, *disjunctive* judgments, that is judgments with well-defined truth values satisfying the law of excluded middle (it is the meaning of "disjunctive"). But Phoronomy's *Erscheinungslehre* concerns the relation of knowledge not to constituted objects but to the subject (observer, *Zuschauer*): its judgments are, in Kant's terminology, *alternative* judgments having no well-defined truth-values. This opens a critical issue for general logic. In modern terms, we could say that the semantic of alternative judgments is ill-defined, and that such judgments are contextual,

[15] "quasi" because of the restrictions emphasized in sections 3.7 and 4.1.

relative to an observer, in some sense "pragmatic", w.r.t. a measurement context (a frame). To be disjunctive, judgments must concern objects of experience *invariant* w.r.t. changes of inertial frames.

So, predicative judgments processing velocities as *properties* of bodies and ascribing them well-defined numerical values are, in Kant's terminology, only "possible" and not "real", possibility and reality being the two first *modal* categories. Using modern formulations, we could say that Kant tackled for the first time an issue of "value definiteness": "to have a value" for an observable does not mean "to possess a property" for an object. There exists no "element of physical reality" (no physical content) associated to velocities.

In Kant's terms, in a disjunctive judgment

> "of two *objectively* opposed predicates, one is assumed to the exclusion of the other for the determination of the object." (556/95).

On the contrary, in an alternative judgment

> "of two judgments objectively equivalent, yet subjectively opposed to one another, one is assumed for the determination of the object without excluding its opposite – and thus by mere choice." (Ibidem)

And, since "K moving with $v_{K/F} = v$" and "K at rest with $v_{F/K} = -v$" are objectively equivalent, the *predicate* of motion "is a merely possible predicate in experience" (Ibidem).It is in this argumentation that Kant introduces a reference to a forthcoming note on logic in his long *General Remark to Phenomenology*. "In logic the *either-or* always signifies a *disjunctive* judgment" because it concerns "the relation of the cognition to the object" (559/99). But in Phoronomy, judgments concern the relation to the subject (the observer) and are therefore alternative:

> "For here the proposition that the body is either moved and the space at rest, or conversely, is not a disjunctive proposition in an objective relation, but only in a subjective one, and the two judgments contained therein are valid alternatively." (560/99).

To be complete, we must mention that Kant introduced also what is called *distributive* judgments. They correspond to the fact that at the level of Mechanics (not at the level of Phoronomy) there exist criteria for *choosing* some privileged inertial frame, in particular, for a closed system, the frame of the center of mass. In such a frame adapted to the forces and interactions under scrutiny (according to the physical laws of conservation, causality and equality of action and reaction), the physical magnitudes are "distributed" between the material bodies:

> "Wherever the motion is considered *mechanically* (as when a body approaches another seemingly at rest), then the formally disjunctive judgment must be used *distributively* with respect to the object, so

that the motion must not be attributed *either* to one *or* the other, but rather an equal share of it to each." (560/99, Remark to the General Remark to Phenomenology).

6 Relativity groupoid and Phoronomy

We want now specify the notion of alternative judgment using modern concepts while remaining as close as possible to Kant's version of Galilean relativity, and respecting

(i) the difference in Phoronomy between the concatenation of moving frames and the addition of velocities;

(ii) the fact that Phenomenology concerns the semantic status of alternative judgments in Phoronomy. [16]

6.1 Congruences and categories of isomorphisms

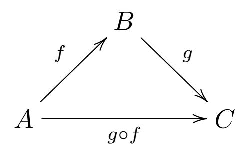
As we have seen, Kant's relativity concerns in particular the subgroup $\mathbb{G}_U$ of $\mathbb{G}$ of Galilean boosts (LUMs). But we cannot use the general notion of a group of relativity since it did not exist at that time. Yet in Phoronomy there is a surrogate of the concept of group, namely congruences as isomorphisms. As we have seen again, Kant was specifically the philosopher of symmetries and had a very acute consciousness of the issue of "congruences". All changes of inertial frames are congruences. So it is in terms of isomorphisms that we want to formalize Kant's relativity and his clear comprehension of the relation between changes of frames as transformations and the algebraic-arithmetic structure for the calculus of velocities.

As far as changes of frames are space-time transformations, it is natural to formulate all these issues in the language of the general formal theory of transformations, namely *category theory.* Remember that a category is a class of objects A with transformations, called "morphisms", $A \xrightarrow{f} B$, such that[17]

(i) transformations can be composed, that is if $A \xrightarrow{f} B$ and $B \xrightarrow{g} C$ are transformations such that the goal of f is the source of g (condition of composability), then there exists a composed transformation $A \xrightarrow{g \circ f} C$

[16] This groupoid and functorial approach of Phoronomy has been presented for the first time the 30th of March 2015 at the seminar *Philosophie de la Gravitation Quantique Canonique* organized by Gabriel Catren

[17] Classes of objects (groups, topological spaces, etc.) can be too big to be sets. When the class of objects is a set, the category is called "small". We will be concerned only with small categories.



and the composition is associative;

(ii) any object A has an identity transformation Id_A such that for every transformation $A \xrightarrow{f} B$ we have $f = f \circ Id_A = Id_B \circ f$.

In a category, a transformation $A \xrightarrow{f} B$ is called an *isomorphism* if it is invertible, that is if there exists a transformation $B \xrightarrow{f^{-1}} A$ such that $f^{-1} \circ f = Id_A$ and $f \circ f^{-1} = Id_B$. If $A = B$, an isomorphism f is called an automorphism. The set $\mathrm{Aut}(A)$ of automorphisms of A is a group. A category where all the morphisms are isomorphisms is called a *groupoid*.

It is justified to say that, conceptually if not formally, Kant pointed out a relation between

(i) the groupoid of frame changes, and

(ii) the addition of vectors.

So the notion of group was not there, but groupoid + addition was a good equivalent. Kant analyzed in a detailed way the fact that every point a of space $\mathcal{E}$ can be an observer and that, attached to this observer there is an exemplar of the relativity group $\mathbb{G}_{a,U}$. But this $\mathbb{G}_{a,U}$ acts also on the space $\mathcal{E}$ as a change of observer. So we need in fact a copy of $\mathbb{G}_U$ at every point a. This is linked to the fact that Kant worked in the *tangent bundle* $T\mathcal{E}$ of $\mathcal{E}$ gluing all the tangent spaces $T_a\mathcal{E}$ (see above section 3.7).

6.2 Groups as groupoids

If G is a group (with elements f, g, h, etc., composition law hg noted multiplicatively[18], neutral element e, and inverses noted g^{-1}), it is easy to associate to it a groupoid. We consider a category $\mathbf{G}$ with a single object $\bullet$ and consider G as the group of automorphisms of $\bullet$. If $g, h \in G$, their composition $h \circ g$ as automorphisms of G is simply their product hg in G.

Now, suppose that G is the group of relativity of a background structure for a certain class of phenomena, $\bullet$ being identified with an *observer*. As $e = Id_\bullet$, e is the context attached to the observer. Suppose we consider a phenomenon K of this class and look at an observable and measurable quantity m_K having values in a set M and suppose that, w.r.t. to the observer context $g = e$, m_K takes the

[18] We leave open the possibility that G would be noncommutative. It is not an irrelevant generality since the Galilean group includes the group of rotations of $\mathbb{R}^3$ which is noncommutative.

value $\mu_{K,e} = \mu_K$. As this value is context-dependent, it must be transformed in a certain way when the context $g \in G$ changes. So G must act on the set M and m_K has therefore a value $\mu_{K,g} = \widetilde{g}(\mu_K)$ w.r.t. any $g \in G$, these values being possibly different but necessarily coherent. To formulate this coherence, that is *G-covariance*, let us mimic Kant's "Proposition" for relative velocities. Let us begin with the rule (where K means F_K): $(K/F_0) * (F_0/F_1) = (K/F_1)$. If (K/F_0) corresponds to g and (F_0/F_1) to h, then $(K/F_0) * (F_0/F_1) = (K/F_1)$ corresponds to $h \circ g$,[19] and $(F_0/F_1) * (F_1/F_0) = Id_{F_0}$ means that $h^{-1} \circ h = e$.

Consider now the main rule:

$$v_{K/F_0} + v_{F_0/F_1} = v_{(K/F_0)*(F_0/F_1)}$$

M is the vector space of velocities and to add v_{F_0/F_1} defines the action $\widetilde{h}$ of $(F_0/F_1) \equiv h \in G$ (change of inertial frame) on M. So the rule can be written $\widetilde{h}(\mu_{K,g}) = \widetilde{h \circ g}(\mu_K)$, that is, since $\mu_{K,g} = \widetilde{g}(\mu_K)$:

$$\widetilde{h \circ g}(\mu_K) = \widetilde{h}(\widetilde{g}(\mu_K)) = \left(\widetilde{h} \circ \widetilde{g}\right)(\mu_K)$$

This means that the observable m_K is a *functor* from the category **G** to the category **Set** of sets, which associates to the single object $\bullet$ (the observer) the set of values M of m_K and to any morphism g of **G** (i.e. to any $g \in G$) the automorphism $\widetilde{g}$ of M (it is an automorphism since $\widetilde{g^{-1}}$ is an inverse $\widetilde{g}^{-1}$ of g), the property of functoriality meaning precisely that $\widetilde{h \circ g} = \widetilde{h} \circ \widetilde{g}$ (G-covariance).

This functorial formulation means that the set of values M of any observable m of the theory must be endowed with what is called a *G-set structure*, that is an action of the relativity group G over M. In this categorical formalism, the difference between the background structure (the relativity groupoid **G**) and the values of an observable (the set M) becomes very salient and we see that, due to functoriality, a value is not an element μ_K of M but the *orbit* $\widetilde{G}(\mu_K)$ of μ_K in M w.r.t. the action of G, that is the whole set $\{\mu_{K,g} = \widetilde{g}(\mu_K)\}_{g \in G}$. For an observable m to have a *physical* content is to be not only G-covariant but also *G-invariant*, that is to be a *constant* functor with all the $\widetilde{g} = Id_M$.

In the special case of velocities, the categorical formulation can be strengthened. Indeed the set of values M is a system of numbers endowed with an additive group structure correlated to G:

(i) the $\widetilde{g}$ are group automorphisms of M, and

(ii) there exists a morphism of groups $G \to M$, $h \mapsto \mu_h$ such that $\widetilde{h \circ g} = \widetilde{g} + \mu_h$.

So, we have

$$\mu_{K,h \circ g} = \widetilde{h \circ g}(\mu_K) = \mu_{K,g} + \mu_h = \widetilde{g}(\mu_K) + \mu_h$$

[19] The composition $*$ is written from left to right wile the composition $\circ$ is written from right to left. Hence the inversion of order.

6.3 Groupoids and groups acting on themselves

In the previous groupoid formulation, the category of isomorphisms $\mathbf{G}$ has a single object $\bullet$, which corresponds to a single observer, and a group of automorphisms of $\bullet$ expressing the changes of context (frames) relative to this observer. The observer $\bullet$ with its identity $Id_\bullet = e$ is like an ego-centered context, to which all relative measures can be refered. Measures are context-dependent but not observer-dependent. It is analogous to the arithmetic Galilean group (see section 4.1) relative to a choice of a point $a \in \mathbb{A}^3 = \mathcal{E}$ and frames at a and in $T_a\mathcal{E}$. But the structure of full-fledged Phoronomy is richer since it takes into account the *changes of observers*. We must take it into account to understand what are alternative judgments. To do justice to this issue while remaining in the general framework of groups and groupoids, we will translate Kant's intensive-extensive game in terms of groupoids. We have seen in section 3.7 that in Phoronomy Kant identified the intensive "internal" instantaneous velocity v at $a \in \mathcal{E}$ with the extensive vector $b - a$ in $\mathbb{R}^3$ and externalized v to connect a with $b = a + v$. The way of understanding this in terms of a relativity group G is to use the fact that any group G acts canonically on itself by (left) *translations*. If $g \in G$, the left translation is the map $L_g : G \to G$ defined by $L_g(h) = gh$. It is not a group morphism since $e \mapsto g$ and e is not fixed. This natural action of G on itself corresponds to the the set-theoretic product $G \times G$ with the law of composition $(g, h)(g', h') = (ghg', hh') = (gL_h(g'), hh')$. It is not a group.

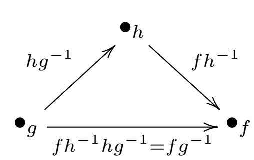
There is a groupoid $\mathfrak{G}$ associated to this action.[20]. $\mathfrak{G}$ is the category fibered over the underlying set $|G|$ of G and having as fibers the groupoid $\mathbf{G}$. More precisely, the objects $\bullet_g$ of $\mathfrak{G}$ are labelled by the elements of G. They are the different possible observers.

In what concerns morphisms, there are privileged "horizontal" or "external" morphisms between any pair of objects namely (left) translations corresponding to changes of observers:

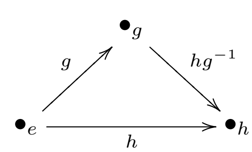
(i) Between $\bullet_e$ and $\bullet_g$ there exists the single translation $\bullet_e \xrightarrow{g} \bullet_g$.

(ii) Between $\bullet_g$ and $\bullet_h$ there exists the single (left) translation $\bullet_g \xrightarrow{hg^{-1}} \bullet_h$. Indeed the left translation $L_x(g) = h$ is given by $xg = h$ and hence $x = hg^{-1}$.

(iii) Of course, the composition of morphisms is coherent as shown by the commutative diagram:



in particular



[20] For the groupoid associated to the tangent bundle of a smooth manifold, see the section II.5 of Alain Connes' *Noncommutative Geometry* [13].

But there are also "vertical" or "internal" morphisms $\mathrm{Aut}(\bullet_g) = G_g = G$, the object $\bullet_g$ endowed with $\mathrm{Aut}(\bullet_g) = G_g$ being a copy $\mathbb{G}_g$ of the groupoid $\mathbf{G}$. These automorphisms correspond to changes of contexts relative to the observer $\bullet_g$. They can be translated. Indeed, let f_e be an element of $\mathrm{Aut}(\bullet_e)$ noted $\bullet_e^{\circlearrowleft f_e}$; when we translate it to g, we get an element f_g of $\mathrm{Aut}(\bullet_g)$ according to the diagram

$$\bullet_e^{\circlearrowleft f_e} \xrightarrow{g} \bullet_g^{\circlearrowleft f_g}$$

Coherence requires $f_g = g \circ f_e \circ g^{-1}$, which means that f_g is the *conjugate* of f_e by g. Note that the translated f_g of f_e by g is an automorphism of $\bullet_g$ and not a morphism $\bullet_e \to \bullet_g$ (indeed $g \circ f_e = f_g \circ g \neq f_g$). More generally, if $f_g \in \mathrm{Aut}(\bullet_g)$ and if we translate it to $\bullet_h$ by hg^{-1}, we get $f_h \in \mathrm{Aut}(\bullet_h)$

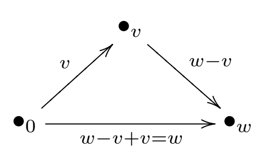
$$\bullet_g^{\circlearrowleft f_g} \xrightarrow{hg^{-1}} \bullet_h^{\circlearrowleft f_h}$$

defined by

$$\begin{aligned} hg^{-1} \circ f_g &= h \circ g^{-1} \circ g \circ f_e \circ g^{-1} \\ &= h \circ f_e \circ g^{-1} = h \circ f_e \circ h^{-1} \circ h \circ g^{-1} \\ &= f_h \circ hg^{-1} \end{aligned}$$

and so, as expected, $f_h = hg^{-1} \circ f_g \circ gh^{-1}$ is the conjugate of f_g by hg^{-1}.

As we have seen, Kant tackled this issue for the affine space with the groupoid associated to linear uniform motions. Vectors $a \to b$ between points $a, b \in \mathcal{E}$ are identified with extensive-like "external" velocities v acting during a unit time and the fibers $\mathrm{Aut}(\bullet_v) = G_v$ are identified with intensive "internal" velocities:



7 Contextual truth values and "alternative" judgments

7.1 Topos theory and transcendental logic

A great advantage of formulating Phoronomy using the groupoid $\mathfrak{G}$ is to clarify the structural meaning of "alternative" contextual judgments using very general categorical constructions correlating a special class of categories to logic. This topic, called the "*internal logic of toposes*", is an immense and technical subject developed in the early 1970s by Bill Lawvere [41] in the framework of topos theory excogitated in the 1960s by Alexander Grothendieck for highly technical problems of cohomology in algebraic geometry [4]. But behind these very high order technicalities, there are some fundamental intuitions of universal scope.[21] To understand why, let us return to the most trivial aspects of set-theoretic semantics of predicate calculus. Let L be a certain logical language, $p(x)$ a predicate of L with one free variable x, and E_p the extension $E_p = \{x : p(x) \text{ is true}\}$ of p in a certain semantic interpretation of L in a set S. The E_p are subsets of S and it is well known that the logical operations of L are reflected in operations on sets according to the translation (where $\vee$ is the logical connective "or", $\wedge$ the connective "and", $\neg$ the negation, $\Longrightarrow$ the implication, $\exists$ the existential quantifier "there exists", $\forall$ the universal quantifier "for all", $\cup$ the union of sets, $\cap$ the intersection of sets, $Y \subset X$ the inclusion of sets, $X - Y$ the complement of Y in X when $Y \subset X$, and $\varnothing$ the empty set): $E_{p\vee q} = E_p \cup E_q$, $E_{p\wedge q} = E_p \cap E_q$, $E_{\neg p} = S - E_p$, $\exists x\ p(x)$ is true iff $E_p \neq \varnothing$, $\forall x\ p(x)$ is true iff $E_p = S$. As for the implication, $p \Longrightarrow q$ means $\neg(p \wedge \neg q)$ and is therefore represented by $S - (E_p \cap (S - E_q))$ and therefore $p \Longrightarrow q$ is true iff $S - (E_p \cap (S - E_q)) = S$, iff $E_p \subset E_q$.

Furthermore, truth can be also expressed in terms of set-theoretic operations. Let $\{1, 0\}$ be the truth values $True$ and $False$. Any subset E of a set S can be retrieved from a characteristic map $\chi : S \to \{1, 0\}$ by $E = \chi^{-1}(1)$. The predicate $p(x)$ is then true for $x \in S$ iff the following diagram is a pull-back (Cartesian fiber product):

$$\begin{array}{ccc} E & \longrightarrow & \{*\} \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle \mathbf{True}} \\ S & \xrightarrow{\ \chi\ } & \{1,0\} \end{array}$$

where $\{*\}$ is a set with a single element $*$ and $True : \{*\} \to \{1, 0\}$ maps $*$ onto 1.

This correlation between syntactic logical operations and semantic set operations is fundamental: *it ensures that a language is able to speak of a world of objects*. We could say that, with its operations, set theory is an "object framework". Classical predicate calculus is adapted to the set-theoretic object framework concerning any type of objects. For instance, the law of excluded middle $\forall x\ (p(x) \vee \neg p(x))$ reflects the trivial fact that $E_p \cup (S - E_p) = S$. Likewise, the rule $\neg\neg p = p$ (a double negation is an affirmation) reflects the trivial fact that $S - (S - E_p) = E_p$.

But set operations are not the unique way of correlating the logical grammar of a language with semantic operations. If we change the object framework of the world of objects, then the logic must change. The best known example is that of *intuitionistic* logic whose logical properties reflect the fact that the extensions E_p of predicates are *open* subsets of a *topological* space S. The main change concerns the negation: indeed, if E_p is open, the complementary set $S - E_p$ is a closed subset and, in general will not be open. So the extension of

[21] For an introduction, see [53] and its bibliography, for a technical source see the classical book of Saunders Mac Lane and Ieke Moerdijk *Sheaves in Geometry and Logic* [43].

$E_{\neg p}$ must be interpreted no longer by $S - E_p$ but by the interior $[S - E_p]^\circ$ of $S - E_p$. So the law of excluded middle fails since in general $E_p \cup [S - E_p]^\circ \neq S$, and the rule $\neg\neg p = p$ also fails since $\left[S - [S - E_p]^\circ\right]^\circ \neq E_p$ in general.

The question is then to decide what pole has the priority: the logical pole or the object pole. We think that as science is composed of judgments about *constituted* objects (objects of experience in Kant's sense), *the object pole is prior to the logical pole*. To remain faithful to Kant, we call *transcendental* an *object-oriented* logic, that is a logic whose grammar is constrained by the object framework with which it is correlated. General logic is the logic of *non* constituted objects supposed to be given *an sich*. As said Ferdinand Gonseth [24], "logic is the physics of the arbitrary object".

Now, the thesis is that object frameworks must be defined in terms of categories. So the natural question to ask is: what are the conditions on a category $\mathcal{C}$ to be able to interpret coherently the basic logical operations $\vee$, $\wedge$, $\neg$, $\Rightarrow$, $\exists$, $\forall$. For that, the category must share with the category **Set** of sets some operations analogous to $\cup$, $\cap$, $\subset$, $-$, $\varnothing$. Lawvere's great achievement (see e.g. [41]) has been to identify this categorical structure: it is the structure of topos. This structure is too technical to be presented here (see [53] for some informations). We will use it in a so elementary way that the technicalities will disappear, but the epistemological relevance will fully remain because, in a word, *topos theory is the modern form of transcendental logic as object-oriented logic.*

7.2 Observables and values

Let us generalize to the groupoid $\mathfrak{G}$ what we have seen for the simpler groupoid $\mathbf{G}$. The objects $\bullet_g$ are the different "observers" and to each observer is associated a copy $\mathbf{G}_g$ of the groupoid $\mathbf{G}$ of changes of contexts or frames. An observable and measurable quantity m having values in a set M will be described by a functor $\mathfrak{M} : \mathfrak{G} \to \mathbf{Set}$, that is by observer-dependent and context-dependent measures. It is convenient to suppose that $\mathfrak{M}$ is contravariant. To every g is associated a set of values $M_g = \mathfrak{M}(\bullet_g)$ and to the "horizontal" isomorphisms $\bullet_g \xrightarrow{hg^{-1}} \bullet_h$ are associated "horizontal" isomorphisms $M_h \xrightarrow{\varphi_{hg^{-1}}} M_g$. To $\bullet_e$ is associated $M_e = M$ and to $\bullet_e \xrightarrow{g} \bullet_g$ is associated $M_g \xrightarrow{\varphi_g} M$. The contravariant convention ensures that all observers can map directly their measures of m into the same set M of values.[22] The functoriality of compositions implies that, since

$$\bullet_e \xrightarrow{g} \bullet_g \xrightarrow{hg^{-1}} \bullet_h = \bullet_e \xrightarrow{h} \bullet_h,$$

we must have

$$M_h \xrightarrow{\varphi_{hg^{-1}}} M_g \xrightarrow{\varphi_g} M = M_h \xrightarrow{\varphi_h} M$$

that is $\varphi_g \circ \varphi_{hg^{-1}} = \varphi_h$, that is $\varphi_{hg^{-1}} = (\varphi_g)^{-1} \circ \varphi_h$.

[22] As we are in a groupoid, the covariant convention $M_g \xrightarrow{\varphi_{hg^{-1}}} M_h$ would be equivalent.

The observables being defined as (contravariant) functors expressing the effect of relativity on their values, we can formulate what is a predicate such as $\mu \in \Delta$, $\Delta \subset M$ being a subset of M and μ the measured value of the quantity m. Δ is described by a subfunctor $\mathfrak{D}$ of $\mathfrak{M}$ associating to each observer $\bullet_g$ a subset $\Delta_g \subset M_g$ of M_g, the Δ_g being transformed functorially by the φ_g. In particular μ will be described by a set of values $\mu_g \in M_g$ varying functorially, that is by an "orbit" $\boldsymbol{\mu}$ of the action of $\mathfrak{G}$ on M.

The main point is that in the groupoid $\mathfrak{G}$ *truth is relativized to observers*. There is a functor $\Omega : \mathfrak{G} \rightarrow \mathbf{Set}$ associating to each observer $\bullet_g$ a set of truth values $\{1_g, 0_g\} = \{True_g, False_g\}$ functorially transformed. As in set theory, a subfunctor $\mathfrak{D}$ of $\mathfrak{M}$ is defined by a "characteristic function" (i.e. a natural transformation of functors) $\chi : \mathfrak{M} \rightarrow \Omega$ such that $\chi(\bullet_g) : \mathfrak{M}(\bullet_g) \rightarrow \Omega(\bullet_g)$, i.e. $\chi_g : M_g \rightarrow \{1_g, 0_g\}$, yields Δ_g by $\Delta_g = (\chi_g)^{-1}(1_g)$. Let $\mathbf{1}$ be the constant functor on a single object 1 and **True** be the functor picking 1_g for each $\bullet_g$: $\mathbf{True}(\bullet_g) = 1_g$. For every functor $\mathfrak{M} : \mathfrak{G} \rightarrow \mathbf{Set}$ there exists a unique natural transformation $\mathfrak{M} \rightarrow \mathbf{1}$. Consider now the inclusion $j : \boldsymbol{\mu} = \{\mu_g\}_{g \in G} \hookrightarrow \mathfrak{M}$ of an orbit $\boldsymbol{\mu}$ in $\mathfrak{M}$. The predicate $\mu \in \Delta$ is true iff $\chi \circ j = \mathbf{True}_{\boldsymbol{\mu}}$, $\boldsymbol{\mu} \overset{j}{\hookrightarrow} \mathfrak{M} \overset{\mathbf{True}}{\rightarrow} \Omega$ with $\mathbf{True}_{\boldsymbol{\mu}} : \boldsymbol{\mu} \rightarrow \mathbf{1} \overset{\mathbf{True}}{\rightarrow} \Omega$.

Once again, the *functoriality* of $\boldsymbol{\mu}$ does not imply that $\boldsymbol{\mu}$ is *invariant* and has a physical content.

Functoriality without invariance means kinematic relativity and includes a "subjective" component of knowledge. It applies to phenomenal properties as positions or velocities. On the contrary, invariance means physical content and concerns the objective component of knowledge. It applies to properties of objects of experience as accelerations.

8 Conclusion: from metaphysics to mathematics

In this investigation, we tried to argue that in Phoronomy Kant presents *metaphysically* the main part of Galilean relativity for CM. In the *MFNS* metaphysics anticipates many modern formulations.

1. The a priori components correspond to the background structures of CM.

2. These background structures include a conception of relativity based on congruences and non-conceptual equivalences (hence the role of groupoids as categories of isomorphisms in our formalization).

3. A key originality of Kant was to prove the need of two frames for defining the addition of motions.and to link the composition (concatenation) of Galilean changes of inertial frames enforced by the composition law "$(A/B) * (C/D)$ is possible iff $B = C$" with the additive composition law of velocities as vectors. This algebraization allows to convert motions

as “quanta” into measurable quantities. Hence, in our formalization, the functor from the relativity groupoid to the group of vectors.

4. The affine structure of space $\mathcal{E}$ and time $\mathcal{T}$ (places and dates).

5. The vector structure of instantaneous velocities at each point (a, t) of space-time $\mathcal{E} \times \mathcal{T}$ as the structure of the tangent space $T_{(a,t)}(\mathcal{E} \times \mathcal{T})$.

6. A very primitive form of connection correlating intensive velocities with extensive paths (the path described at constant speed during a unit of time).

7. A theory of alternative judgments violating the “either ... or” law of excluded middle and lacking truth values. The explanation that motion cannot be a predicate for a material body considered as an object “carrier” of properties. The metaphysical justification of this absence of “value definiteness” and “element of reality”.

8. The metaphysical explanation of the applicability to kinematic phenomena of a mathematical structure in the modern (Bourbakist) sense.

In a nutshell, we could say that *the MFNS are a metaphysical anticipation of the Newton-Cartan structure of CM*. It is for this very reason that our evaluation of the *MFNS* is opposite of the traditional one. As indicated in their very title, the *MFNS* are foundational and metaphysical. But this has been interpreted as a thesis on the absolute and forever lasting status of Newton’s theory and of its mathematics, which is completely absurd since Kant criticized some aspect of Newton’s conception (in particular absolute space and time), introduced deep and far reaching cosmological speculations going completely beyond Newtonian Mechanics, considered that the status of chemistry was still completely unsolved at his time, asked in the Third Critique for a theory of biological morphogenesis, and in the *Opus Postumum* for a theory of the primitive internal forces of matter beyond the purely “derivative” external mechanical forces. So nothing is most opposed to Kant’s spirit than the metaphysical “freezing” of Newton’s Mechanics. His problematic is a problematic of constitution of objectivity and construction of objects of scientific experience.

A *transcendental* investigation analogous to the *MFNS* can be worked out for every physical theory and in the works mentioned in section 2.3 we sketched the transcendental path from Lagrangian Mechanics to General Relativity, Cartan and Weyl. A very interesting aspect of the history of these successive metaphysical foundations is that Kant’s *MFNS* were progressively *internalized* in the successive *mathematical* formalisms, which means that, ultimately, we can dispense with Kant and substitute more and more modern mathematics to transcendental metaphysics, at least if we consider that the mathematical status of physics is no longer a problem.

But as soon as the issue of this mathematical status is reopened, metaphysics comes again on the stage. A beautiful example is quantum mechanics where the new basic phenomena are no longer positions and motions with velocities but *spectral rays* with frequencies. In his 1925 groundbreaking paper "Über quantentheoretische Umdeutung kinematischer und mechanischer Beziehungen", Heisenberg worked out a radical "reset" of physics and "reinterpreted" from scratch the foundations of mathematical physics, and especially the relations between kinematics and mechanics, in a remarkably Kantian way. He starts with the principle of reduction to observables (i.e. to phenomena) and wants to

> "establish a basis for theoretical quantum mechanics founded exclusively upon relationships between quantities which in principle are observable."

He emphasizes strongly the change of regional object:

- one does not know how to associate a position to an electron, while
- one knows how to associate a frequency to an electron.

So the basic observables are frequencies. Now, as in Kant's Phoronomy, the key problem is to interpret the *composition* of these basic phenomena in terms of *algebraic* operations on *quantities* and systems of numbers for measuring them. The Balmer and Rydberg empirical formulae leads to the Ritz-Rydberg combination principle for frequencies ν_{ij}:

$$\begin{cases} \nu_{ij} = \nu_i - \nu_j \\ \nu_{ik} = \nu_{ij} + \nu_{jk} \end{cases}$$

There exists a *groupoid* Φ of frequencies: they are indexed by pairs (i, j) fulfilling the composition law $(i, j) * (j, k) = (i, k)$.

Alain Connes strongly emphasized that, going to the root of the difficulties and shifting from Fourier analysis of geometric circular motions in Bohr's model of atoms to an "algebraic calculation" of purely observable frequencies, Heisenberg discovered that

> "physical quantities are governed by non commutative algebra" ([13], 34)
> "the spectrum is naturally endowed with a partially defined law of composition." (Idem, 37)
> "Heisenberg's rules of algebraic calculation were imposed on him by the experimental results of spectroscopy." (Idem, 38)

And one of the main contributions of Alain Connes to mathematical physics has been precisely to do the inverse of Bohr: instead of trying to reconstruct spectra from geometry, and generalizing Gelfand's spectral theory of commutative C^*-algebras to noncommutative C^*-algebras he was able to reconstruct all classical Riemann-Cartan geometries (differentiable manifolds, differential forms, Riemannian manifold, spin structures, etc.) from the primitive concept of spectrum and to couple these spectrally reinterpreted classical geometries with completely new forms of noncommutative geometries. He was then able to retrieve the Standard Model of Quantum Field Theory and to interpret Higgs fields as the purely noncommutative part of the model. In [57] we have shown how a transcendental foundational metaphysics remains still adapted to such a breakthrough.

A last word. In QM the obstruction to ascribe truth values to judgments ascribing numerical values to magnitudes is radical. It no longer derives from relativity but from the noncommutativity of the algebra of observables. The problem is well known since Bell and Kochen-Specker no-go theorems (1967, see [37]) later improved by Adan Cabello (1996): it is impossible to ascribe definite values to the observables a under the two hypotheses

1. "to have a value $V(a)$ for an observable a" means "to possess for an object an intrinsic (non contextual) property whose measure gives $V(a)$", and

2. for functions f, we have the functional rule $V(f(a, b) = f(V(a), V(b))$ even just for compatible (i.e. commuting) observables.

So, no-go theorems show that "measure" is not "valuation".and that there exists an irreducible gap between:

(i) "the measure of the observable a gives the result r when the system is in state ρ", and

(ii) "when the system is in state ρ, the quantity a has the value r ."

In other words, there exists no "element of physical reality" associated to a measure.

The Kochen-Specker theorem and the problem of how ascribing truth values to judgments in QM has been interpreted in the late 1990s in a foundational, structural and conceptual way by Chris Isham, Jeremy Butterfield [30], Andreas Döring, Chris Heunen, Bas Spitters, using the "interplay between spatial and logical structures inherent to topos theory." They coined for that the term *Bohr topos*. But it is another story.

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