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# Neurogeometry of neural functional architectures

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ABSTRACT

The term "neurogeometry" denotes the geometry of the functional architecture of visual areas. The paper reviews some elements of the neurogeometry of the functional architecture of the first visual area *V*1 and explains why contact geometry, sub-Riemannian geometry, and noncommutative harmonic analysis are brought in as natural tools. It emphasizes the fact that these geometries are radically different from Riemannian geometries.



## 1. Introduction

In the late 1990s, we coined the term "neurogeometry" to denote the geometry of the functional architecture of visual areas. In general, in neural net models, few hypotheses are made concerning the precise geometry of the connectivity defined by the synaptic weights and yet it is this geometry which explains the structure of percepts.

In this paper, we review some elements of a neurogeometrical model of the functional architecture of the first visual area *V*1. We show that the underlying neurogeometry belongs to what is called in mathematics *contact geometry*, *sub-Riemannian geometry*, and *noncommutative harmonic analysis*. We emphasize the fact that these geometries are radically different from Riemannian geometries. Finally, we show how a process as fundamental as diffusion depends heavily upon them.

## 2. The functional architecture of *V*1

Let us begin with a very rough outline of *V*1's functional architecture. In the linear approximation, "simple" neurons of *V*1 (there exist also "complex" ones) operate as filters on the optic signal coming from the retina. Their receptive fields, that is the bundle of photoreceptors they are connected with via the retino-geniculo-cortical pathways, have receptive profiles (or transfert function) with a characteristic shape, highly anisotropic and elongated along a preferential orientation. They can be modeled either by second order derivatives of Gaussians, $\varphi(x,y) = \frac{\partial^2 G}{\partial x^2}$ (with $G = \exp(-(x^2+y^2))$) or by Gabor wavelets $\varphi(x,y) = \exp(i2x)\exp(-(x^2+y^2))$ (real part) (see Fig. 1).[1]

Linear receptive fields operate by convolution on the visual signal and perform a *wavelet analysis* (for an introduction to wavelets, see [18]). Due to their characteristic shape, they detect a preferential orientation. They measure, at a certain scale, pairs $(a,p)$ of a spatial (retinal) position $a$ and a local orientation $p$ at $a$. Pairs $(a,p)$ are called *contact elements* in differential geometry.

The discovery of the first part of the functional architecture of *V*1 won David Hubel and Torsten Wiesel the Nobel Prize in 1981. For a given position $a = (x_0, y_0)$ in the visual field or the retina $R$, the simple neurons with variable orientations $p$ constitute an anatomically definable micromodule called an "hypercolumn". Hypercolumns associate retinotopically to each position $a$ of the retina $R$ a full exemplar $P_a$ of the space $P$ of orientations $p$ at $a$. So, this part of the functional architecture implements what is called in geometry the *fibration* $\pi: V = R \times P \to R$ with base $R$, fiber $P$, and total space $V = R \times P$.

Fibrations mathematize Hubel's concept of "engrafting" "secundary" variables (orientation, ocular dominance, color, direction of movement, etc.) on the basic retinal variables $(x,y)$:

[1] True RFs are far more sophisticated. They are adapted to the processing of natural images and not bars and gratings as in the classical experimental protocols.

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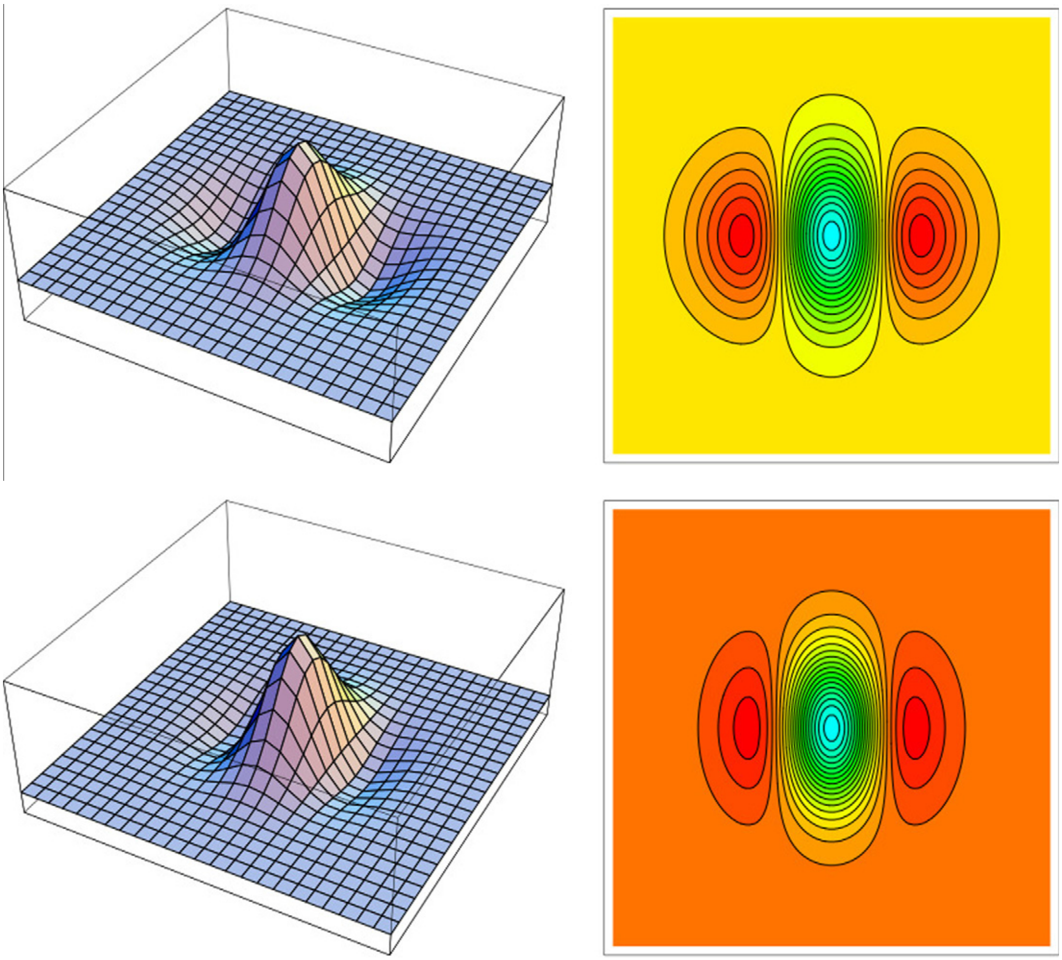

**Fig. 1.** Two models for the RF of a "simple" neuron of $V1$: a second order derivative of a Gaussian $\varphi(x,y) = \frac{\partial^2 G}{\partial x^2}$ $(G = \exp(-(x^2 + y^2)))$ or an even Gabor wavelet $\varphi(x,y) = \exp(i2x)\exp(-(x^2 + y^2))$ (real part).

> What the cortex does is map not just two but many variables on its two-dimensional surface. It does so by selecting as the basic parameters the two variables that specify the visual field coordinates (...), and on this map it engrafts other variables, such as orientation and eye preference, by finer subdivisions (Hubel [15], p. 131).

But, as is well known, the "vertical" retinotopic structure is not sufficient. To implement a global coherence, the visual system must be able to compare two retinotopically neighboring fibers $P_a$ and $P_b$ over two neighboring points $a$ and $b$. This is what is called in geometry a problem of *parallel transport*. It has been confirmed at the empirical level by the discovery of "horizontal" cortico-cortical connections, which are slow ($\approx 0.2$ m/s) and weak and connect neurons of almost similar orientation in neighboring hypercolumns. This means that the system is able to know, for $b$ near $a$, if the orientation $q$ at $b$ is the same as the orientation $p$ at $a$.

So, the retino-geniculo-cortical "vertical" connections implement the relations between $(a,p)$ and $(a,q)$ (different orientations $p$ and $q$ at the same point $a$), while the "horizontal" cortico-cortical connections implement the relations between $(a,p)$ and $(b,p)$ (same orientation $p$ at different points $a$ and $b$). Moreover cortico-cortical connections connect neurons coding pairs $(a,p)$ and $(b,p)$ such that $p$ is approximately the orientation of the axis $ab$. This fascinating functional architecture is summarized by Bosking et al. [6]:

> The system of long-range horizontal connections can be summarized as preferentially linking neurons with co-oriented, co-axially aligned receptive fields.

So, the well known Gestalt law of "good continuation" is neurally implemented. We meet here a foundational element of geometry.

Many other experiments have confirmed and improved this functional architecture. In particular, a certain amount of *curvature* is allowed in alignments and the pure co-alignment constraint is too strict.

## 3. Neural dynamics and entoptic vision

Encoding a functional architecture into the synaptic weights of a neural net enables to explain many puzzling visual phenomenal. Building on previous works of Ermentrout and Cowan [10], a beautiful example has been worked out by Bressloff, Cowan, Golubitsky, Thomas, and Wiener for explaining hallucinations in entoptic vision (see [7] and [8]).

Entoptic vision concerns geometrical patterns of phosphenes which are perceived after a strong pressure on the eyeballs (mechanical stimulation), electro-magnetic stimulations (transcranial magnetic stimulation, electrical stimulation via implanted micro-electrodes), exposures to a violent flickering light, headaches, absorptions of substances such as mescaline, LSD, psilocybin, ketamine, some alkaloids (peyote) (neuropharmaco stimulation), or near death experiences. They depend upon an increased abnormal excitability of the photo-receptors and of $V1$. Subjects see spontaneously and vividly typical morphological patterns such as tunnels and funnels, spirals, lattices (honeycombs, triangles), cobwebs. As was explained by Frégnac in [11],

> Such visual imagery is dynamic and the illusory contours usually explode from the center of gaze to the periphery, appearing initially in black and white before bright colors take over, and eventually pulsate and rotate in time as the experience progresses.

In the case of ingestion of a drug, a qualitative explanation is that the substance shifts the balance of activity of the brain away from its ground state, by a vector representing the profile of binding affinities at different receptors. The bifurcation of the brain state explains the hallucination.

The key move of Bressloff et al. [7] is to take the Hopfield equations of a neural net and to encode the functional architecture of $V1$ into its synaptic weights. The authors take $R=\mathbb{R}^2, P=\mathbb{S}^1$ and work therefore in the fibration $\pi : \mathbb{V}_S=\mathbb{R}^2\times\mathbb{S}^1\to\mathbb{R}^2$ with coordinates $v=(a,\theta)$ ($a=(x,y)$, $\theta$ = the angle of the preferred orientation $p$) labelling the "simple" neurons. Let $u(a,\theta,t)$ be the activity of $V1$. They look for the integro-differential equation governing the evolution of $a$:

$$\frac{\partial u(a,\theta,t)}{\partial t}=-\alpha u(a,\theta,t)+\frac{\mu}{\pi}\int_0^{\pi}\int_{\mathbb{R}^2} w\langle a,\theta|a',\theta'\rangle\sigma(u(a',\theta',t))da'd\theta'+h(a,\theta,t)$$

where $\sigma$ is a non linear gain function (with $\sigma(0)=0$), $h$ an external input, $w\langle a,\theta|a',\theta'\rangle$ the weight of the connection between the neuron $v=(a,\theta)$ and the neuron $v'=(a',\theta')$, $\alpha$ a decay parameter ($\alpha$ can be taken = 1), and $\mu$ a parameter of excitability of $V1$. The increasing of $\mu$ models an increasing of the excitability of $V1$ due to the action of substances on the nuclei which produce specific neurotransmitters such as serotonin or noradrenalin. After having made the encoding, the authors suppose that there exist no external input ($h=0$) and that $\mu=0$: that is the subject is in a dark room and see nothing, his $V1$ activity being in its "ground state" (which can be very complex due to endogeneous activity, spontaneous noise, etc.). In the model, the "ground state" state is the homogeneous state $u\equiv 0$. It is stable and the activity $u$ measures therefore in fact the shift of the $V1$ state away from the "ground state" when $h=0$ but $\mu\neq 0$.

Now, the analysis of the evolution equation shows that, as the parameter $\mu$ increases, this initial activation state $u\equiv 0$ can become unstable and bifurcate for critical values $\mu_c$ of $\mu$. The bifurcations can be analyzed using classical methods:

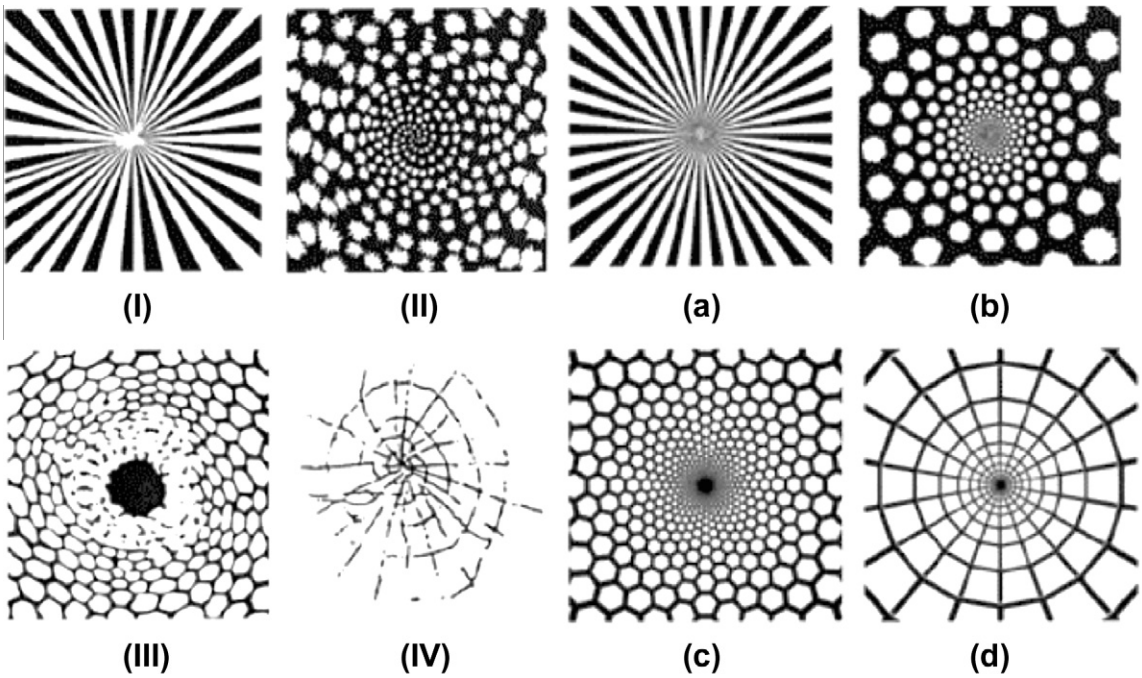


**Fig. 2.** Left (*I, II, III, IV*): visual hallucinations drawn by the neurophysiologist Heinrich Klüver in 1926 after ingestion of mescal. Right (*a,b,c,d*): neurogeometrical models of Klüver's data in Bressloff et al. [7].

- linearization of the equation near the solution $u \equiv 0$ and the critical value $\mu_c$;
- spectral analysis of the linearized equation;
- computation of its eigenvectors (eigenmodes);
- hypothesis of periodicity w.r.t. a lattice $\Lambda$ of $\mathbb{R}^2$.

The last step is to reconstruct from eigenmodes in $V1$ virtual retinal images in $\mathbb{R}^2$ by using the retinotpic map between the retina and $V1$. Fig. 2 shows how well the mathematical model (on the right) fits with the empirical data (on the left).

## 4. 1-jets and contact structure

It is therefore very important to understand better the neurogeometry of the functional architecture of $V1$. The first idea we introduced in the late 1990s was that the experimental results mean essentially that the *contact structure* of the fibration $\pi: R \times P \to R$ is neurally implemented.

### 4.1. The first model: the space of 1-jets of curves $\gamma$ in $\mathbb{R}^2$

Let us begin with the model $\pi_J : \mathbb{V}_J = \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ with $R = \mathbb{R}^2$ the visual plane and $P = \mathbb{R}$ the line of the tangent $p = \tan(\theta)$ of the orientations $\theta$. This fibration underlies the concept of *jet*, due to Charles Ehresmann, which generalizes the classical notion of Taylor expansion and confers it an intrinsic geometric meaning. Consider in $\mathbb{R}^2$ endowed with coordinates $(x,y)$ a smooth curve $\gamma$ that is the graph $\{x,f(x)\}$ of a real function $f$ on $\mathbb{R}$. The first order jet of $f$ at $x$, $j^1f(x)$, is characterized by three slots: the coordinate $x$, the value $y = f(x)$ of $f$ at $x$, and the value $p = f'(x)$ of the first derivative of $f$ at $x$. So, a 1-jet is nothing else than a pair $c = (a,p)$, that is a contact element. Conversely, to every pair $c = (a,p)$, one can associate the set of smooth functions $f$ whose graph is tangent to $c$ at $a$. $J^1(\mathbb{R},\mathbb{R})$ denotes the fiber bundle $\pi_J : \mathbb{V}_J = \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ of the 1-jets of smooth curves in $\mathbb{R}^2$.

One can give a more geometric version of the fibration $\pi_J$. In général, if $M$ is a smooth $n$-manifold, one can consider at every point $a$ of $M$ the set $C_aM$ of the hyperplanes of its tangent space $T_aM$. $C_aM$ is isomorphic to the projective space $\mathbb{P}^{n-1}$. The total space $CM$ gluing these fibers is called the *contact bundle* of $M$. In our case, $n = 2, M = \mathbb{R}^2$, and the hyperplanes are the lines of $\mathbb{R}^2$ through the origin 0. So $CM = \mathbb{R}^2 \times \mathbb{P}^1$. We will denote it by $\mathbb{V}_P$.

$\mathbb{V}_P$ is the compactification of the space of 1-jets $J^1(\mathbb{R},\mathbb{R}) = \mathbb{V}_J$ associated to the choice of coordinates $(x,y)$. To see this, it is enough to interpret the coordinate in its fibers $C_a\mathbb{R}^2$ in terms of $T_a\mathbb{R}^2$. Let $(\xi,\eta)$ be the coordinates of $T_a\mathbb{R}^2$ associated to the base $\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right)$. Then, over any open set which does not contain the "vertical" line $\xi = 0$, an admissible local coordinate for $C_a\mathbb{R}^2$ is $p = \frac{\eta}{\xi}$ and, in the neighborhood of $\xi = 0$, another admissible local coordinate is $q = \frac{\xi}{\eta}$. An element $c$ of $C\mathbb{R}^2 = \mathbb{V}_P$ is therefore individuated by coordinates $(x,y,p) = (a,p)$ or $(x,y,q) = (a,q)$ with $p,q \in \mathbb{P}^1$, and so $\mathbb{V}_P = \mathbb{R}^2 \times \mathbb{P}^1$.

The difference between $\mathbb{V}_P$ and $\mathbb{V}_J = J^1(\mathbb{R},\mathbb{R})$ is that the fiber of $J^1(\mathbb{R},\mathbb{R})$ is not the $\mathbb{P}^1$ of the angles $\theta$ (mod$\pi$) but the $\mathbb{R}$ of the values of $\tan(\theta)$, $\theta$ being measured w.r.t. the $x$-axis and remaining $\neq \frac{\pi}{2}$. The fiber $\mathbb{P}^1$ of $\mathbb{V}_P$ corresponds to that of $\mathbb{V}_J$ via the stereographic projection $\mathbb{P}^1 \to \mathbb{R}, \theta \mapsto \tan(\theta)$.

### 4.2. The functionality of jets

Jan Koenderink strongly emphasized the functional importance of the concept of jet. Without jets, it would be impossible to understand how the visual system could extract geometric features such as the tangent or the curvature of a curve. Indeed, neurons are "point processors" that can only measure the numerical value encoded in their firing rate. And it is impossible to compute *directly* differential features using only point processors. As Koenderink claimed,

> geometrical features become multilocal objects, i.e. in order to compute boundary curvature the processor would have to look at different positions simultaneously, whereas in the case of jets it could establish a format that provides the information by addressing a single location. Routines accessing a single location may aptly be called points processors, those accessing multiple locations array processors. The difference is crucial in the sense that point processors need no geometrical expertise at all, whereas array processors do (e.g. they have to know the environment or neighbours of a given location). ([17], p. 374).

The main functional interest of jet spaces is that they allow to implement differential features in networks of "point processors" provided a functional architecture is introduced.

The key idea is

1. to add *new independent variables* describing local features such as orientation;
2. to introduce an *integrability condition* enabling to integrate them into global structures.

Neurophysiologically, it comes to add differential feature detectors and to couple them via a functional architecture in order to ensure a binding process that integrates them.

It is the beginning of neurogeometry.

### 4.3. The contact structure and its integral curves

In the following, we will work in $\mathbb{V}_J$ with coordinates $(x,y,p) = (a,p), p = \tan(\theta) = \frac{dy}{dx}$ and, to have a bijection with $\mathbb{V}_P$ with coordinates $(x,y,\theta(\text{mod}\pi)) = (a,\theta(\text{mod}\pi))$, we allow the value $p = \pm\infty$ (i.e. $\theta = \pm\frac{\pi}{2}$).

If $\gamma$ is a parametrized smooth curve $a(s) = (x(s),y(s))$ in the base plane $\mathbb{R}^2$ (a visual contour), it can be lifted to $\mathbb{V}_J$. The lifting $\Gamma$ is the 1-jet map $j^1\gamma(a(s))$ of $\gamma$ that associates to $a(s) = (x(s),y(s))$ the contact element $(a(s),p_{a(s)})$ with $p_{a(s)} = \frac{y'(s)}{x'(s)}$ (the slope of the tangent to $\gamma$ at $a(s)$). So $\Gamma = \upsilon(s) = (a(s),p_{a(s)})$.

This lift $\Gamma$ – called in differential geometry a *Legendrian* lift – represents $\gamma$ as the *envelope* of its tangents: biological evolution of the visual system found out projective duality!.

To every smooth curve $\gamma$ in $\mathbb{R}^2$ is associated a skew curve $\Gamma$ in $\mathbb{V}_J$. But the converse is of course completely false. If

$$\Gamma = v(s) = (a(s), p(s)) = (x(s), y(s), p(s))$$

is a curve in $\mathbb{V}_J$, the projection $a(s) = (x(s),y(s))$ of $\Gamma$ is a curve $\gamma$ in $\mathbb{R}^2$. But $\Gamma$ is the Legendrian lift of $\gamma$ if and only if $p(s) = p_{a(s)}$. In other words, if $\Gamma$ is locally defined by equations $y = f(x)$, $p = g(x)$, there exists a curve $\gamma$ in $\mathbb{R}^2$ such that $\Gamma = j^1\gamma$ iff $g(x) = f'(x)$, that is iff $p = y'$.

This condition is called an *integrability condition*. Let us denote by

$$t = (a, p; \alpha, \pi) = (x, y, p; \xi, \eta, \pi)$$

the tangent vectors to $\mathbb{V}_J$ at the point $v = (a,p) = (x,y,p)$. Along $\gamma$ (we suppose $x$ is the independent variable) $t = (x,y,p; 1,y',p')$ and the integrability condition $p = y'$ means that we have in fact $t = (x,y,p; 1,p,p')$. It is straightforward to verify that this condition is equivalent to the fact that $t$ is in the kernel of the 1-form $\omega_J = dy - pdx$, $\omega_J = 0$ meaning simply that $p = \frac{dy}{dx}$. Indeed, to compute the value of a 1-form $\varpi$ on a tangent vector $t = (\xi,\eta,\pi)$ at $(x,y,p)$, one applies the rules $dx(t) = \xi$, $dy(t) = \eta$, $dp(t) = \pi$. So, if $t_i$ and $\varpi_i$ are the components of $t$ and $\varpi$ w.r.t. the bases of $T\mathbb{V}_J$ and $T^*\mathbb{V}_J$ associated to the coordinates $(x,y,p)$, one gets $\varpi(t) = \sum \varpi_i t_i$ and therefore, in our case, $\omega_J(t) = -p.1 + 1.p + 0.p' = -p + p = 0$ since $\omega_J = -pdx + 1.dy + 0.dp$.

So the kernel of the 1-form $\omega_J$ is the field $\mathscr{K}$ of *planes* $K_v$ – called the *contact planes* – with equation $-p\xi + \eta = 0$, and the tangent vectors $X_1 = \frac{\partial}{\partial x} + p\frac{\partial}{\partial x} = (\xi = 1, \eta = p, \pi = 0)$ and $X_2 = \frac{\partial}{\partial p} = (\xi = 0, \eta = 0, \pi = 1)$ are evident generators.

Now we can express purely geometrically the integrability condition: a curve $\Gamma$ in $\mathbb{V}_J$ is a Legendrian lift iff it is everywhere tangent to the field $\mathscr{K}$ of contact planes. The field $\mathscr{K}$ is called the *contact structure* of $\mathbb{V}_J$ and the Legendrian lifts $\Gamma$ are called its *integral curves*.

It must be emphasized that the vertical component $\pi = p'$ of the tangent vector of $\Gamma$ at $v = (a,p)$ is the *curvature* of the projection $\gamma$ at $a$. Indeed, $p = y'$ implies $p' = y''$ and therefore $\pi = p' = y''$.

### 4.4. SE(2)-invariance of the contact structure

The contact structure is invariant under the action of the special Euclidean group $SE(2)$ of rigid motions in the plane, which is the semi-direct product $SE(2) = \mathbb{R}^2 \rtimes SO(2)$ of the rotation group $SO(2)$ and the translation group $\mathbb{R}^2$. If $(p,r_\theta)$ is an element of $SE(2)$, it acts on a point $a$ of $\mathbb{R}^2$ by

$$(p, r_\theta)(a) = p + r_\theta(a).$$

If $(p,r_\theta)$ and $(q,r_\varphi)$ are two elements of $SE(2)$, their (non commutative) product is given by the formula:

$$(q, r_\varphi) \circ (p, r_\theta) = (q + r_\varphi(p), r_{\varphi+\theta}).$$

This product is noncommutative since $(p,r_\theta)\circ(q,r_\varphi) = (p + r_\theta(q),r_{\theta+\varphi})$ and in general $q + r_\varphi(p) \neq p + r_\theta(q)$.

The rotation $r_\theta$ acts on the fibration $\mathbb{V}_J$ by $r_\theta(a,p(\varphi)) = (r_\theta(a),p(\varphi + \theta))$, this very particular form of action expressing the fact that the alignment of preferential directions is invariant.

### 4.5. The non integrability of the contact structure

A fundamental point must be underlined concerning the contact structure $\mathscr{K}$. It is defined as the field of planes $v \in \mathbb{V}_J \mapsto K_v \subset T_v\mathbb{V}_J$ which are the kernels of the 1-form $\omega_J = dy - pdx$ and the Legendrian lifts are its integral curves. There exist therefore a lot of 1-dimensional integrals. But, nevertheless, there exist *no* 2-dimensional integrals, no *surfaces* $S$ of $\mathbb{V}_J$ which are tangent to $K_v$ at every point $v \in S$, i.e. such that $T_vS = K_v$. This is due to the fact that the field $K_v$ spins too rapidly with $p$ to be integrable: $K_v$ is the "vertical" plane above the "horizontal" line of slope $p$ and, when $p$ varies along the fiber $\mathbb{R}_a$ above $a$, it rotates with $p$ (see Fig. 3).

More precisely, the non integrability of $\mathscr{K}$ results from the violation of the *Frobenius integrability condition* saying that a 1-form $\varpi$ admits integral surfaces $S$ iff $\varpi \wedge d\varpi = 0$ (that is $d\varpi(t,t') = 0$ for all tangent vectors $t$ and $t'$ such that $\varpi(t) = \varpi(t') = 0$). This condition follows itself from the theorem saying that $\varpi$ is integrable iff for every local basis $\{t_1,t_2\}$ of the kernels $K_v$ of $\varpi$, the Lie bracket $[t_1,t_2]$ belongs

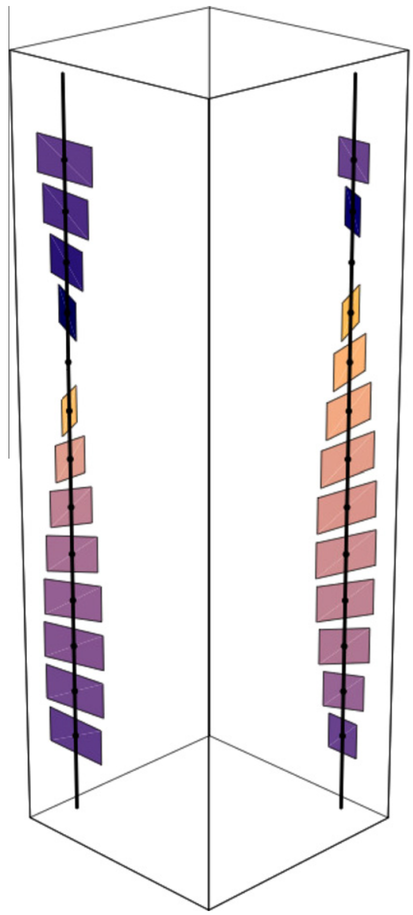

**Fig. 3.** The field of contact planes $K_v$ spins too rapidly to be integrable.

to $K_v$ or, in other words, iff $K_v$ is a Lie subalgebra of $T_v\mathbb{V}_J$. Now, for $\omega_J = dy - pdx$ one gets

$$\begin{aligned} d\omega_J &= -\left(\frac{\partial p}{\partial x}dx\wedge dx + \frac{\partial p}{\partial y}dy\wedge dx + \frac{\partial p}{\partial p}dp\wedge dx\right) + d^2y - pd^2x \\ &= -dp\wedge dx = dx\wedge dp \end{aligned}$$

and therefore

$$\begin{aligned} \omega_J\wedge d\omega_J &= (-pdx + dy)\wedge dx\wedge dp = dy\wedge dx\wedge dp \\ &= -dx\wedge dy\wedge dp. \end{aligned}$$

But this 3-form is a *volume* form of $\mathbb{V}_J$ and vanishes nowhere. By the way, for the basis

$$\left\{t_1 = \frac{\partial}{\partial x} + p\frac{\partial}{\partial y} = (1,p,0), t_2 = \frac{\partial}{\partial p} = (0,0,1)\right\}$$

of $K_v$ one has $[t_1,t_2] = t_3 = -\frac{\partial}{\partial y} = (0,-1,0)$ and the vector $t_3 = (0,-1,0) \notin K_v$ since it satisfies $-p\xi + \eta = -0 + (-1) = -1 \neq 0$.

The non-integrability of the contact structure $\mathscr{K}$ shows that $\mathscr{K}$ is functionally dedicated to the integration of *curves*. Moreover, it implies that the neurogeometry of the functional architecture of $V1$ is completely different from Euclidean geometry, even at the infinitesimal level, and is therefore even no Riemannian. We will see that it is in fact a typical *sub-Riemannian* geometry.

## 5. The contact structure as a group structure

A key point concerning the contact structure on $\mathbb{V}_J$ is that it is left-invariant for a noncommutative group structure which is isomorphic to the Heisenberg group and called the *polarized* Heisenberg group.

### 5.1. The polarized Heisenberg group

The product law is given by the formula:

$$(x,y,p)\cdot(x',y',p') = (x+x', y+y'+px', p+p').$$

It is straightforward to verify that this law is associative, that the origin $(0,0,0)$ of $\mathbb{V}_J$ is its neutral element, and that the inverse of $v = (x,y,p)$ is $v^{-1} = (-x,-y+px,-p)$. Due to the asymmetry of the coupling term $px'$, the product is noncommutative. $\mathbb{V}_J$ is a semi-direct product $\mathbb{V}_J = \mathbb{R}^2 \rtimes \mathbb{R}$. The base plane $\mathbb{R}^2$ of the fibration $\pi_J : \mathbb{V}_J = \mathbb{R}^2\times\mathbb{R} \to \mathbb{R}^2$ is the commutative subgroup of translations and the center $Z$ of $\mathbb{V}_J$ is the $y$-axis. Indeed, $v' = (x',y',p')$ commutes with all $v \in \mathbb{V}_J$ iff for every $v = (x,y,p)$ we have $px' = p'x$, which implies $x' = p' = 0$.

If $t = (\xi,\eta,\pi)$ are vectors of the Lie algebra $\mathscr{V}_J = T_0\mathbb{V}_J$ of $\mathbb{V}_J$, then $\mathscr{V}_J$ has Lie brackets

$$[t,t'] = [(\xi,\eta,\pi),(\xi',\eta',\pi')] = (0,\xi'\pi - \xi\pi',0)$$

and is generated as a Lie algebra by the basis of $K_v$: $X_1 = \frac{\partial}{\partial x} + p\frac{\partial}{\partial y} = (\xi = 1, \eta = p, \pi = 0)$ and $X_2 = \frac{\partial}{\partial p} = (\xi = 0, \eta = 0, \pi = 1)$ at $v = 0$. Indeed, at 0, $X_1 = (1,0,0)$, $X_2 = (0,0,1)$ and $[X_1,X_2] = (0,-1,0) = -X_3$ (the other brackets = 0).

It is essential to emphasize the fact that the basis $\{X_1,X_2\}$ of the field of planes $\mathscr{K}$ Lie-generates the whole tangent bundle $T^*\mathbb{V}_J$. $\mathscr{K}$ is said to be *bracket generating* or to satisfy *Hörmander condition*.

Computations in $\mathbb{V}_J$ become very easy if we use the matrix representation

$$v = (x,y,p) = \begin{pmatrix} 1 & p & y \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$t = (\xi,\eta,\pi) = \begin{pmatrix} 0 & \pi & \eta \\ 0 & 0 & \xi \\ 0 & 0 & 0 \end{pmatrix}.$$

Indeed, the product in $\mathbb{V}_J$ becomes the matrix product $v.v'$ and the Lie product in $\mathscr{V}_J$ becomes the commutator $[t,t'] = t.t' - t'.t$. Using this trick, let us summarize the basic elements of the theory of $\mathbb{V}_J$ as a Lie group. Consider first the left translation $L_v$ of $\mathbb{V}_J$ defined by $L_v(v') = v.v'$. It is a diffeomorphism of $\mathbb{V}_J$ whose tangent map at 0 is the linear map

$$\begin{aligned} &T_0L_v : \mathscr{V}_J = T_0\mathbb{V}_J \to T_v\mathbb{V}_J \\ &\qquad t = (\xi,\eta,\pi) \mapsto T_0L_v(t) = (\xi,\eta+p\xi,\pi). \end{aligned}$$

The matrix of $T_0L_v$ is

$$T_0L_v = \begin{pmatrix} 1 & 0 & 0 \\ p & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

This shows that the coordinate basis $\{\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial p}\}$ – also called an *holonomic* basis – of the tangent bundle $T\mathbb{V}_J$ associated to the coordinates $\{x,y,p\}$ is *not* left-invariant. This is the source of non holonomy. To get a left-invariant basis, we must translate via the $L_v$ the basis $\left\{\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial p}\right\}_0$ at 0. We get the *non holonomic* basis $\{\frac{\partial}{\partial x} + p\frac{\partial}{\partial y},\frac{\partial}{\partial y},\frac{\partial}{\partial p}\}$, that is $\{t_1,-t_3,t_2\} = \{X_1,X_3,X_2\}$ (remember that $[t_1,t_2] = t_3$, the other brackets vanish).

### 5.2. Left invariance of the contact structure

Now let $t$ be a vector of the contact plane $K_0$ at 0. Since $\eta = p\xi$ and $p = 0$, we have $\eta = 0$. Its translated $T_0L_v(t)$ is therefore $(\xi,p\xi,\pi)$, and since $\eta = p\xi$, $T_0L_v(t)$ is an element of the contact plane $K_v$ and the contact structure $\mathscr{K} = \{K_v\}$ is nothing else than the left-invariant field of planes left-translated from $K_0$. In fact, the 1-form $\omega_J$ itself is left-invariant. Indeed, at the origin $\omega_J = dy - pdx$ is simply $(\omega_J)_0 = dy$. If we put $(\omega_J)_0 = \omega_{J,0}$ and translate $\omega_{J,0}$ to the point $v$, we get $\omega_{J,v} = T_0L_v^*(\omega_{J,0})$ defined by the formula $\omega_{J,v}(t) = \omega_{J,0}\left(T_0L_v^{-1}(t)\right)$ for $t = (\xi,\eta,\pi) \in T_v\mathbb{V}_J$. But

$$T_0L_v^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -p & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$T_0L_v^{-1}(t) = \begin{pmatrix} 1 & 0 & 0 \\ -p & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \xi \\ \eta \\ \pi \end{pmatrix} = \begin{pmatrix} \xi \\ -p\xi+\eta \\ \pi \end{pmatrix}.$$

So

$$\omega_{J,v}(t) = dy(\xi, -p\xi + \eta, \pi) = -p\xi + \eta = dy(t) - pdx(t) = (\omega_J)_v(t).$$

### 5.3. Adjoint and coadjoint representations

The left-translation $L_v$ translates the situation at 0 to the equivalent situation at $v$. One can come back to 0 using the right-translation $R_{v^{-1}}$. One gets that way what is called an inner automorphism of the Lie group $\mathbb{V}_J$ (it is trivial to verify that it is a group morphism):

$$\begin{aligned} A_v : v' &\mapsto v.v'.v^{-1} \\ (x', y', p') &\mapsto (x', y' + px' - p'x, p') \end{aligned}$$

As 0 is a fix point of $A_v$, the tangent map $Ad_v = T_0A_v$ of $A_v$ at 0 is well defined and is an automorphism of the Lie algebra $\mathcal{V}_J = T_0\mathbb{V}_J$. We have the formula:

$$Ad_v = \begin{pmatrix} 1 & 0 & 0 \\ p & 1 & -x \\ 0 & 0 & 1 \end{pmatrix}$$

which yields what is called the *adjoint representation* of the Lie group $\mathbb{V}_J$ on its Lie algebra $\mathcal{V}_J$.

If, in a second step, we look at the tangent map of $Ad_v$ itself, we get a morphism of Lie algebra $ad_t$ from the Lie algebra $\mathcal{V}_J$ into the Lie algebra $\mathrm{End}(\mathcal{V}_J)$ (endomorphisms of $\mathcal{V}_J$) of the Lie group $\mathrm{Aut}(\mathcal{V}_J)$ (automorphisms of $\mathcal{V}_J$). If $t = (\xi, \eta, \pi) \in \mathcal{V}_J$, the matrix of $ad_t$ is

$$ad_t = \begin{pmatrix} 0 & 0 & 0 \\ \pi & 0 & -\xi \\ 0 & 0 & 0 \end{pmatrix}.$$

We have therefore $ad_t(t') = (0, \xi'\pi - \xi\pi', 0) = [t, t']$ and the Lie bracket of $\mathbb{V}_J$ can be retrieved from the adjoint representation.

To express diffusion along the contact structure, we will need the orbits of the adjoint representation. They are easy to compute. If $v = (x,y,p)$ varies in $\mathbb{V}_J$ and if $t = (\xi, \eta, \pi) \in \mathcal{V}_J$ is fixed, then $Ad_v(t) = (\xi, p\xi + \eta - x\pi, \pi)$ generates the line $\tilde{t} = (\xi, \mathbb{R}, \pi)$ if $\xi \neq 0$ or $\pi \neq 0$. When $\xi = \pi = 0$, then $Ad_v(t) = t$ and all elements $t = (0, \eta, 0)$ are fixed points: $\tilde{t} = \{t\}$.

It is straightforward to *dualize* these constructions. For the *co-adjoint representation*, take the basis $\{dx, dy, dp\}$ of the cotangent bundle $T^*\mathbb{V}_J$ of $\mathbb{V}_J$. At 0 we get the space $\mathcal{V}_J^*$ of 1-forms on $\mathcal{V}_J$ which is the dual vector space of $\mathcal{V}_J$. If $\varpi$ is a 1-form on $\mathcal{V}_J$, it can be written as $\varpi = \xi^*dx + \eta^*dy + \pi^*dp = (\xi^*, \eta^*, \pi^*)$.[2] If $t \in \mathcal{V}_J$, the convention is to denote $\langle \varpi, t \rangle$ the value $\varpi(t)$. The co-adjoint representation is then defined by the duality $\langle Ad_v^*(\varpi), t \rangle = \langle \varpi, Ad_{-v}(t) \rangle$. It is easy to see that it is a representation of the Lie group $\mathbb{V}_J$ on $\mathcal{V}_J^*$. As

$$\begin{aligned} \langle \varpi, Ad_{-v}(t) \rangle &= \Big\langle \xi^*dx + \eta^*dy + \pi^*dp, \\ &\qquad \xi\frac{\partial}{\partial x} + (-p\xi + \eta + x\pi)\frac{\partial}{\partial y}\pi\frac{\partial}{\partial p} \Big\rangle \\ &= \xi^*\xi + \eta^*(-p\xi + \eta + x\pi) + \pi^*\pi \\ &= (\xi^* - \eta^*p)\xi + \eta^*\eta + (\pi^* + \eta^*x)\pi, \end{aligned}$$

one get $Ad_v^*(\varpi) = (\xi^* - \eta^*p, \eta^*, \pi^* + \eta^*x)$. For $\eta^* \neq 0$, the orbits are the planes $(\mathbb{R}, \eta^*, \mathbb{R})$ and for $\eta^* = 0$, every point of the $(\xi^*, 0, \pi^*)$ plane is an orbit.

If we take the tangent map of the co-adjoint representation, we get the adjoint map $ad^*$ of the map $ad$. It is the morphism of Lie algebras from $\mathcal{V}_J$ into $\mathrm{End}(\mathcal{V}_J^*)$ defined by

$$\begin{aligned} ad_t^*(\varpi)(t') &= \langle ad_t^*(\varpi), t' \rangle = \langle \varpi, ad_{-t}(t') \rangle = \langle \varpi, -[t, t'] \rangle \\ &= \langle \xi^*dx + \eta^*dy + \pi^*dp, (0, -\xi'\pi + \xi\pi', 0) \rangle \\ &= \eta^*(-\xi'\pi + \xi\pi'). \end{aligned}$$

But as $\xi' = dx(t')$ and $\pi' = dp(t')$, we get $ad_t^*(\varpi) = (-\eta^*\pi, 0, \eta^*\xi)$.

## 6. Subjective contours and sub-Riemannian geometry

### 6.1. Kanizsa illusory contours

We now focus on a fundamental visual phenomenon, namely the construction by the visual system of long range *subjective contours* completing incomplete ones. Their existence is well known to the specialists of Gestalttheorie since the celebrated works of Kanizsa [16]. Their function is to complete lacunar sense data. We consider *curved* Kanizsa contours because their exact shape yields a fundamental information on the neural mechanisms of contour integration. Fig. 4 gives an example of a curved Kanizsa illusory contour.

That the functional architecture of $V1$ implements the contact structure of $\mathbb{V} = R \times P$ will enable us to understand the strange formation of long range subjective contours. Indeed, as a first approximation, we can consider that the pacmen define boundary conditions $(a,p)$ and $(b,q)$ and

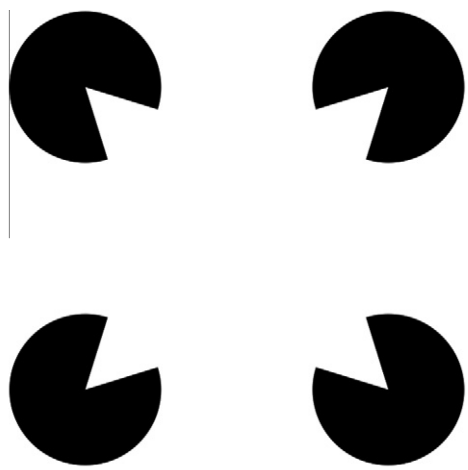

**Fig. 4.** An example of a curved Kanizsa illusory contour.

[2] We chose the notation $\xi^*$, etc. to maintain the relation between roman and greek letters while indicating the dual nature of a covector w.r.t. a tangent vector.

that the subjective contour is a solution to a *variational problem*.

### 6.2. A first sub-Riemannian model

In our neuro-geometrical framework, we can easily interpret the variational process giving rise to illusory contours. We still work in the fibration of 1-jets $\pi_J : \mathbb{V}_J = \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ idealizing $V1$. The pacmen define two contact elements $(a,p)$ and $(b,q)$ and an illusory contour interpolating between $(a,p)$ and $(b,q)$ is a skew curve $\Gamma$ in $\mathbb{V}_J$ from $(a,p)$ to $(b,q)$ which is at the same time:

1. a Legendrian lift of a curve $\gamma$ in the base plane $\mathbb{R}^2$, that is an integral curve of the contact structure, a curve satisfying the integrability condition $p(x) = y'(x)$;
2. "as straight as possible", that is "geodesic" in $\mathbb{V}_J$ for a certain metric, which means, since the variation of $p$ measures the curvature $\kappa$ of $\gamma$, to minimize at the same time the length and the curvature of the projection $\gamma$.

Such an idea has already been introduced, after pioneering works of Hoffman [14] and Ullman [26], by David Mumford in 1992 in his celebrated paper "Elastica and Computer Vision" [21], Euler's *elastica* being curves $\gamma$ in $\mathbb{R}^2$ minimizing at the same time the length and the integral of the squared curvature, that is an energy of the type

$$E = \int_\gamma (\alpha\kappa^2 + \beta)ds.$$

Mumford's idea was that a virtual contour constitutes a chain of contact elements $(a_i,p_i)$ along which the energy leakage is minimal. But there are two types of leakages:

- a leakage proportional to the number $N$ of elements in the chain;
- a leakage due to the curvature, which is the sum of the orientation deflections between successive elements. If $\theta_i$ is the angle of the slope $p_i$, one can take for instance $\sum_{i=1}^{i=N-1}(\theta_{i+1} - \theta_i)^2$.

At the limit $N \to \infty$, the first term tends towards the length $\int_\gamma ds$ in the Euclidean plane $\mathbb{R}^2$, while the second term tends towards the integral of the squared curvature $\int_\gamma \kappa^2 ds$ since $\kappa = \frac{d\theta}{ds}$. Minimizing the leakages leads therefore to the variational problem:

$$\min\left(\int_\gamma (\alpha\kappa^2 + \beta)ds\right)$$

which is well known in elasticity theory and comes back to Euler. Its solutions are transcendent curves (the *elastica*), which can be explicitly represented using elliptic functions.

We have reformulated Mumford's elastica model as a variational model in $\mathbb{V}_J$ and no longer in the Euclidean plane $\mathbb{R}^2$ (see e.g. [22] and [23]). The key idea was to use a *geodesic* model in a *sub-Riemannian* geometry naturally associated to the contact structure. Indeed, if $\mathcal{K}$ is the contact structure on $\mathbb{V}_J$ and if one considers only curves $\Gamma$ in $\mathbb{V}_J$ which are integral curves of $\mathcal{K}$, then metrics $g_{\mathcal{K}}$ defined only on the contact planes $K_v$ enable to compute the length of $\Gamma$. Such metrics are called sub-Riemannian metrics. They are extremely different from Riemannian ones. For an introduction to their theory, see e.g. [25] and [27].

As $\mathcal{K}$ is bracket generating and satisfies Hörmander condition, a celebrated theorem of Chow says that every pair of points $(v,v')$ of $\mathbb{V}_J$ can be connected by an integral curve of $\mathcal{K}$ (and therefore by a geodesic, which is a minimizing integral curve).

So the Kanizsa problem becomes to find an integral curve $\Gamma$ of $\mathcal{K}$ from $(a,p)$ to $(b,q)$ which is a geodesic for a natural sub-Riemannian metric on $\mathcal{K}$. It is quite difficult to solve. As claimed Richard Beals, Bernard Gaveau and Peter Greiner who solved it for the Heisenberg group ([4], p. 634):

> The results indicate how complicated a control problem can become, even in the simplest situation.

Strangely, a search for a solution is quite recent. In 1977 Gaveau still said ([12], p. 114):

> The variational problem is to minimize the energy of a curve in the base manifold under the Lagrange condition that its lifting is given in the fiber bundle. It seems that this problem has not yet been studied.

In the polarized Heisenberg group $\mathbb{V}_J = J^1(\mathbb{R},\mathbb{R})$ with coordinates $(x,y,p = \tan(\theta))$, product

$$(x,y,p).(x',y',p') = (x + x', y + y' + px', p + p')$$

and contact planes generated by $X_1 = \frac{\partial}{\partial x} + p\frac{\partial}{\partial y} = (1,p,0)$ and $X_2 = \frac{\partial}{\partial p} = (0,0,1)$ with Lie brackets $[X_1,X_2] = -X_3 = (0,-1,0) = -\frac{\partial}{\partial y}$ and 0, the geodesic problem can be easily formulated as a *control* problem. If $\Gamma = \{v(s)\}$ is a smooth parametrized curve in $\mathbb{V}_J$, to say that it is an integral curve of the contact structure is to say that $\dot{v}(s) = u_1X_1 + u_2X_2$ for appropriate controls $u_1$ and $u_2$ or, in other words, that

$$\dot{x} = u_1, \quad \dot{y} = pu_1, \quad \dot{p} = u_2,$$

the integrability condition $\frac{\dot{y}}{\dot{x}} = p$ being a priori satisfied.

To find the geodesics, we have to minimize the kinetic energy $\frac{1}{2}\|\dot{v}\|^2_{SR}$ along such curves for a well chosen sub-Riemannian metric $SR$ with scalar product $\langle\circ,\circ\rangle_{SR}$ and norm $\|\circ\|_{SR}$. Using Legendre transform, this Lagrangian can be transformed into the Hamiltonian

$$h(v,\varpi) = \langle\varpi, \dot{v}\rangle - \frac{1}{2}\|\dot{v}\|^2_{SR}$$
$$= \varpi(u_1X_1 + u_2X_2) - \frac{1}{2}\|u_1X_1 + u_2X_2\|^2_{SR}.$$

If $\varpi = \xi^*dx + \eta^*dy + \pi^*dp = (\xi^*,\eta^*,\pi^*)$, then

$$h(v,\varpi) = \xi^*u_1 + \eta^*u_1p + \pi^*u_2$$
$$- \frac{1}{2}\left(u_1^2\|X_1\|^2_{SR} + 2u_1u_2\langle X_1,X_2\rangle_{SR} + u_2^2\|X_2\|^2_{SR}\right).$$

It is natural to choose a *left-invariant* metric, and namely the sub-Riemannian metric $SR_J$ making $\{X_1,X_2\}$an orthonormal basis of the contact plane $K_v$ since $\{X_1,X_2\}$ is the left-invariant basis translating the standard Euclidean orthonormal basis of $K_0$. It must be strongly emphasized that this metric is not the Euclidean metric $\langle\circ,\circ\rangle_E$, $\|\circ\|_E$. Due to non holonomy, Euclidean metric is not left-invariant. By the way, even if $\|X_2\|_E = 1$ and $\langle X_1,X_2\rangle_E = 0$ as in the Euclidean case, $\|X_1\|_E = 1 + p^2 \neq 1$ if $p \neq 0$. It is only on the $(x,y)$ plane that the two metrics are the same.

If we chose $SR_J$, then $\|X_1\|_{SR_J} = \|X_2\|_{SR_J} = 1$ and $\langle X_1, X_2\rangle_{SR_J} = 0$, and $h(v,\varpi)$ simplifies:

$$h(v,\varpi) = \xi^* u_1 + \eta^* u_1 p + \pi^* u_2 - \frac{1}{2}\left(u_1^2 + u_2^2\right).$$

One can then apply a fundamental result of control theory called the Pontryagin *maximum principle*, which generalizes the classical method of variational calculus using Euler–Lagrange equations and Lagrange multipliers (see [3]). We have a family of Hamiltonians $h_u(v,\varpi) = \varpi(u_1X_1 + u_2X_2) - \frac{1}{2}\left(u_1^2 + u_2^2\right), h_u : T^*\mathbb{V}_J \to \mathbb{R}$. We want to find the controls $u_j$ that maximize them and solve the corresponding Hamilton equations. These controls $u_{j,\max}$ satisfy $\frac{\partial h}{\partial u_j} = 0$. Let $H = \mathrm{Sup}_u(h_u)$ be the Hamiltonian $h_{u_{\max}}$. Then the projections on $\mathbb{V}_J$ of the trajectories of $H$ are trajectories "in optimal time". As emphasized by Gamkrelidze and Agrachev [3]:

> The maximum principle is a first order optimality condition, an elaboration of the classical Lagrange multipliers rule, where $\varpi(t)$ plays the role of the Lagrange multiplier.[3]

In our case, "optimal time" means "minimal length" and we can express the sub-Riemannian geodesic problem in $\mathbb{V}_J$ as an Hamiltonian problem defined on the cotangent bundle $T^*\mathbb{V}_J$. The maximization conditions are

$$\begin{cases} \frac{\partial h_u}{\partial u_1} = \varpi(X_1) - u_1 = 0 \\ \frac{\partial h_u}{\partial u_2} = \varpi(X_2) - u_2 = 0 \end{cases}$$

and therefore $u_{1,\max} = \varpi(X_1)$ and $u_{2,\max} = \varpi(X_2)$. So

$$\begin{aligned} H(v,\varpi) &= u_1\varpi(X_1) + u_2\varpi(X_2) - \frac{1}{2}\left(u_1^2 + u_2^2\right) = \frac{1}{2}\left(u_1^2 + u_2^2\right) \\ &= \frac{1}{2}\left(\langle\varpi, X_1\rangle^2 + \langle\varpi, X_2\rangle^2\right) \end{aligned}$$

and in terms of coordinates:

$$H(x,y,p,\xi^*,\eta^*,\pi^*) = \frac{1}{2}\left[(\xi^* + p\eta^*)^2 + \pi^{*2}\right].$$

The structure of geodesics implies that the sub-Riemannian sphere $S$ (the ends of the geodesics starting at 0 and of sub-Riemannian length = 1 that are *global* minimizers) and the wave front $W$ (the ends of geodesics starting at 0 and of sub-Riemannian length = 1 that are not necessarily global minimizers) are rather strange. We will compute them explicitly adapting computations already done by Beals, Gaveau, and Greiner in [4] for the Heisenberg group. We will see that a fundamental feature of this geometry, compared with Euclidean geometry, is that the *cut locus* of 0 (that is the ends of geodesics that stop being globally minimizing), and the *conjugate locus* or *caustic* of 0 (that is the singular locus of the exponential map $\mathscr{E}$ integrating geodesics) are rather complex.

### 6.3. Carnot groups and fractality of sub-Riemannian geometry

Before integrating Hamilton equations, let us strongly emphasize the fact that the sub-Riemannian metric $g_{\mathscr{K}}$ is extremely anisotropic and inhomogeneous, singular and even fractal. Indeed, $\mathbb{V}_J$ is of dimension 3, but its Hausdorff dimension w.r.t. $g_{\mathscr{K}}$ is 4!.

We meet here a very deep geometrical property shared by a special class of groups – called *Carnot groups* – which are (simply connected) Lie groups $N$, nilpotent and stratified, whose Lie algebra $\mathscr{N}$ decomposes into a direct sum $\mathscr{N} = \oplus_1^m \mathscr{V}_i$, $m < n$ = dimension of $N$, of vector sub-spaces $\mathscr{V}_i$ satisfying $\mathscr{V}_{i+1} = [\mathscr{V}_1, \mathscr{V}_i]$ and $[\mathscr{V}_1, \mathscr{V}_m] = 0$. $\mathscr{V}_1$ generates therefore through left-translations a field $\mathscr{K}$ of tangent subspaces which is bracket generating. In our case, $\mathscr{V}_J = \mathscr{V}_1 \oplus \mathscr{V}_2$ with $\mathscr{V}_1 = \mathrm{Span}\{X_1, X_2\} = K_0$ and $\mathscr{V}_2 = \mathrm{Span}\{X_3\}$. The number $m$ is called the "step" (or degree of non-holonomy) of $N$ and

$$D_h = \sum_1^m i\dim(\mathscr{V}_i) \neq n = \sum_1^m \dim(\mathscr{V}_i)$$

is called the *homogeneous dimension* of $N$. In our case, $D_h = 1 \times 2 + 2 \times 1 = 4$ while $n = 3$. In the formula giving $D_h$, the factor $i$ in front of $\dim(\mathscr{V}_i)$ means that the components of $\mathscr{V}_i$ are of "weight" $i$, which explains the strong anisotropy of the Carnot groups $N$. In our case, the variables $x,p$ are of weight 1, and the variable $y$ of weight 2, $pdx$ is of weight $1 + 1 = 2$, $dy$ of weight 2, and the contact 1-form $\omega = dy - pdx$ is therefore homogeneous of weight 2.

The strong anisotropy of Carnot groups is reflected in their metric structure. For instance, a celebrated "ball-box theorem" due to Gromov (see [13]) says that the ball $B(0,r)$ of center 0 and radius $r$ is encapsulated into boxes Box $(r)$ with $k$-th side of order $r^i$ if the $k$-axis belongs to $\mathscr{V}_i$: there exist two constants $c$ and $C$ such that for every $r > 0$

$$\mathrm{Box}(cr) \subset B(0,r) \subset \mathrm{Box}(Cr)$$

This theorem explains why the Hausdorff dimension of $N$ is $D_h = \sum_1^m i\dim(V_i)$ since the (Euclidean) volume of $B(0,r)$ is of order $r^{D_h}$. Intuitively, it means that the infinitesimal balls $B(0,\varepsilon)$ are flattened against $\mathscr{V}_1$: they are of order $\varepsilon$ along $\mathscr{V}_1$ but of ordre $\varepsilon^i$ along the directions of homogeneity degree $i > 1$. Even if, according to Chow theorem, any point can be reached from 0 by an integral curve of $\mathscr{K}$, these curves are tangent to $\mathscr{K}$ and can go in another direction only using Lie brackets.

We will now construct explicitly the sphere $S(0,r)$ and the wave front $W$ of $\mathbb{V}_J$.

### 6.4. Geodesics of the polarized Heisenberg group (a variant of Beals, Gaveau, Greiner computations)

The Hamilton equations on $T^*\mathbb{V}_J$ derived from the Hamiltonian

$$H(x,y,p,\xi^*,\eta^*,\pi^*) = \frac{1}{2}\left[(\xi^* + p\eta^*)^2 + \pi^{*2}\right]$$

[3] A supplementary second order condition is needed to warrant that the so defined extrema are "good" maxima. It leads to the theory of Jacobi curves.

are the following

$$\begin{cases} \dot{x}(s) = \frac{\partial H}{\partial \xi^*} = \xi^* + p\eta^* \\ \dot{y}(s) = \frac{\partial H}{\partial \eta^*} = p(\xi^* + p\eta^*) = p\dot{x}(s) \\ (\text{i.e. } p = \frac{\dot{y}}{\dot{x}} = \frac{dy}{dx}, \text{integrability condition}) \\ \dot{p}(s) = \frac{\partial H}{\partial \pi^*} = \pi^* \\ \dot{\xi}^*(s) = -\frac{\partial H}{\partial x} = 0 \\ \dot{\eta}^*(s) = -\frac{\partial H}{\partial y} = 0 \\ \dot{\pi}^*(s) = -\frac{\partial H}{\partial p} = -\eta^*(\xi^* + p\eta^*) = -\eta^*\dot{x}(s). \end{cases}$$

As $H$ is independent from $x$ and $y$, the derivatives $\dot{\xi}^*(s) = -\frac{\partial H}{\partial x}$ and $\dot{\eta}^*(s) = -\frac{\partial H}{\partial y}$ vanish and the moments $\xi^*$ and $\eta^*$ are therefore constant along any geodesic: $\xi^* = \xi_0^*$ and $\eta^* = \eta_0^*$. This fact simplifies the equations since $\dot{x}(s) = \xi_0^* + p\eta_0^*, \dot{y}(s) = p(\xi_0^* + p\eta_0^*)$, and $\dot{\pi}^*(s) = -\eta_0^*(\xi_0^* + p\eta_0^*)$. We emphasize the relations $\ddot{p} = \dot{\pi}^* = -\eta^*\dot{x}$ and $\ddot{x} = \eta^*\dot{p}$, or $(\ddot{x}, \ddot{p}) = \eta_0^*(\dot{p}, -\dot{x})$, which mean that in the $(x,p)$ plane the acceleration is orthogonal to the velocity and geodesics are circles whose radius increases when $\eta_0^*$ decreases (at the limit $\eta_0^* = 0$ the circle becomes a straight line). By the way, $H(x, y, p, \xi^*, \eta^*, \pi^*) = \frac{1}{2}(\dot{x}^2 + \dot{p}^2)$ and the Hamiltonian $H$ is therefore the kinetic energy of the projection of the trajectories on the $(x,p)$ plane.

For structuring computations, we introduce, as Beals et al. did in [4] for the Heisenberg group, the complex variable $z = (x,p)$ in the contact plane $K_0$, the complex moment $\chi^* = (\xi^*,\pi^*)$, and the new variable $\varsigma^* = (\xi^* + p\eta^*, \pi^*) = \chi^* + \eta^*\Lambda(z) = (\dot{x}, \dot{p})$, with

$$\Lambda = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \Lambda^T = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$-\Lambda^T + \Lambda = J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$J^2 = -\mathbf{1}.$$

We verify that $2H$ is the norm of $\varsigma^*$:

$$H(z, y, \chi^*, \eta^*) = \frac{1}{2}\langle \varsigma^*, \varsigma^* \rangle = \frac{1}{2}\|\varsigma^*\|^2 = H_0 = \frac{1}{2}\langle \varsigma^*(0), \varsigma^*(0) \rangle = \frac{1}{2}\|\varsigma^*(0)\|^2$$

and that Hamilton equations become:

$$\begin{cases} \dot{z}(s) = \frac{\partial H}{\partial \chi^*} = \varsigma^*(s) \\ \dot{y}(s) = \frac{\partial H}{\partial \eta^*} = p(\xi^* + p\eta^*) = \langle \varsigma^*, \mathbf{\Lambda}(z) \rangle \\ \dot{\chi}^*(s) = -\frac{\partial H}{\partial z} = (0, -\eta^*(\xi^* + p\eta^*)) = -\eta^*\mathbf{\Lambda}^T(\varsigma^*) \\ \dot{\eta}^*(s) = 0, \quad \eta^* = \text{cst} = \eta_0^*. \end{cases}$$

The differential equation for $\varsigma^*$ is:

$$\begin{aligned} \dot{\varsigma}^*(s) &= \dot{\chi}^*(s) + \eta^*\Lambda\dot{z}(s) \ (\text{since } \eta^* = \text{cste}) \\ &= -\eta^*\mathbf{\Lambda}^T\varsigma^*(s) + \eta^*\Lambda\varsigma^*(s) = \eta^*J\varsigma^*(s) \\ (\ddot{x}, \ddot{p}) &= \eta^*J(\dot{x}, \dot{p}), \end{aligned}$$

$J$ being the complex multiplication by $i$ in the $z$-plane. The solution is (the exponential being, as $\Lambda$, a $2 \times 2$ matrix acting on $\varsigma^*(0)$):

$$\varsigma^*(s) = e^{-s\eta^*J}\varsigma^*(0).$$

This integration of $\varsigma^*$ yields explicit formulae for geodesics.[4]

For $\eta^* = 0$, we have $\dot{\varsigma}^* = \dot{\chi}^* = \mathbf{0}$ and therefore $\chi^* = \chi_0^* = (\xi_0^*, \pi_0^*)$ and $\varsigma^* = \varsigma_0^*$ are constant. The variable $z = s\chi_0^* + z_0$ moves along a straight line in the $(x,p)$ plane and we get

$$\begin{cases} x(s) = s\xi_0^* + x_0 \\ p(s) = s\pi_0^* + p_0 \\ \dot{y}(s) = \langle \varsigma^*, \mathbf{\Lambda}(z) \rangle = \langle \varsigma_0^*, \Lambda(s\chi_0^* + z_0) \rangle = s\xi_0^*\pi_0^* + \xi_0^*p_0 \\ y(s) = \frac{1}{2}s^2\xi_0^*\pi_0^* + s\xi_0^*p_0 + y_0. \end{cases}$$

If $z_0 = 0$, then $z = s\chi_0^*$ and $y(s) = \frac{1}{2}s^2\xi_0^*\pi_0^* + y_0$. If moreover $y_0 = 0$, then $y(s) = \frac{1}{2}s^2\xi_0^*\pi_0^*$.

For $\eta^* \neq 0$ $(\eta^* = \text{cst} = \eta_0^*)$, the situation is of course more complex. Suppose that $z_0 = 0$. Then, $\dot{z}(s) = \varsigma^*(s)$ gives

$$\begin{aligned} z(s) &= \int_0^s \varsigma^*(r)dr = \int_0^s e^{-r\eta_0^*J}\varsigma^*(0)dr = \frac{-\varsigma^*(0)}{J\eta_0^*}\left(e^{-s\eta_0^*J} - 1\right) \\ &= \frac{J}{\eta_0^*}(\varsigma^*(s) - \varsigma^*(0)). \end{aligned}$$

This value provides explicit formulae for the geodesics connecting in time $\tau$ an origin $(x_0 = 0, y_0, p_0 = 0)$ to an end point $(x_1 = x(\tau), y_1 = y(\tau), p_1 = p(\tau))$. Indeed,

$$z(\tau) = z_1 = \frac{J}{\eta_0^*}\varsigma^*(0)\left(e^{-\tau\eta_0^*J} - 1\right)$$

gives $\varsigma^*(0)$ and then $\chi_0^* = (\xi_0^*, \pi_0^*)$. If $e^{-i\tau\eta_0^*} - 1 \neq 1$, that is if $\tau\eta_0^* \neq 2k\pi$ (if $\tau\eta_0^* = 2k\pi$ then $z(\tau) = z_0$), we get

$$\begin{aligned} z(s) &= \frac{e^{-s\eta_0^*J} - 1}{e^{-\tau\eta_0^*J} - 1}z_1 = \frac{e^{-is\eta_0^*} - 1}{e^{-i\tau\eta_0^*} - 1}z_1 = \frac{e^{-i\frac{s}{2}\eta_0^*}\left(e^{-i\frac{s}{2}\eta_0^*} - e^{i\frac{s}{2}\eta_0^*}\right)}{e^{-i\frac{\tau}{2}\eta_0^*}\left(e^{-i\frac{\tau}{2}\eta_0^*} - e^{i\frac{\tau}{2}\eta_0^*}\right)}z_1 \\ &= e^{i\frac{(\tau-s)}{2}\eta_0^*}\frac{\sin\left(\frac{s}{2}\eta_0^*\right)}{\sin\left(\frac{\tau}{2}\eta_0^*\right)}z_1. \end{aligned}$$

In other words, we get for the $(x,p)$ part of the geodesics the formulae:

$$\begin{cases} x(s) = \frac{\sin\left(\frac{s}{2}\eta_0^*\right)}{\sin\left(\frac{\tau}{2}\eta_0^*\right)}\left(\cos\left(\frac{(\tau-s)}{2}\eta_0^*\right)x_1 - \sin\left(\frac{(\tau-s)}{2}\eta_0^*\right)p_1\right) \\ p(s) = \frac{\sin\left(\frac{s}{2}\eta_0^*\right)}{\sin\left(\frac{\tau}{2}\eta_0^*\right)}\left(\sin\left(\frac{(\tau-s)}{2}\eta_0^*\right)x_1 + \cos\left(\frac{(\tau-s)}{2}\eta_0^*\right)p_1\right). \end{cases}$$

These are the equations of a circle in the $z = (x,p)$ plane. Indeed, we have

$$x^2 + p^2 - x(x_1 + p_1\cot(\frac{\eta_0^*\tau}{2})) - p(p_1 - x_1\cot(\frac{\eta_0^*\tau}{2})) = 0$$

which is the equation of a circle passing through 0 and $z_1$ and whose center is

$$x_c = \frac{1}{2}\left(x_1 + p_1\cot\left(\frac{\eta_0^*\tau}{2}\right)\right), \quad y_c = \frac{1}{2}\left(p_1 - x_1\cot\left(\frac{\eta_0^*\tau}{2}\right)\right)$$

and the radius

$$r^2 = \frac{1}{4}(x_1^2 + p_1^2)\left(1 + \cot\left(\frac{\eta_0^*\tau}{2}\right)\right) = \frac{1}{4\sin^2\left(\frac{\eta_0^*\tau}{2}\right)}|z_1|^2$$

[4] The main difference between Beals–Gaveau–Greiner computations and ours is that in the Heisenberg group case, $\Lambda$ is antisymmetric and $-\Lambda^T + \Lambda = 2\Lambda$.

We notice that the equality $H = H_0 = \frac{1}{2}\langle \varsigma^*(0), \varsigma^*(0)\rangle$ implies that the constant value of the Hamiltonian along a trajectory is:

$$H_0 = \frac{1}{2}\|\varsigma^*(0)\|^2 = \frac{\eta_0^{*2}}{8\sin^2\left(\frac{\eta_0^*\tau}{2}\right)}|z_1|^2 = \frac{\eta_0^{*2}}{2}r^2.$$

Indeed,

$$z_1 = \frac{i}{\eta_0^*}e^{-i\frac{\tau}{2}\eta_0^*}\left(e^{-i\frac{\tau}{2}\eta_0^*} - e^{i\frac{\tau}{2}\eta_0^*}\right)\varsigma^*(0) = \frac{2}{\eta_0^*}e^{-i\frac{\tau}{2}\eta_0^*}\sin\left(\frac{\tau}{2}\eta_0^*\right)\varsigma^*(0).$$

For $y(s)$, computations are more involved. We have $\dot{y}(s) = \langle \varsigma^*, \mathbf{\Lambda}(z)\rangle = p(\xi^* + p\eta^*)$ and we already know $\varsigma^*(s)$ and $z(s)$. Using the given values $x_1, p_1, \eta_0^*$ and $\tau$ and the classical integrals:

$$\begin{cases} \int_0^s \cos(\eta^* r)dr = \frac{\sin(\eta^* s)}{\eta^*} \\ \int_0^s \sin(\eta^* r)dr = \frac{1-\cos(\eta^* s)}{\eta^*} \\ \int_0^s \cos^2(\eta^* r)dr = \frac{2\eta^* s + \sin(2\eta^* s)}{4\eta^*} \\ \int_0^s \sin^2(\eta^* r)dr = \frac{2\eta^* s - \sin(2\eta^* s)}{4\eta^*} \\ \int_0^s \cos(\eta^* r)\sin(\eta^* r)dr = \frac{\sin^2(\eta^* s)}{2\eta^*} \end{cases}$$

we get

$$\begin{aligned} y(s) - y_0 = \frac{1}{8(\cos(\eta_0^*\tau) - 1)}&\left[-2\eta_0^* s(x_1^2 + p_1^2)\right. \\ &- 4x_1p_1\cos\left(\eta_0^*(s-\tau)\right) + 2\left(x_1^2 - p_1^2\right)\sin\left(\eta_0^*(s-\tau)\right) \\ &+ 2x_1p_1\cos\left(\eta_0^*(2s-\tau)\right) - \left(x_1^2 - p_1^2\right)\sin\left(\eta_0^*(2s-\tau)\right) \\ &+ 2x_1p_1\cos\left(\eta_0^*\tau\right) + \left(x_1^2 - p_1^2\right)\sin\left(\eta_0^*\tau\right) \\ &\left.+ 2\left(x_1^2 + p_1^2\right)\sin\left(\eta_0^* s\right)\right] \end{aligned}$$

In terms of $\xi_0^*, \pi_0^*, \eta_0^*$ and $\tau$, the formula writes:

$$\begin{aligned} y(s) - y_0 = {} & \xi_0^{*2}\frac{2\eta_0^* s + \sin\left(2\eta_0^* s\right)}{4\eta_0^{*2}} - \xi_0^{*2}\frac{\sin\left(\eta_0^* s\right)}{\eta_0^{*2}} + \xi_0^*\pi_0^* \\ & \times \frac{\sin^2\left(\eta_0^* s\right)}{\eta_0^{*2}} - \xi_0^*\pi_0^*\frac{1 - \cos\left(\eta_0^* s\right)}{\eta_0^{*2}} + \pi_0^{*2} \\ & \times \frac{2\eta_0^* s - \sin\left(2\eta_0^* s\right)}{4\eta_0^{*2}} \end{aligned}$$

and we notice that, if we expand the expression to the third order, we retrieve the limit $y(s) - y_0 = \frac{1}{2}\xi_0^*\pi_0^* s^2$ when $\eta_0^* \to 0$.

These equations explain the origin of the *multiplicity* of sub-Riemannian geodesics connecting two points. Let us indeed compute $y_1 - y_0 = y(\tau) - y_0$. As $\cos\left(\eta_0^*\tau\right) - 1 = -2\sin^2\left(\frac{\eta_0^*\tau}{2}\right)$, we find

$$\begin{aligned} y_1 - y_0 &= \frac{-1}{16\sin^2\left(\frac{\eta_0^*\tau}{2}\right)}\left[-2\eta_0^*\tau(x_1^2 + p_1^2)\right. \\ &\quad - 4x_1p_1 + 2x_1p_1\cos\left(\eta_0^*\tau\right) - \left(x_1^2 - p_1^2\right)\sin\left(\eta_0^*\tau\right) \\ &\quad + 2x_1p_1\cos\left(\eta_0^*\tau\right) + \left(x_1^2 - p_1^2\right)\sin\left(\eta_0^*\tau\right) \\ &\quad \left.+ 2\left(x_1^2 + p_1^2\right)\sin\left(\eta_0^*\tau\right)\right] \\ &= \frac{1}{8\sin^2\left(\frac{\eta_0^*\tau}{2}\right)}\left[2x_1p_1\left(1 - \cos\left(\eta_0^*\tau\right)\right)\right. \\ &\quad \left.+ \left(x_1^2 + p_1^2\right)\left(\eta_0^*\tau - \sin(\eta_0^*\tau)\right)\right] \\ &= \frac{1}{2}x_1p_1 + \frac{x_1^2 + p_1^2}{4}\left[\frac{\left(\frac{\eta_0^*\tau}{2}\right)}{\sin^2\left(\frac{\eta_0^*\tau}{2}\right)} - \frac{\cos\left(\frac{\eta_0^*\tau}{2}\right)}{\sin\left(\frac{\eta_0^*\tau}{2}\right)}\right] \end{aligned}$$

If we introduce the new variable $\varphi = \frac{\eta_0^*\tau}{2}$, we see that we must solve the equation

$$4\left(y_1 - y_0 - \frac{1}{2}x_1p_1\right) = \mu(\varphi)\|z_1\|^2$$

where $\mu(\varphi)$ is the function

$$\mu(\varphi) = \frac{\varphi}{\sin^2(\varphi)} - \cot(\varphi)$$

It is the function $\mu(\varphi)$ which is the key of the strange behavior of sub-Riemannian geodesics. $\mu(\varphi)$ is an odd function that diverges for $\varphi = k\pi$ $(k \neq 0)$ (i.e. $\eta_0^*\tau = 2k\pi$, we already met this condition) and presents critical points when $\varphi = \tan(\varphi)$. But when $\varphi = \tan(\varphi)$, we have

$$\mu(\varphi) = \frac{\tan(\varphi)}{\sin^2(\varphi)} - \cot(\varphi) = \frac{1 - \cos^2(\varphi)}{\cos(\varphi)\sin(\varphi)} = \tan(\varphi) = \varphi$$

and the minima of $\mu(\varphi)$ are on the diagonal. The graph of $\mu(\varphi)$ is represented at Fig. 5.

To get geodesics connecting $(0, y_0, 0)$ to the same end point $(x_1 = x(\tau), y_1 = y(\tau), p_1 = p(\tau))$, we must solve the equation $4(y_1 - y_0 - \frac{1}{2}x_1p_1) = \mu(\varphi)\|z_1\|^2$ in $\varphi = \frac{\eta_0^*\tau}{2}$ (we can take $\eta_0^* = 2$ and $\tau = \varphi$). For instance, for $y_0 = 0$, $x_1 = 2$, $p_1 = 4$, $y_1 = 104$, we find $\mu(\varphi) = 20$, which gives 11 solutions: $\varphi_1 = 2.737$, $\varphi_2 = 3.552$, $\varphi_3 = 5.695$, $\varphi_4 = 6.885$, $\varphi_5 = 8.680$, $\varphi_6 = 10.195$, $\varphi_7 = 11.672$, $\varphi_8 = 13.506$, $\varphi_9 = 14.656$, $\varphi_{10} = 16.845$, $\varphi_{11} = 17.608$.

We can now display the strange structure of the sub-Riemannian sphere $S$ and wave front $W$. We must first compute the length of geodesics. Let $\gamma$ be a geodesic starting at 0 and ending at time $\tau$ at $(x_1, y_1, p_1) = (z_1, y_1)$. If $L$ is its length, we have $L = \int_0^\tau \ell ds$ with $\ell^2 = (\xi^* + p\eta^*)^2 + \pi^{*2}$ the squared norm of $\dot{\gamma}$ in the contact plane endowed with the orthonormal basis $\{X_1 = \partial_x + p\partial_y, X_2 = \partial_p\}$. But $\ell^2 = 2H = 2H_0$ since the Hamiltonian is constant along its trajectories, and we know that

$$H_0 = \frac{\eta_0^{*2}}{8}\frac{1}{\sin^2\left(\frac{\eta_0^*\tau}{2}\right)}\|z_1\|^2.$$

So, with $\frac{\eta_0^*\tau}{2} = \varphi$,

$$L = \sqrt{2}\left(\frac{\eta_0^*\tau}{2}\right)\frac{1}{\left|\sin\left(\frac{\eta_0^*\tau}{2}\right)\right|}\|z_1\| = \sqrt{2}\frac{\varphi}{|\sin(\varphi)|}\|z_1\|$$

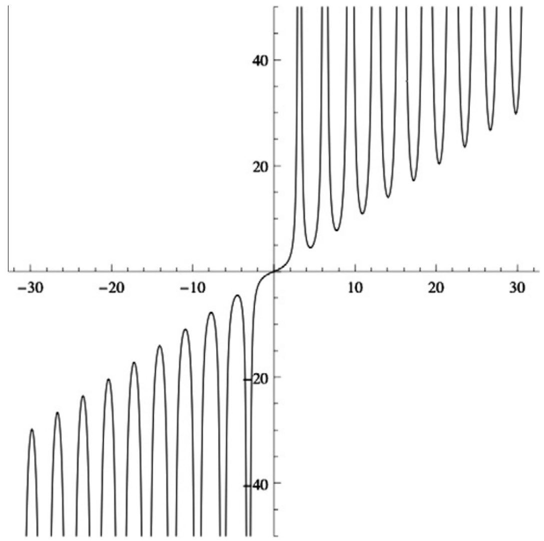


**Fig. 5.** The function $\mu(\varphi)$ occuring in the construction of sub-Riemannian geodesics of the polarized Heisenberg group $\mathbb{V}_J$ (the scale of the two axes are not the same).

In the sub-Riemannian geometry of $\mathbb{V}_J$, the sphere $S$ and the wave front $W$ with radius $\sqrt{2}$ are given by the fundamental equation

$$\|z_1\| = \frac{|\sin(\varphi)|}{\varphi}.$$

We get therefore

$$\begin{cases} x_1 = \frac{|\sin(\varphi)|}{\varphi}\cos(\theta) \\ p_1 = \frac{|\sin(\varphi)|}{\varphi}\sin(\theta) \\ y_1 = \frac{1}{2}x_1p_1 + \frac{\varphi - \sin(\varphi)\cos(\varphi)}{4\varphi^2} = \frac{\varphi + 2\sin^2(\varphi)\cos(\theta)\sin(\theta) - \cos(\varphi)\sin(\varphi)}{4\varphi^2} \end{cases}$$

We show at Figs. 6 and 7 images of $S$ and $W$. The external surface is the sub-Riemannian sphere $S$. The internal part is $W - S$. It presents smaller and smaller circles of cusp singularities which converge towards 0. Such a complex behavior is impossible in Riemannian geometry.

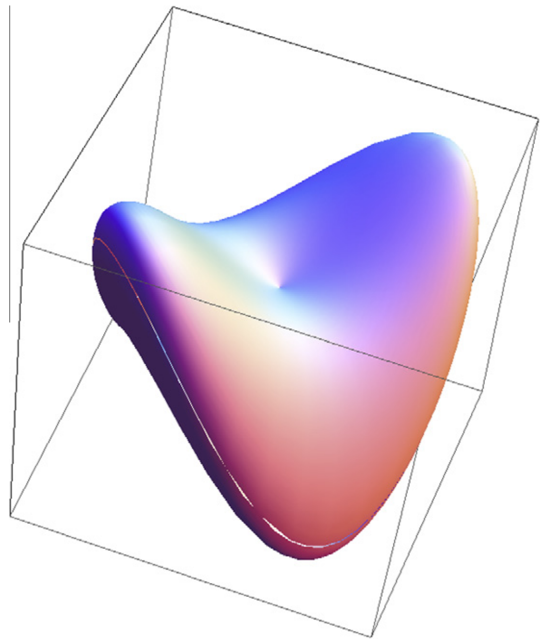

**Fig. 6.** The sub-Riemannian sphere $S$. It has a saddle form with singularities at the intersections with the $y$-axis. The white line corresponds to $\varphi = 0$.

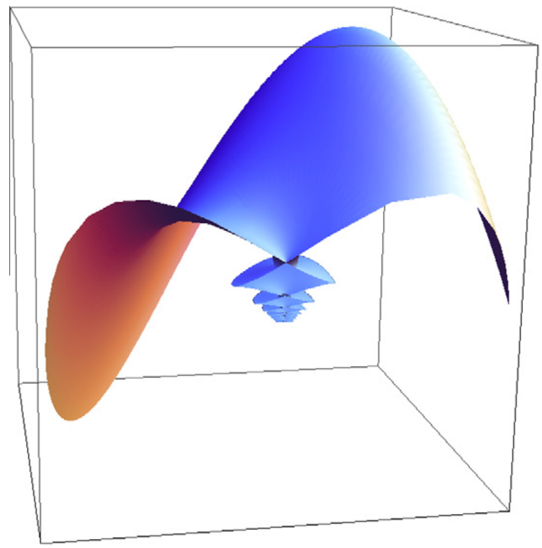

**Fig. 7.** A part of the sub-Riemannian wave front $W$. The external surface is the sub-Riemannian sphere $S$. The internal part is $W - S$. It presents smaller and smaller circles of cusp singularities which converge towards 0.

## 7. Irreducible representations and noncommutative harmonic analysis

### *7.1. Unirreps and Stone-von Neumann theorem*

We have presented the sub-Riemannian geometry of $\mathbb{V}_J$ which is left-invariant under its group structure. But such a geometry acts as an infrastructure for dynamical processes of transport along geodesics such as *diffusion*. It is therefore relevant to study the *heat kernel* on $\mathbb{V}_J$. As the central tool is *noncommutative harmonic analysis* and noncommutative Fourier transform, we must first define the *dual* $\mathbb{V}_J^*$ of the group $\mathbb{V}_J$, that is the set of its unitary irreducible representations (unirreps).

According to a variant of a celebrated result concerning the Heisenberg group and called the Stone–von Neumann theorem, every unirrep of the polarized Heisenberg group $\mathbb{V}_J$ in an Hilbert space which is not trivial on its center $Z$ (the $y$-axis) is equivalent to a Schrödinger representation $\rho_\lambda(x,y,p)$ acting on the infinite dimensional Hilbert space $\mathcal{H} = L^2(\mathbb{R},\mathbb{C})$ via

$$\rho_\lambda(x,y,p)u(s) = e^{i\lambda(y+xs)}u(s+p), \text{ with } \lambda \neq 0 \text{ and } u(y+s) \text{ an element of } \mathcal{H} = L^2(\mathbb{R},\mathbb{C}).$$

For $\lambda = 0$ these unirreps degenerate into trivial representations of dimension 1 called "characters":

$$\rho_{\mu,\nu}(x,y,p)u(s) = e^{i(\mu x+\nu p)}u(s).$$

We must notice the relation between these unirreps and the orbits of the coadjoint representation of $\mathbb{V}_J$ described above: the planes $(\mathbb{R},\eta^*,\mathbb{R})$ for $\eta^* \neq 0$ and every point of the $(\xi^*,0,\pi^*)$ plane for $\eta^* = 0$. In fact, it is a necessity due to a deep theorem of Kirillov.

### 7.2. Agrachev–Boscain–Gauthier–Rossi theorem

As shown in 2008 by Agrachev et al. [2], in the case of a unimodular Lie group $G$ of dimension 3 endowed with a left-invariant sub-Riemannian geometry, one can use the noncommutative generalized Fourier transform (GFT) defined on the dual $G^*$ of $G$ to compute the heat kernel associated to the *hypoelliptic* Laplacian $\Delta_{\mathscr{K}} = X_1^2 + X_2^2$ equal to the sum of squares of the generators $\{X_1,X_2\}$ of the field of planes $\mathscr{K}$. The Laplacian is hypoelliptic due to the fact that $\mathscr{K}$ is bracket generating and satisfies Hörmander condition.

We have seen that the unirreps of $\mathbb{V}_J$ into the groups of automorphisms $\mathscr{U}(\mathscr{H})$ are parametrized by $\lambda$:

$$\begin{aligned}\rho_\lambda : \mathbb{V}_J &\to \mathscr{U}(\mathscr{H})\\ v &\mapsto \rho_\lambda(v) : \mathscr{H} \to \mathscr{H}\\ &\qquad u(s) \mapsto e^{i\lambda(y+xs)}u(s+p).\end{aligned}$$

There exists a measure on $\mathbb{V}_J^*$, called the *Plancherel measure*, given by $dP(\lambda) = \lambda d\lambda$, which enables to make integrations. To compute the Fourier transform of the sub-Riemannian Laplacian $\Delta_{\mathscr{K}}$, we have to look at the action of the differential of the unirreps on the left-invariant vector fields $X$ on $\mathbb{V}_J$, which are given by the left translation of vectors $X(0)$ of the Lie algebra $\mathscr{V}_J$ of $\mathbb{V}_J$. By definition,

$$d\rho_\lambda : X \to d\rho_\lambda(X) := \left.\frac{d}{dt}\right|_{t=0}\rho_\lambda(e^{tX})$$

and we get the Fourier transform $\widehat{X_i}^\lambda = d\rho_\lambda(X_i)$.

It is easy to apply these formulae to our case.

$$\begin{aligned}&X_1(0) = (1,0,0)\\ &e^{tX_1} = (t,0,0)\\ &\rho_\lambda(e^{tX_1})u(s) = e^{i\lambda ts}u(s)\\ &\widehat{X_1}^\lambda u(s) = d\rho_\lambda(X_1)u(s) = \left.\frac{d}{dt}\right|_{t=0}\rho_\lambda(e^{tX_1})u(s) = \left.\frac{d}{dt}\right|_{t=0}e^{i\lambda ts}u(s) = i\lambda s u(s),\\ &X_2(0) = (0,0,1)\\ &e^{tX_2} = (0,0,t)\\ &\rho_\lambda(e^{tX_2})u(s) = u(s+t)\\ &\widehat{X_2}^\lambda u(s) = d\rho_\lambda(X_2)u(s) = \left.\frac{d}{dt}\right|_{t=0}\rho_\lambda(e^{tX_2})u(s) = \left.\frac{d}{dt}\right|_{t=0}u(s+t) = \frac{du(s)}{ds}.\end{aligned}$$

The GFT of the sub-Riemannian Laplacian is therefore the Hilbert sum (integral on $\lambda$ with the Plancherel measure $dP(\lambda) = \lambda d\lambda$) of the $\widehat{\Delta_{\mathscr{K}}}^\lambda$ with

$$\widehat{\Delta_{\mathscr{K}}}^\lambda u(s) = ((\widehat{X_1}^\lambda)^2 + (\widehat{X_2}^\lambda)^2)u(s) = \frac{d^2u(s)}{ds^2} - \lambda^2 s^2 u(s)$$

This equation is nothing else than the equation of the *harmonic oscillator*.

The heat kernel is then given by the integral

$$P(v,t) = \int_{\mathbb{V}_J^*} Tr(e^{t\widehat{\Delta_{\mathscr{K}}}^\lambda}\rho_\lambda(v))dP(\lambda), t \geqslant 0.$$

If the $\widehat{\Delta_{\mathscr{K}}}^\lambda$ have discrete spectrum and a complete set of normalized eigenfunctions $\{u_n^\lambda\}$ with eigenvalues $\{\alpha_n^\lambda\}$ then

$$P(v,t) = \int_{\mathbb{V}_J^*}\left(\sum_n e^{\alpha_n^\lambda t}\langle u_n^\lambda, \rho_\lambda(v)(u_n^\lambda)\rangle\right)dP(\lambda), \quad t \geqslant 0.$$

It is the case here. The eigenfunctions of the harmonic oscillator are well known and satisfy:

$$\frac{d^2u_n^\lambda(s)}{ds^2} - \lambda^2 s^2 u_n^\lambda(s) = \alpha_n^\lambda u_n^\lambda(s)$$

with $\alpha_n^\lambda = -\frac{2n+1}{\lambda}$. They are essentially the Hermite functions scaled by $\lambda$:

$$u_n^\lambda(s) = (2^n n!\sqrt{\pi})^{-\frac{1}{2}}\lambda^{\frac{1}{4}}e^{-\lambda\frac{s^2}{2}}H_n(\sqrt{\lambda}s)$$

$H_n$ being the $n$th Hermite polynomial.

In place of giving now an example of sub-Riemannian diffusion we prefer to present a second model, more natural and richer, of the neurogeometry of the first visual area $V1$.

## 8. The *SE*(2) model

### 8.1. A natural hypothesis

With Alessandro Sarti and Giovanna Citti, we emphasized the fact that it is more natural to avoid any privileged $x$-axis and to work using the fibration $\pi_S : \mathbb{V}_S = \mathbb{R}^2 \times \mathbb{S}^1 \to \mathbb{R}^2$ with the contact form $\omega_S = -\sin(\theta)dx + \cos(\theta)dy$, which is $\cos(\theta)(dy - pdx) = \cos(\theta)\omega_J$ (see [9] and [24]). The contact planes are spanned by the tangent vectors $X_1 = \cos(\theta)\partial_x + \sin(\theta)\partial_y$ and $X_2 = \partial_\theta$ with Lie bracket $[X_1,X_2] = \sin(\theta)\partial_x - \cos(\theta)\partial_y = -X_3$. Contrary to the polarized Heisenberg case, the $X_j$ constitute an Euclidean orthonormal basis and are therefore more natural. The distribution $\mathscr{K}$ of contact planes is still bracket generating (Hörmander condition) and maximally non integrable since

$$d\omega_S = \cos(\theta)dx \wedge d\theta + \sin(\theta)dy \wedge d\theta$$

and $\omega_S \wedge d\omega_S = -dx \wedge dy \wedge d\theta$ is a volume form which cannot vanish. The Frobenius condition $\omega_S \wedge d\omega_S = 0$ is still not satisfied and there exists no integral surface of $\mathscr{K}$ in $\mathbb{V}_S$ (but there exist a lot integral curves of $\mathscr{K}$: all the Legendrian lifts $\Gamma$ in $\mathbb{V}_S$ of curves $\gamma$ in the base plane $\mathbb{R}^2$). As for the vector field $X_3$, it is called the "characteristic field" (or Reeb field) of the contact field $K_v$. It is orthogonal to $K_v$ for the Euclidean metric and defines a *scale* through

$$\begin{aligned}\omega_S(X_3) &= (-\sin(\theta)dx + \cos(\theta)dy)(X_3) = \sin^2(\theta) + \cos^2(\theta)\\ &= 1.\end{aligned}$$

The two contact structures on $\mathbb{V}_J = \mathbb{R}^2 \times \mathbb{R}$ and $\mathbb{V}_S = \mathbb{R}^2 \times \mathbb{S}^1$ seem to be alike but are nevertheless very different. Indeed, let us look at their respective Lie algebras. For $\mathbb{V}_J$ we have the algebra generated by

$\{t_1, t_2, t_3\}\left(t_1 = \frac{\partial}{\partial x} + p\frac{\partial}{\partial y}, t_2 = \frac{\partial}{\partial p}, t_3 = -\frac{\partial}{\partial y}\right)$ with $[t_1,t_2] = t_3$ and $[t_1,t_3] = [t_2,t_3] = 0$. This is a *nilpotent* algebra. On the contrary, for $\mathbb{V}_S$ we have the algebra generated by $\{X_1,X_2,X_3\}$ ($X_1 = \cos(\theta)\partial_x + \sin(\theta)\partial_y, X_2 = \partial_\theta, X_3 = -{}_y, X_2 = \partial_\theta, X_3 = -\sin(\theta)\partial_x + \cos(\theta)\partial_y$) with $[X_1,X_2] = -X_3$, $[X_1,X_3] = 0$ and $[X_2,X_3] = -X_1$. Due to the non vanishing of $[X_2,X_3]$, this Lie algebra is not nilpotent. Nevertheless, we can notice that for small $\theta$, we have at first order $p \sim \theta$, $\sin(\theta) \sim \theta$ and $\cos(\theta) \sim 1$, and so $\omega_S = -\sin(\theta)dx + \cos(\theta)dy$ can be approximated by $\omega = -\theta dx + dy$ which is nothing else than the 1-form $\omega_J = dy - pdx$. $\mathbb{V}_J$ is in some sense "tangent" to $\mathbb{V}_S$. In fact it is called the *tangent cone* of $\mathbb{V}_S$. For this important concept and its role in sub-Riemannian geometry, see e.g. [5,19].

As a Lie group, the group $\mathbb{V}_S = \mathbb{R}^2 \times \mathbb{S}^1$ is in fact isomorphic to the special Euclidean group $SE(2) = \mathbb{R}^2 \rtimes SO(2)$. Its product is given by the formula

$$\begin{pmatrix} x_1 \\ y_1 \\ \theta_1 \end{pmatrix} \cdot \begin{pmatrix} x_2 \\ y_2 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2\cos(\theta_1) - y_2\sin(\theta_1) \\ y_1 + x_2\sin(\theta_1) + y_2\cos(\theta_1) \\ \theta_1 + \theta_2 \end{pmatrix}$$

The origin $(0,0,0)$ is the neutral element and the inverse of $(x,y,\theta)$ is

$$(-x\cos(\theta) - y\sin(\theta), x\sin(\theta) - y\cos(\theta), -\theta)$$

The Lie algebra of $\mathbb{V}_S$ is the vector space $\mathscr{V}_S = T_0\mathbb{V}_S \simeq \mathbb{R}^3$ endowed with the Lie bracket:

$$[t, t'] = [(\xi, \eta, \tau), (\xi', \eta', \tau')] = (-\tau\eta' + \eta\tau', \tau\xi' - \xi\tau', 0)$$

The left-translation $L_v$ defined by $L_v(v') = v.v'$ is a diffeomorphism of $\mathbb{V}_S$ whose tangent map at 0 is the linear map:

$$\begin{aligned} T_0L_v : T_0\mathbb{V}_S \to T_v\mathbb{V}_S \qquad & t = (\xi, \eta, \tau) \mapsto T_0L_v(t) \\ &= (\xi\cos(\theta) - \eta\sin(\theta), \xi\sin(\theta) + \eta\cos(\theta), \tau). \end{aligned}$$

In the basis $(x,y,\theta)$, the matrix of $T_0L_v$ is therefore

$$T_0L_v = \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} r_\theta & 0 \\ 0 & 1 \end{pmatrix}$$

with $r_\theta$ the rotation of angle $\theta$ in the $(x,y)$ plane. One verifies immediately that the basis $\{\partial_x,\partial_y,\partial_\theta\}_0$ of $T_0\mathbb{V}_S$ is translated at $v$ by $T_0L_v$ on the basis of $T_v\mathbb{V}_S$

$$\{\cos(\theta)\partial_x + \sin(\theta)\partial_y = X_1, -\sin(\theta)\partial_x + \cos(\theta)\partial_y = X_3, \partial_\theta = X_2\}_v$$

which shows that the basis $\{X_1,X_2,X_3\}$ is left-invariant.

In the same way, if we left-translate the value at 0, $\omega_{S,0} = dy$, of the 1-form $\omega_S = -\sin(\theta)\, dx + \cos(\theta)dy$, we get, since $\omega_{S,v} = T_0L_v^*(\omega_{S,0})$ is defined by $\omega_{S,v}(t') = \omega_{S,0}(T_0L_v^{-1}(t'))$ for $t' = (\xi', \eta', \tau') \in T_v\mathbb{V}_S$, since

$$T_0L_v^{-1} = \begin{pmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and since

$$\begin{aligned} T_0L_v^{-1}(t') &= \begin{pmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \xi' \\ \eta' \\ \tau' \end{pmatrix} \\ &= \begin{pmatrix} \xi'\cos(\theta) + \eta'\sin(\theta) \\ -\xi'\sin(\theta) + \eta'\cos(\theta) \\ \tau' \end{pmatrix}, \end{aligned}$$

the left-invariant 1-form

$$\begin{aligned} \omega_v(t') = dy(T_0L_v^{-1}(t')) &= -\xi'\sin(\theta) + \eta'\cos(\theta) \\ &= -\sin(\theta)dx(t') + \cos(\theta)dy(t'), \end{aligned}$$

which is nothing else than $\omega_S$.

Coming back from $v$ to 0 using the right-translation $R_{v^{-1}}$, we get the inner automorphism of $\mathbb{V}_S$:

$$A_v : v' \mapsto v \cdot v' \cdot v^{-1}$$

$$v \cdot v'.v^{-1} = \begin{pmatrix} x + (x'\cos(\theta) - x\cos(\theta')) - (y'\sin(\theta) - y\sin(\theta')) \\ y + (x'\sin(\theta) - x\sin(\theta')) + (y'\cos(\theta) - y\cos(\theta')) \\ \theta' \end{pmatrix}.$$

Of course 0 is a fix point of $A_v$ and the tangent map $Ad_v = T_0A_v$ of $A_v$ at 0 is an automorphism of the Lie algebra $\mathscr{V}_S = T_0\mathbb{V}_S$, which defines the adjoint representation. We have

$$Ad_v = \begin{pmatrix} \cos(\theta) & -\sin(\theta) & y \\ \sin(\theta) & \cos(\theta) & -x \\ 0 & 0 & 1 \end{pmatrix}$$

Taking again the tangent map, we get a morphism of Lie algebras, $ad_t$, from the Lie algebra $\mathscr{V}_S$ into the Lie algebra End($\mathscr{V}_S$) of the automorphism group Aut($\mathscr{V}_S$). If $t = (\xi, \eta, \tau) \in \mathscr{V}_S = T_0\mathbb{V}_S$, the matrix of $ad_t$ is

$$ad_t = \begin{pmatrix} 0 & -\tau & \eta \\ \tau & 0 & -\xi \\ 0 & 0 & 0 \end{pmatrix}.$$

We retrieve the Lie bracket through $ad_t(t') = (-\tau\eta' + \eta\tau', \tau\xi' - \xi\tau', 0) = [t,t']$.

### 8.2. The SE(2) sub-Riemannian geodesics (Agrachev)

After some discussions with us, Citti and Sarti on the special Euclidean group endowed with the sub-Riemannian metric making $\{X_1,X_2\}$ an orthonormal basis of $K_v$, Andrei Agrachev and his group at the SISSA of Trieste (Sachkov, Boscain, Moiseev) solved the problem of geodesics (see [1] and [20]).

The formulation of the problem in terms of control theory yields the differential system:ce:display>

$$\begin{cases} \dot{x} = u_1\cos(\theta) \\ \dot{y} = u_1\sin(\theta) \\ \dot{\theta} = u_2 \end{cases}$$

**Remark 1.** Mumford's elastica model corresponds to the particular case $\dot{x} = \cos(\theta), \dot{y} = \sin(\theta), \dot{\theta} = \kappa =$ curvature. □

Agrachev applies again Pontryagin maximum principle, starting with the kinetic energy defined on the tangent bundle $T\mathbb{V}_S$

$$\frac{1}{2}\dot{\nu}^2 = \frac{1}{2}\left((u_1\cos(\theta))^2 + (u_1\sin(\theta))^2 + u_2^2\right) = \frac{1}{2}\left(u_1^2 + u_2^2\right)$$

and takes the Legendre transform defined on the cotangent bundle $T^*\mathbb{V}_S$:

$$h(\nu,\varpi) = \langle\varpi,\dot{\nu}\rangle - \frac{1}{2}\dot{\nu}^2,$$

$\varpi$ being a covector with components $\varpi = (\xi^*,\eta^*,\vartheta^*)$ in the basis $dx$, $dy$, $d\theta$ of $T^*\mathbb{V}_S$. One verifies immediately that

$$h(\nu,\varpi) = \langle\varpi, u_1X_1(\nu) + u_2X_2(\nu)\rangle - \frac{1}{2}\left(u_1^2 + u_2^2\right).$$

According to the maximum principle, we get the Hamiltonian of geodesics by maximizing $h(\nu,\varpi)$ w.r.t. controls $u_1$ and $u_2$. We make $\frac{\partial h}{\partial u_1} = \frac{\partial h}{\partial u_2} = 0$, whose solutions are

$$\begin{cases} u_1(\nu,\varpi) = \langle\varpi, X_1(\nu)\rangle = \xi^*\cos(\theta) + \eta^*\sin(\theta) \\ u_2(\nu,\varpi) = \langle\varpi, X_2(\nu)\rangle = \vartheta^*, \end{cases}$$

the $u_i(\nu,\varpi) = \langle\varpi,X_i(\nu)\rangle$ being the natural coordinates on the cotangent bundle $T^*\mathbb{V}_S$, together with

$$u_3(\nu,\varpi) = \langle\varpi, X_3(\nu)\rangle = -\xi^*\sin(\theta) + \eta^*\cos(\theta)$$

The Hamiltonian for geodesics on $T^*\mathbb{V}_S$ is then

$$\begin{aligned} H(\nu,\varpi) &= \frac{1}{2}\left(u_1^2 + u_2^2\right) = \frac{1}{2}\left(\langle\varpi, X_1(\nu)\rangle^2 + \langle\varpi, X_2(\nu)\rangle^2\right) \\ &= \frac{1}{2}\left(\varpi_1^2 + \varpi_2^2\right) = \frac{1}{2}\left((\xi^*\cos(\theta) + \eta^*\sin(\theta))^2 + \vartheta^{*2}\right) \end{aligned}$$

where $\{\varpi_1,\varpi_2,\varpi_3\}$ are the components of the covector $\varpi$ in the dual basis of $\{X_1,X_2,X_3\}$.

Hence the Hamilton equations:

$$\begin{cases} \dot{x} = \frac{\partial H}{\partial \xi^*} = \xi^*\cos^2(\theta) + \eta^*\cos(\theta)\sin(\theta) \\ \dot{y} = \frac{\partial H}{\partial \eta^*} = \eta^*\sin^2(\theta) + \xi^*\cos(\theta)\sin(\theta) \\ \dot{\theta} = \frac{\partial H}{\partial \vartheta^*} = \vartheta^* \\ \dot{\xi}^* = -\frac{\partial H}{\partial x} = 0 \\ \dot{\eta}^* = -\frac{\partial H}{\partial y} = 0 \\ \dot{\vartheta}^* = -\frac{\partial H}{\partial \theta} = (\xi^*\cos(\theta) + \eta^*\sin(\theta))(-\xi^*\sin(\theta) + \eta^*\cos(\theta)) \end{cases}$$

The sub-Riemannian geodesics are the projections of the $H$-trajectories on $\mathbb{V}_S$. As $\xi^*$ and $\eta^*$ are constant, write them $(\xi^*,\eta^*) = \rho e^{i\beta}$. Then

$$\dot{\vartheta}^* = \frac{1}{2}\rho^2\sin(2(\theta-\beta))$$

and the constant Hamiltonian $H = \frac{1}{2}(\rho^2\cos^2(\theta-\beta) + \vartheta^{*2})$ yields the energy first integral:

$$\rho^2\cos^2(\theta-\beta) + \vartheta^{*2} = c$$

(with $c = 1$ if $H = \frac{1}{2}$) and the ODE for $\dot{\theta}$ ($c$, $\rho$ and $\beta$ are constant):

$$\dot{\theta}^2 = \vartheta^{*2} = c - \rho^2\cos^2(\theta-\beta).$$

For $\beta = 0$ (rotation invariance), the equations become:

$$\begin{cases} \dot{x} = \rho\cos^2(\theta) \\ \dot{y} = \rho\cos(\theta)\sin(\theta) = \frac{1}{2}\rho\sin(2\theta) \\ \dot{\theta} = \vartheta^* \\ \ddot{\theta} = \dot{\vartheta}^* = \frac{1}{2}\rho^2\sin(2\theta) \end{cases}$$

For $\rho = 1$, $2\theta = \pi - \mu$, and $\mu = 2\varphi = \pi - 2\theta$, we get a *pendulum equation* $\ddot{\mu} = -\sin(\mu)$ with first integral $\dot{\varphi}^2 + \sin^2(\varphi)$. As

$$dt = \pm\frac{1}{\sqrt{c}}\frac{d\varphi}{\sqrt{1 - \frac{1}{c}\sin^2(\varphi)}}$$

the system can be explicitly integrated using elliptic functions. The computation of the sub-Riemannian sphere and wave front are extremely involved (see [20]).

### 8.3. The SE(2) heat kernel

In [2], Agrachev, Boscain, Gauthier and Rossi computed the heat kernel for $\mathbb{V}_S$, the hypoelliptic Laplacian being again $\Delta_{\mathscr{K}} = X_1^2 + X_2^2$. The sub-Riemannian diffusion on $\mathbb{V}_S$ is highly anisotropic since it is restricted to an angular diffusion of $\theta$ and a spatial diffusion only along the $X_1$ direction. It is strongly constrained by the "good continuation" Gestalt law and its difference with classical (Euclidean) diffusion is striking. The Fig. 8, due to Jean-Paul Gauthier, starts with the image of an eye masked by a white grid and applies the diffusion until the grid has vanished. In spite of this very important diffusion the geometry of the image remains quite excellent. The Fig. 9 shows how bad would be a classical diffusion.

To compute the sub-Riemannian diffusion, the authors use the noncommutative generalized Fourier transform (GFT). The dual $\mathbb{V}_S^*$ of $\mathbb{V}_S$ is this time the set of unitary irreducible representations (unirreps) of $\mathbb{V}_S$ in the Hilbert space $\mathscr{H} = L^2(\mathbb{S}^1,\mathbb{C})$, and these unirreps are parametrized by a positive real $\lambda$:

$$\begin{aligned} &\rho_\lambda : \mathbb{V}_S \to \mathscr{U}(\mathscr{H}) \\ &\qquad \nu \mapsto \rho_\lambda(\nu) : \mathscr{H} \to \mathscr{H} \\ &\qquad\qquad \psi(\theta) \mapsto e^{i\lambda(x\sin(\theta)+y\cos(\theta))}\psi(\theta+\alpha) \end{aligned}$$

The Plancherel measure on $\mathbb{V}_S^*$ is still $dP(\lambda) = \lambda d\lambda$ and to compute the Fourier transform of the sub-Riemannian Laplacian we have to look at the action of the differential of the unirreps on the left-invariant vector fields $X$. As previously explained,

$$d\rho_\lambda : X \to d\rho_\lambda(X) := \left.\frac{d}{dt}\right|_{t=0}\rho_\lambda(e^{tX})$$

and we have $\widehat{X_i}^\lambda = d\rho_\lambda(X_i)$.

It is easy to apply these formulae to the $\mathbb{V}_S = SE(2)$ case.

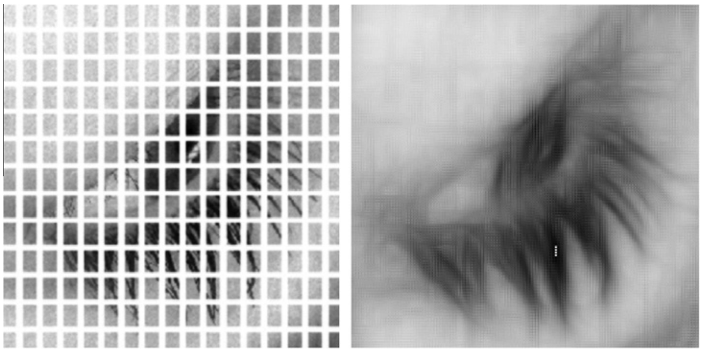

**Fig. 8.** Diffusion in $\mathbb{V}_S$ according to Jean-Paul Gauthier. The initial image is an eye masked by a white grid. Sub-Riemannian diffusion is applied until the grid has vanished. In spite of this very important diffusion the geometry of the image remains quite excellent.

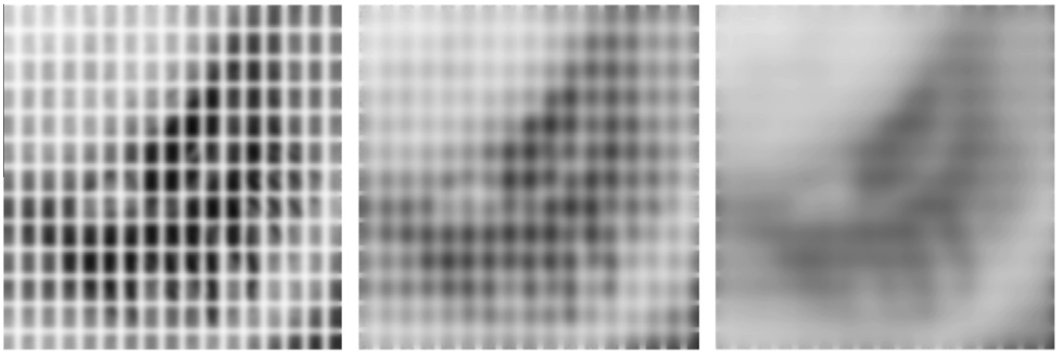

**Fig. 9.** To make the grid of Fig. 8 disappear using classical isotropic diffusion (Gaussian blurring), the image has to be completely blurred.

$$
\begin{aligned}
&X_1(0) = (1,0,0)\\
&e^{tX_1} = (t,0,0)\\
&\rho_\lambda(e^{tX_1})\psi(\theta) = e^{i\lambda t\sin(\theta)}\psi(\theta),\\
&\widehat{X_1}^{\lambda}\psi(\theta) = d\rho_\lambda(X_1)\psi(\theta) = \left.\frac{d}{dt}\right|_{t=0}\rho_\lambda(e^{tX_1})\psi(\theta)\\
&= \left.\frac{d}{dt}\right|_{t=0} e^{i\lambda t\sin(\theta)}\psi(\theta) = i\lambda\sin(\theta)\psi(\theta)\\
&X_2(0) = (0,0,1)\\
&e^{tX_2} = (0,0,t)\\
&\rho_\lambda(e^{tX_2})\psi(\theta) = \psi(\theta+t)\\
&\widehat{X_2}^{\lambda}\psi(\theta) = d\rho_\lambda(X_2)\psi(\theta) = \left.\frac{d}{dt}\right|_{t=0}\rho_\lambda(e^{tX_2})\psi(\theta)\\
&= \left.\frac{d}{dt}\right|_{t=0}\psi(\theta+t) = \frac{d\psi(\theta)}{d\theta}.
\end{aligned}
$$

The GFT of the sub-Riemannian Laplacian is therefore the Hilbert sum (integral on $\lambda$ with the Plancherel measure $dP(\lambda) = \lambda d\lambda$) of the $\widehat{\Delta_{\mathscr{K}}}^{\lambda}$ with

$$\widehat{\Delta_{\mathscr{K}}}^{\lambda}\psi(\theta) = ((\widehat{X_1}^{\lambda})^2 + (\widehat{X_2}^{\lambda})^2)\psi(\theta) = \frac{d^2\psi(\theta)}{d\theta^2} - \lambda^2\sin^2(\theta)\psi(\theta)$$

which is nothing else than *Mathieu equation*. The heat kernel is

$$P(v,t) = \int_{\mathbb{V}_S^*} Tr(e^{t\widehat{\Delta_{\mathscr{K}}}^{\lambda}}\rho_\lambda(v))dP(\lambda), \quad t \geqslant 0.$$

If the $\widehat{\Delta_{\mathscr{K}}}^{\lambda}$ have discrete spectrum and a complete set of normalized eigenfunctions $\{\psi_n^\lambda\}$ with eigenvalues $\{\alpha_n^\lambda\}$ then

$$P(v,t) = \int_{\mathbb{V}_S^*}\left(\sum_n e^{\alpha_n^\lambda t}\langle\psi_n^\lambda, \rho_\lambda(v)(\psi_n^\lambda)\rangle\right)dP(\lambda), \quad t \geqslant 0.$$

It is the case here. The $2\pi$-periodic eigenfunctions of the Mathieu equation satisfy:

$$\frac{d^2\psi(\theta)}{d\theta^2} - \lambda^2\sin^2(\theta)\psi(\theta) = E\psi(\theta)$$

and, as $\sin^2(\theta) = \frac{1}{2}(1-\cos(2\theta))$, this means:

$$
\begin{aligned}
&\frac{d^2\psi(\theta)}{d\theta^2} - \frac{\lambda^2}{2}\psi(\theta) - E\psi(\theta) + \frac{\lambda^2}{2}\cos(2\theta)\psi(\theta) = 0\\
&\frac{d^2\psi(\theta)}{d\theta^2} + (a - 2q\cos(2\theta))\psi(\theta) = 0,
\end{aligned}
$$

with $a = -\left(\frac{\lambda^2}{2} + E\right)$ and $q = -\frac{\lambda^2}{4}$. The normalized $2\pi$-periodic eigenfunctions are known: they are even or odd and denoted cen $(\theta,q)$ and sen $(\theta,q)$. The associated $a_n(q)$ and $b_n(q)$ are called characteristic values.

## 9. Conclusion

We have presented two neurogeometrical models of $V1$, $\mathbb{V}_J$ and $\mathbb{V}_S$, whose contact structures and sub-Riemannian geometries explain deep Gestalt phenomena such as subjective contours and anisotropic diffusion. To conclude this survey, we notice that one can construct an interpolation between the two models, which corresponds to a *confluence* of singularities between the two associated equations. Mohammed Zahaf and Manchon [28] constructed such an interpolation given by a family of models $\mathbb{V}^\alpha$ and studied the confluence of the corresponding differentials equations in the Fourier space. By appropriate changes of variable, all these equations can be reduced to the form

$$P_0(t)y''(t) + P_1(t)y'(t) + P_2(t)y(t) = 0$$

for well chosen polynomials $P_0(t)$, $P_1(t)$ and $P_2(t)$. The singularities of such a second order equation (considered as an equation of the complex variable $t$) are the zeroes $t_j$ of $P_0(t)$. They are called "regular" when $(t-t_j)\frac{P_1(t)}{P_0(t)}$ and $(t-t_j)^2\frac{P_2(t)}{P_0(t)}$ are analytic in a neighborhood of $t_j$.

The model $\mathbb{V}^\alpha$ can be summarized by the following table:

$$X_1^\alpha = \cos(\theta)\partial_x + \frac{1}{\alpha}\sin(\alpha\theta)\partial_y$$
$$X_2^\alpha = \partial_\theta$$
$$X_3^\alpha = -\alpha\sin(\alpha\theta)\partial_x + \cos(\theta)\partial_y$$
$$[X_1^\alpha, X_2^\alpha] = -X_3^\alpha$$
$$[X_2^\alpha, X_3^\alpha] = \alpha^2 X_1^\alpha$$
$$[X_1^\alpha, X_3^\alpha] = 0$$
$$\mathbb{V}^\alpha = SE_\alpha(2) \text{ with } \mathbb{S}^1_\alpha = \frac{\mathbb{R}}{2\pi\alpha^{-1}\mathbb{Z}}$$
$$X_1^\alpha(\psi(\theta)) = i\lambda\alpha^{-1}\sin(\alpha\theta)\psi(\theta)$$
$$X_2^\alpha(\psi(\theta)) = \psi'(\theta)$$
$$\widehat{\Delta}^\lambda : \psi''(\theta) - \frac{\lambda^2}{\alpha^2}\sin^2(\alpha\theta)\psi(\theta)$$
$$\psi''(\theta) + (\mu - \frac{\lambda^2}{\alpha^2}\sin^2(\alpha\theta))\psi(\theta) = 0.$$

The change of variable $\frac{\sin^2(\alpha\theta)}{\alpha^2} \to t$ yields the equation

$$t(1-\alpha^2 t)y''(t) + \frac{1}{2}(1-2\alpha^2 t)y'(t) + \frac{1}{4}(\mu - \lambda^2 t)y(t) = 0;$$

3 singularities : $0, \alpha^{-2}$ are regular, $\infty$ is irregular.

For $\alpha = 1$, $\mathbb{V}^1$ yields the $\mathbb{V}_S$ model:

$$X_1 = \cos(\theta)\partial_x + \sin(\theta)\partial_y$$
$$X_2 = \partial_\theta$$
$$X_3 = -\sin(\theta)\partial_x + \cos(\theta)\partial_y$$
$$[X_1, X_2] = -X_3$$
$$[X_2, X_3] = X_1$$
$$[X_1, X_3] = 0$$
$$\mathbb{V}^1 = \mathbb{V}_S = SE(2) \text{ with } \mathbb{S}^1 = \frac{\mathbb{R}}{2\pi\mathbb{Z}}$$
$$X_1(\psi(\theta)) = i\lambda\sin(\theta)\psi(\theta)$$
$$X_2(\psi(\theta)) = \psi'(\theta)$$
$$\widehat{\Delta}^\lambda : \psi''(\theta) - \lambda^2\sin^2(\theta)\psi(\theta)$$
$$\psi''(\theta) + (\mu - \lambda^2)\sin^2(\theta)\psi(\theta) = 0.$$

The change of variable $\sin^2(\theta) \to t$ yields the equation

$$t(1-t)y''(t) + \frac{1}{2}(1-2t)y'(t) + \frac{1}{4}(\mu - \lambda^2 t)y(t) = 0;$$

3 singularities : $0, 1$ are regular, $\infty$ is irregular.

For $\alpha \to 0$ and small $\theta$ (denoted $p$), $\mathbb{V}^0$ yields the $\mathbb{V}_J$ model:

$$X_1^0 = \partial_x + p\partial_y$$
$$X_2^0 = \partial_p$$
$$X_3^0 = \partial_y$$
$$\left[X_1^0, X_2^0\right] = -X_3^0$$
$$[X_2^\alpha, X_3^\alpha] = 0$$
$$[X_1^\alpha, X_3^\alpha] = 0$$
$$\mathbb{V}^0 = \mathbb{V}_J \text{ with } S_0^1 = \mathbb{R}$$
$$X_1^0(y(s)) = i\lambda s y(s)$$
$$X_2^0(y(s)) = y'(s)$$
$$\widehat{\Delta}^\lambda : y''(s) - \lambda^2 s^2 y(s)$$
$$y''(s) + (\mu - \lambda^2)s^2 y(s) = 0.$$

The change of variable $s^2 \to t$ yields the equation

$$ty''(t) + \frac{1}{2}y'(t) + \frac{1}{4}(\mu - \lambda^2 t)y(t) = 0;$$

2 singularities : 0 is regular, $\alpha^{-2} = \infty$ is irregular.

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