

Noncommutative geometry and physics

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This is a compilation of some well known propositions of Alain Connes concerning the use of noncommutative geometry in mathematical physics.

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1 Gelfand theory

To understand NC Geometry we must first come back to Gelfand theory for commutative C^* -algebras.

1.1 C^* -algebras

Recall that a C^* -algebra \mathcal{A} is a (unital) Banach algebra on \mathbb{C} (i.e. a \mathbb{C} -algebra which is normed and complete for its norm) endowed with an involution $x \rightarrow x^*$ s.t. $\|x\|^2 = \|x^*x\|$. The norm (the metric structure) is then deducible from the algebraic structure. Indeed, $\|x\|^2$ is the spectral radius of the ≥ 0 element x^*x , that is, the Sup of the modulus of the spectral values of x^*x :¹

$$\|x\|^2 = \text{Sup} \{|\lambda| : x^*x - \lambda I \text{ is not invertible}\}.$$

(where I is the unit of \mathcal{A}). In a C^* -algebra the norm becomes therefore a purely *spectral* concept.

An element $x \in \mathcal{A}$ is called self-adjoint if $x = x^*$, normal if $xx^* = x^*x$, and unitary if $x^{-1} = x^*$ ($\|x\| = 1$).

In this classical setting, the mathematical interpretation of the fundamental (categorical) concepts of

1. space of states,
2. observable,
3. measure,

is the following:

1. the space of states is a smooth manifold: the phase space M (in Hamiltonian mechanics, $M = T^*N$ is the cotangent bundle of the space of configurations N endowed with its canonical symplectic structure);
2. the observables are functions $f : M \rightarrow \mathbb{R}$ (interpreted as $f : M \rightarrow \mathbb{C}$ with $f = \bar{f}$) which measure some property of states and output a real number;

¹In the infinite dimensional case, the spectral values ($x - \lambda I$ is not invertible) are not identical with the eigenvalues ($x - \lambda I$ has a non trivial kernel). Indeed non invertibility no longer implies non injectivity (a linear operator can be injective and non surjective). For instance, if e_n , $n \in \mathbb{N}$, is a countable basis, the shift $\sum_n \lambda_n e_n \rightarrow \sum_n \lambda_n e_{n+1}$ is injective but not surjective and is not invertible.

3. the measure of f in the state $x \in M$ is the evaluation $f(x)$ of f at x ; but as $f(x) = \delta_x(f)$ (where δ_x is the Dirac distribution at x) a state can be dually interpreted as a continuous linear operator on observables.

The observables constitute a commutative C^* -algebra \mathcal{A} and Gelfand theory explains that the *geometry* of the manifold M can be completely recovered from the *algebraic* structure of \mathcal{A} .

1.2 Gelfand's theorem

Let M be a topological space and let $\mathcal{A} := \mathcal{C}(M)$ be the \mathbb{C} -algebra of continuous functions $f : M \rightarrow \mathbb{C}$ (the \mathbb{C} -algebra structure being inherited from that of \mathbb{C} itself via pointwise addition and multiplication). Under very general conditions (e.g. if M is compact ²), it is a C^* -algebra for complex conjugation $f^* = \bar{f}$.

The possible values of f – that is the possible results of a measure of f – can be defined in a purely algebraic way as the *spectrum* of f that is

$$\text{sp}_{\mathcal{A}}(f) := \{c : f - cI \text{ is not invertible in } \mathcal{A}\}.$$

Indeed, if $f(x) = c$ then $f - cI$ is not invertible in \mathcal{A} . $\text{sp}_{\mathcal{A}}(f)$ is the complementary set of what is called the *resolvent* of f ,

$$r(f) := \{c : f - cI \text{ is invertible in } \mathcal{A}\}.$$

The main point is that the evaluation process $f(x)$ – that is measure – can be interpreted as a *duality* $\langle f, x \rangle$ between the space M and the algebra \mathcal{A} . Indeed, to a point x of M we can associate the *maximal ideal* of the $f \in \mathcal{A}$ vanishing at x :

$$\mathfrak{m}_x := \{f \in \mathcal{A} : f(x) = 0\}.$$

But the maximal ideals \mathfrak{m} of \mathcal{A} constitute themselves a space – called the *spectrum* of the algebra \mathcal{A} . They can be considered as the kernels of the *characters* of \mathcal{A} , that is of the morphisms (multiplicative linear forms) $\chi : \mathcal{A} \rightarrow \mathbb{C}$,

$$\mathfrak{m} = \chi^{-1}(0).$$

A character is by definition a coherent procedure for evaluating the elements $f \in \mathcal{A}$. The evaluation $\chi(f)$ is also a *duality* $\langle \chi, f \rangle$ and its results $\chi(f)$ belong to $\text{sp}_{\mathcal{A}}(f)$. Indeed, as *distributions* (continuous linear forms),

²If M is non compact but only locally compact, then one take $\mathcal{A} = \mathcal{C}_0(M)$ the algebra of continuous functions vanishing at infinity but \mathcal{A} is no longer unital since the constant function 1 doesn't vanish at infinity..

the characters correspond to the Dirac distributions δ_x and if $\chi = \delta_x$, then $\chi(f) = f(x) = c$ and $c \in \text{sp}_{\mathcal{A}}(f)$.

The *spectrum* of the C^* -algebra \mathcal{A} (not to be confused with the spectra $\text{sp}_{\mathcal{A}}(f)$ of the single elements f of \mathcal{A}) is by definition the space of characters $\text{Sp}(\mathcal{A}) := \{\chi\}$ endowed with the topology of simple convergence: $\chi_n \rightarrow \chi$ iff $\chi_n(f) \rightarrow \chi(f) \forall f \in \mathcal{A}$. It is defined uniquely from \mathcal{A} without any reference to the fact that \mathcal{A} is of the form $\mathcal{A} := \mathcal{C}(M)$. It is also the space of irreducible representations of \mathcal{A} (since \mathcal{A} is commutative, they are 1-dimensional).

Now, if $f \in \mathcal{A}$ is an element of \mathcal{A} , using duality, we can associate to it canonically a *function* \tilde{f} on the space $\text{Sp}(\mathcal{A})$

$$\begin{aligned} \tilde{f} : \text{Sp}(\mathcal{A}) &\rightarrow \mathbb{C} \\ \chi &\mapsto \tilde{f}(\chi) = \chi(f) = \langle \chi, f \rangle. \end{aligned}$$

We get that way a map

$$\begin{aligned} \tilde{\cdot} : \mathcal{A} &\rightarrow \mathcal{C}(\text{Sp}(\mathcal{A})) \\ f &\mapsto \tilde{f} \end{aligned}$$

which is called the *Gelfand transform*. For every f we have

$$\tilde{f}(\text{Sp}(\mathcal{A})) = \text{sp}_{\mathcal{A}}(f).$$

The key result is then:

Gelfand-Neimark theorem. If \mathcal{A} is a *commutative* C^* -algebra, the Gelfand transform $\tilde{\cdot}$ is an *isometry* between \mathcal{A} and $\mathcal{C}(\text{Sp}(\mathcal{A}))$.

Indeed, the norm of \tilde{f} is the spectral radius of f , $\rho(f) := \lim_{n \rightarrow \infty} (\|f^n\|^{\frac{1}{n}})$ and we have $\|\tilde{f}\| = \rho(f) = \|f\|$. To see this, suppose first that f is self-adjoint ($f = f^* = \bar{f}$). We have $\|f\|^2 = \|f \cdot f^*\| = \|f^2\|$. So, $\|f\| = \|f^{2^n}\|^{2^{-n}}$ and as $\|f^{2^n}\|^{2^{-n}} \rightarrow \rho(f)$ by definition we have $\|f\| = \rho(f)$. Suppose now that f is any element of \mathcal{A} . Since $f \cdot f^*$ is self-adjoint, we have $\|f\|^2 = \|f \cdot f^*\| = \rho(f \cdot f^*) = \|\widetilde{f \cdot f^*}\|$. But $\|\widetilde{f \cdot f^*}\| = \|\widetilde{f} \cdot \widetilde{f^*}\| = \|\widetilde{f} \cdot \tilde{f}^*\| = \|\tilde{f}\|^2$ and therefore $\|f\|^2 = \|\tilde{f}\|^2$ and $\|f\| = \|\tilde{f}\|$.

Gelfand theory shows that, in the classical case of commutative C^* -algebras $\mathcal{A} := \mathcal{C}(M)$ (M compact), there exists a complete *equivalence* between the geometric and the algebraic perspectives.

1.3 Towards a new kinematics

We think that Gelfand theorem has a deep philosophical meaning. In classical mechanics kinematics concerns the structure of the configuration space N and the phase space $M := T^*N$. Observables and measurements are defined in terms of functions on these basic spaces directly

construed from the geometry of space-time. Gelfand theorem shows than we can *exchange* the primary geometrical background and the secondary algebraic moment of measure, take measure as a primitive fact and reconstruct the geometric background from it as a secondary moment.

1.4 Towards Noncommutative Geometry

In Quantum Mechanics, the basic structure is that of the *noncommutative* C^* -algebras \mathcal{A} of observables. It is therefore natural to wonder if there could exist a *geometric correlate* of this noncommutative algebraic setting. It is the origin of Connes' *Noncommutative Geometry* (NCG) also called *Spectral Geometry* or *Quantum Geometry*. In NCG the basic structure is the NC C^* -algebra \mathcal{A} of observables: any phenomenon is primarily something which is observable in the quantum sense, and not an event in space-time. But observables must be defined for states and are therefore represented in the space of states of the system, which is an Hilbert space and not the classical space. The associated NC space is then the space of *irreducible representations*.

NCG is a fundamentally new step toward a *geometrization* of physics. Instead of beginning with classical differential geometry and try to develop Quantum Mechanics on this background, it begins with Quantum Mechanics and construct a new *quantum geometrical framework*. In that sense, Connes is the Einstein of Quantum Mechanics. The most fascinating aspect of his research program is how he succeeded in reinterpreting all the basic structures of classical geometry inside the framework of NC C^* -algebras operating on Hilbert spaces. *The basic concepts remain almost the same but their mathematical interpretation is significantly complexified*, since their classical meaning becomes a *commutative limit*. We meet here a new very deep example of the conceptual transformation of physical theories through mathematical enlargements, as it is the case in GR or QM. As explained by Daniel Kastler [NCG]:

“Alain Connes’ noncommutative geometry (...) is a systematic quantization of mathematics parallel to the quantization of physics effected in the twenties. (...) This theory widens the scope of mathematics in a manner congenial to physics.”

2 NCG and differential forms

Connes reinterpreted (in an extremely deep and technical way) the six classical levels:

1. Measure theory;

2. Algebraic topology and topology (K -theory);
3. Differentiable structure;
4. Differential forms and De Rham cohomology;
5. Fiber bundles, connections, covariant derivations, Yang-Mills theories;
6. Riemannian manifolds and metric structures.

Let us take as a first example the reinterpretation of the differential calculus.

2.1 A universal and formal differential calculus

How can we interpret differential calculus in the new NC paradigm? We want first to define *derivations* $D : \mathcal{A} \rightarrow \mathcal{E}$, that is \mathbb{C} -linear maps satisfying the *Leibniz rule* (which is the universal formal rule for derivations):

$$D(ab) = (Da)b + a(Db)$$

For that, \mathcal{E} must be endowed with a structure of \mathcal{A} -bimodule (right and left products of elements of \mathcal{E} by elements of \mathcal{A}). It is evident that $D(c) = 0$ for every scalar $c \in \mathbb{C}$ since $D(1.a) = D(1)a + 1D(a) = D(a)$ and therefore $D(1) = 0$.

Let $\text{Der}(\mathcal{A}, \mathcal{E})$ be the \mathbb{C} -vector space of such derivations. In $\text{Der}(\mathcal{A}, \mathcal{E})$ there exist very particular elements, the inner derivatives, associated with the elements m of \mathcal{E} , which express the difference between the right and left \mathcal{A} -module structures of \mathcal{E} :

$$D(a) := \text{ad}(m)(a) = ma - am.$$

Indeed,

$$\begin{aligned} \text{ad}(m)(a).b + a.\text{ad}(m)(b) &= (ma - am)b + a(mb - bm) \\ &= mab - abm \\ &= \text{ad}(m)(ab). \end{aligned}$$

In the case where $\mathcal{E} = \mathcal{A}$, $\text{ad}(b)(a) = [b, a]$ expresses the non commutativity of \mathcal{A} . By the way, $\text{Der}(\mathcal{A}, \mathcal{A})$ is a Lie algebra since $[D_1, D_2]$ is a derivation if D_1, D_2 are derivations.

Now, the fact must be stressed that there exists *a universal derivation* depending only upon the algebraic structure of \mathcal{A} , and having therefore

absolutely nothing to do with the classical “infinitesimal” intuitions underlying the classical concepts of differential and derivation. It is given by

$$\begin{aligned} d : \mathcal{A} &\rightarrow \mathcal{A} \otimes_{\mathbb{C}} \mathcal{A} \\ a &\mapsto da := 1 \otimes a - a \otimes 1. \end{aligned}$$

Let $\Omega^1 \mathcal{A}$ be the sub-bimodule of $\mathcal{A} \otimes_{\mathbb{C}} \mathcal{A}$ generated by the elements $adb := a \otimes b - ab \otimes 1$, i.e. the kernel of the multiplication $a \otimes b \mapsto ab$.³ $\Omega^1 \mathcal{A}$ is isomorphic to the tensorial product $\mathcal{A} \otimes_{\mathcal{A}} \overline{\mathcal{A}}$, where $\overline{\mathcal{A}}$ is the quotient \mathcal{A}/\mathbb{C} , with $adb = a \otimes \bar{b}$. It is called the bimodule of *universal 1-forms* on \mathcal{A} where “universality” means that

$$\text{Der}(\mathcal{A}, \mathcal{E}) \simeq \text{Hom}_{\mathcal{A}}(\Omega^1 \mathcal{A}, \mathcal{E})$$

i.e. that a derivation $D : \mathcal{A} \rightarrow \mathcal{E}$ is the same thing as a morphism of algebras between $\Omega^1 \mathcal{A}$ and \mathcal{E} . If $D : \mathcal{A} \rightarrow \mathcal{E}$ is an element of $\text{Der}(\mathcal{A}, \mathcal{E})$, the associated morphism $\tilde{D} : \Omega^1 \mathcal{A} \rightarrow \mathcal{E}$ is defined by

$$a \otimes b \mapsto aD(b).$$

So $da = 1 \otimes a - a \otimes 1 \mapsto 1.D(a) - a.D(1) = D(a)$ (since $D(1) = 0$).

We can generalize this construction to universal n -forms, which have the symbolic form⁴

$$a_0 da_1 \dots da_n.$$

If $\Omega^n \mathcal{A} := (\Omega^1 \mathcal{A})^{\otimes n} = \mathcal{A} \otimes_{\mathcal{A}} (\overline{\mathcal{A}})^{\otimes n}$ with $a_0 da_1 \dots da_n = a_0 \otimes \bar{a}_1 \otimes \dots \otimes \bar{a}_n$, the differential is then

$$\begin{aligned} d : \Omega^n \mathcal{A} &\rightarrow \Omega^{n+1} \mathcal{A} \\ a_0 da_1 \dots da_n &\mapsto da_0 da_1 \dots da_n \\ a_0 \otimes \bar{a}_1 \otimes \dots \otimes \bar{a}_n &\mapsto 1 \otimes \bar{a}_0 \otimes \bar{a}_1 \otimes \dots \otimes \bar{a}_n \end{aligned}$$

Since $d1 = 0$, it is easy to verify the fundamental cohomological property $d^2 = 0$ of the graduate differential algebra $\Omega \mathcal{A} := \bigoplus_{n \in \mathbb{N}} \Omega^n \mathcal{A}$. Some technical difficulties must be overcome (existence of “junk” forms) to transform this framework into a “good” formal differential calculus.

2.2 Noncommutative differential calculus or “quantized” calculus

Connes wanted to *represent* this universal differential algebra in spaces of physical states. Let us suppose therefore that the C^* -algebra \mathcal{A} acts upon an Hilbert space \mathcal{H} . We want to interpret in this representation

³For $a \otimes b - ab \otimes 1$ the multiplication gives $ab - ab = 0$. Reciprocally if $ab = 0$ then $a \otimes b = a \otimes b - ab \otimes 1$ and $a \otimes b$ belongs to $\Omega^1 \mathcal{A}$.

⁴ $da_1 \dots da_n$ is the exterior product of 1-forms, classically denoted $da_1 \wedge \dots \wedge da_n$.

the universal, formal, and purely symbolic differential calculus of the previous section. For that, we must interpret the differential df of the elements $f \in \mathcal{A}$, these f being now represented as *operators* on \mathcal{H} . Connes' main idea was to use the well-known formula of QM

$$\frac{df}{dt} = \frac{2i\pi}{h} [F, f]$$

where F is the Hamiltonian of the system and f any observable.

Consequently, he interpreted the symbol df as

$$df := [F, f]$$

for an appropriate self-adjoint operator F . We want of course $d^2 f = 0$. But $d^2 f = [F^2, f]$ and therefore F^2 must commute with all observables.

The main constraint is that, once interpreted in \mathcal{H} , the symbol df must correspond to an *infinitesimal*. *The classical concept of infinitesimal ought to be reinterpreted in the NC framework*. Connes' definition is that an operator T is infinitesimal if it is *compact*, that is if the eigenvalues $\mu_n(T)$ of its absolute value $|T| = (T^*T)^{1/2}$ – called the *characteristic values* of T – converge to 0, that is if for every $\varepsilon > 0$ the norm $\|T\|$ of T is $< \varepsilon$ outside a subspace of *finite* dimension. If $\mu_n(T) \xrightarrow[n \rightarrow \infty]{} 0$ as $\frac{1}{n^\alpha}$ then T is an infinitesimal of order α (α not necessarily an integer).

If T is compact, let ξ_n be a complete orthonormal basis of \mathcal{H} associated to $|T|$, $T = U|T|$ the polar decomposition of T ⁵ and $\eta_n = U\xi_n$. Then T is the sum

$$T = \sum_{n \geq 0} \mu_n(T) |\eta_n\rangle \langle \xi_n|$$

If T is a positive infinitesimal of order 1, its trace $\text{Trace}(T) = \sum_n \mu_n(T)$ has a logarithmic divergence. If T is of order > 1 , its trace is finite > 0 . It is the basis for *NC integration* which uses the *Dixmier trace*, a technical tool for constructing a new trace extracting the logarithmic divergence of the classical trace. Dixmier trace is a technical way for giving a sense to the formula $\lim_{N \rightarrow \infty} \frac{1}{\ln N} \sum_{n=0}^{n=N-1} \mu_n(T)$. It vanishes for infinitesimals of order > 1 .

Therefore, we interpret the differential calculus in the NC framework through triples $(\mathcal{A}, \mathcal{H}, F)$ where $[F, f]$ is compact for every $f \in \mathcal{A}$. Such a structure is called a *Fredholm module*.

The differential forms $a_0 da_1 \dots da_n$ can now be interpreted as operators on \mathcal{H}

$$a_0 da_1 \dots da_n := a_0 [F, a_1] \dots [F, a_n]$$

⁵The polar decomposition $T = U|T|$ is the equivalent for operators of the decomposition $z = |z|e^{i\theta}$ for a complex number. In general U cannot be unitary but only a partial isometry.

It must be emphasized that the NC generalization of differential calculus is a wide and wild generalization since it enables us to extend differential calculus to fractals!

3 NC Riemannian geometry, Clifford algebras, and Dirac operator

Another great achievement of Alain Connes was the complete and deep reinterpretation of the ds^2 in Riemannian geometry. Classically, $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$. In the NC framework, dx must be interpreted as $dx = [F, x]$ (where $(\mathcal{A}, \mathcal{H}, F)$ is a Fredholm module), and the matrix $(g_{\mu\nu})$ as an element of the $n \times n$ matrix algebra $M_n(\mathcal{A})$. The ds^2 must therefore become a *compact* and *positive* operator of the form

$$G = [F, x^\mu]^* g_{\mu\nu} [F, x^\nu].$$

3.1 A redefinition of distance

Connes' idea is to reinterpret the classical definition of distance $d(p, q)$ between two points p, q of a Riemannian manifold M as the Inf of the length $L(\gamma)$ of the paths $\gamma : p \rightarrow q$

$$d(p, q) = \text{Inf}_{\gamma: p \rightarrow q} L(\gamma)$$

$$L(\gamma) = \int_p^q ds = \int_p^q (g_{\mu\nu} dx^\mu dx^\nu)^{1/2}.$$

Using the equivalence between a point x of M and the pure state δ_x on the commutative C^* -algebra $\mathcal{A} := C^\infty(M)$, an elementary computation shows that this definition of the distance is equivalent to the dual algebraic definition using only concepts concerning the C^* -algebra \mathcal{A}

$$d(p, q) = \text{Sup} \{ |f(q) - f(p)| : \|\text{grad}(f)\|_\infty \leq 1 \}$$

where $\|\dots\|_\infty$ is the L^∞ norm, that is the Sup on $x \in M$ of the norms on the tangent spaces $T_x M$.⁶

⁶Let $\gamma : I = [0, 1] \rightarrow M$ be a C^∞ curve in M from p to q . $L(\gamma) = \int_p^q |\dot{\gamma}(t)| dt = \int_0^1 g(\dot{\gamma}(t), \dot{\gamma}(t))^{1/2} dt$. If $f \in C^\infty(M)$, using the duality between df and $\text{grad} f$ induced by the metric, we find $f(q) - f(p) = \int_0^1 df_{\gamma(t)}(\dot{\gamma}(t)) dt = \int_0^1 g_{\gamma(t)}(\text{grad}_{\gamma(t)} f, \dot{\gamma}(t)) dt$. This shows that $|f(q) - f(p)| \leq \int_0^1 |\text{grad}_{\gamma(t)} f| |\dot{\gamma}(t)| dt \leq \|\text{grad} f\|_\infty L(\gamma)$. Therefore, if $\|\text{grad}(f)\|_\infty \leq 1$ we have $|f(q) - f(p)| \leq d(p, q)$. When we take the Sup we retrieve $d(p, q)$ using the special function $f_p(x) = d(p, x)$ since $|f_p(q) - f_p(p)| = d(p, q)$.

3.2 Clifford algebras

Now the core of the NC definition of distance uses the *Dirac operator*. In order to explain this key point, *which makes distance a quantum concept*, the so called *Clifford algebra* of a Riemannian manifold must be introduced.

Recall that the formalism of Clifford algebras relates *the differential forms and the metric* on Riemannian manifolds. In the classical case of the Euclidean space \mathbb{R}^n , the main idea is to encode the isometries $O(n)$ in an algebra structure. Since every isometry is a product of reflections (Cartan), we can associate to any vector $v \in \mathbb{R}^n$ the reflection \bar{v} relative to the orthogonal hyperplane v^\perp and introduce a multiplication $v.w$ which is nothing else than the composition $\bar{v} \circ \bar{w}$. We are then naturally led to the anti-commutation relations

$$\{v, w\} := v.w + w.v = -2(v, w)$$

where (v, w) is the Euclidean scalar product.

More generally, let V be a \mathbb{R} -vector space endowed with a quadratic form g . Its Clifford algebra $\text{Cl}(V, g)$ is its tensor algebra $\mathcal{T}(V) = \bigoplus_{k=0}^{k=\infty} V^{\otimes k}$ quotiented by the relations

$$v \otimes v = -g(v)1, \quad \forall v \in V$$

(where $g(v) = g(v, v) = \|v\|^2$). In $\text{Cl}(V, g)$ the tensorial product $v \otimes v$ becomes a product $v.v = v^2$. It must be stressed that there exists always in $\text{Cl}(V, g)$ the constants \mathbb{R} which correspond to the 0th tensorial power of V .

Using the scalar product

$$2g(v, w) = g(v + w) - g(v) - g(w)$$

one gets the *anti-commutation* relations

$$\{v, w\} = -2g(v, w)$$

Elementary examples are given by the $\text{Cl}_n = \text{Cl}(\mathbb{R}^n, g_{\text{Euclid}})$.

- $\text{Cl}_0 = \mathbb{R}$.
- $\text{Cl}_1 = \mathbb{C}$ ($V = i\mathbb{R}$, $i^2 = -1$, $\text{Cl}_1 = \mathbb{R} \oplus i\mathbb{R}$).
- $\text{Cl}_2 = \mathbb{H}$ ($V = i\mathbb{R} + j\mathbb{R}$, $ij = k$, $\text{Cl}_2 = \mathbb{R} \oplus i\mathbb{R} \oplus j\mathbb{R} \oplus k\mathbb{R}$).
- $\text{Cl}_3 = \mathbb{H} \oplus \mathbb{H}$.

- $\text{Cl}_4 = \mathbb{H}[2]$ (2×2 matrices with entries in \mathbb{H}).
- $\text{Cl}_5 = \mathbb{C}[4]$.
- $\text{Cl}_6 = \mathbb{R}[8]$.
- $\text{Cl}_7 = \mathbb{R}[8] \oplus \mathbb{R}[8]$.
- $\text{Cl}_{n+8} = \text{Cl}_n \otimes \mathbb{R}[16]$ (Bott periodicity theorem).

If $g(v) \neq 0$ (which would always be the case for $v \neq 0$ if g is non degenerate) v is invertible in this algebra structure and

$$v^{-1} = -\frac{v}{g(v)}.$$

The multiplicative Lie group $\text{Cl}^\times(V, g)$ of the invertible elements of $\text{Cl}(V, g)$ act through *inner automorphisms* on $\text{Cl}(V, g)$. This yields the *adjoint representation*

$$\begin{aligned} \text{Ad} : \text{Cl}^\times(V, g) &\rightarrow \text{Aut}(\text{Cl}(V, g)) \\ v &\mapsto \text{Ad}_v : w \mapsto v.w.v^{-1}. \end{aligned}$$

But ⁷

$$v.w.v^{-1} = -w + \frac{2g(v, w)v}{g(v)} = \text{Ad}_v(w).$$

As $-\text{Ad}_v$ is the reflection relative to v^\perp , this means that reflections act through the adjoint representation of the Clifford algebra. The derivative ad of the adjoint representation enables to recover the Lie bracket of the Lie algebra $\text{cl}^\times(V, g) = \text{Cl}(V, g)$ of the Lie group $\text{Cl}^\times(V, g)$

$$\begin{aligned} \text{ad} : \text{cl}^\times(V, g) = \text{Cl}(V, g) &\rightarrow \text{Der}(\text{Cl}(V, g)) \\ v &\mapsto \text{ad}_v : w \mapsto [v, w] \end{aligned}$$

Now there exists a fundamental relation between the Clifford algebra $\text{Cl}(V, g)$ of V and its exterior algebra Λ^*V . If $g = 0$ and if we interpret $v.w$ as $v \wedge w$, the anti-commutation relations become simply $\{v, w\} = 0$, that is the classical antisymmetry $w \wedge v = -v \wedge w$ of differential 1-forms. Therefore

$$\Lambda^*V = \text{Cl}(V, 0).$$

In fact, $\text{Cl}(V, g)$ can be considered as a way of *quantizing* Λ^*V using the metric g in order to get *non trivial* anti-commutation relations.

⁷ $v.w.v^{-1} = -v.w.\frac{v}{g(v)} = -(-w.v - 2g(v, w))\frac{v}{g(v)} = w.\frac{v^2}{g(v)} + \frac{2g(v, w)v}{g(v)} = -w + \frac{2g(v, w)v}{g(v)}.$

Due to the relations $v^2 = -g(v)1$ which decrease the degree of a product by 2, $\text{Cl}(V, g)$ is no longer a \mathbb{Z} -graded algebra but only a $\mathbb{Z}/2$ -graded algebra, the $\mathbb{Z}/2$ -gradation corresponding to the even/odd elements. But we can reconstruct a \mathbb{Z} -graded algebra $\mathcal{C} = \bigoplus_{k=0}^{k=\infty} C^k$ associated to $\text{Cl}(V, g)$, the C^k being the homogeneous terms of degree k : $v_1 \cdots v_k$.

Theorem. The map of graded algebras $\mathcal{C} = \bigoplus_{k=0}^{k=\infty} C^k \rightarrow \Lambda^*V = \bigoplus_{k=0}^{k=\infty} \Lambda^k$ given by $v_1 \cdots v_k \rightarrow v_1 \wedge \cdots \wedge v_k$ is a *linear isomorphism* (but not an *algebra isomorphism*).

We consider now 2 operations on the *exterior algebra* Λ^*V :

1. The outer multiplication $\varepsilon(v)$ by $v \in V$:

$$\varepsilon(v) \left(\bigwedge_i u_i \right) = v \wedge \left(\bigwedge_i u_i \right).$$

We have $\varepsilon(v)^2 = 0$ since $v \wedge v = 0$.

2. The contraction (inner multiplication) $\iota(v)$ induced by the metric g :⁸

$$\iota(v) \left(\bigwedge_i u_i \right) = \sum_{j=1}^{j=k} (-1)^j g(v, u_j) u_1 \wedge \cdots \wedge \widehat{u}_j \wedge \cdots \wedge u_k.$$

We have also $\iota(v)^2 = 0$. The inner multiplication $\iota(v)$ is a supplementary structure involving the metric structure.

One shows that the following anti-commutations relations obtain:

$$\{\varepsilon(v), \iota(w)\} = -g(v, w)1$$

Let now $c(v) = \varepsilon(v) + \iota(v)$. We get the anti-commutation relations of the Clifford algebra

$$\{c(v), c(w)\} = -2g(v, w)1$$

and $\text{Cl}(V, g)$ is therefore generated in $\text{End}_{\mathbb{R}}(\Lambda^*V)$ by the $c(v)$ (identified with v).

⁸In the following formula \widehat{u}_j means that the term u_j is deleted.

3.3 Spin groups

The isometry group $O(n)$ is canonically embedded in $\text{Cl}(V, g)$ since every isometry is a product of reflections. In fact $\text{Cl}(V, g)$ contains also the *pin group* $\text{Pin}(n)$ which is a 2-fold covering of $O(n)$. If we take into account the orientation and restrict to $SO(n)$, the 2-fold covering becomes the *spin group* $\text{Spin}(n)$. $\text{Spin}(n)$ is generated by the even products of v s.t. $g(v) = \pm 1$, $SO(n)$ is generated by even products of $-Ad_v$ and the covering $\text{Spin}(n) \rightarrow SO(n)$ is given by $v \mapsto -Ad_v$. By restriction of the Clifford multiplication and of the adjoint representation $w \mapsto v.w.v^{-1}$ to $\text{Spin}(n)$, we get therefore a representation γ of $\text{Spin}(n)$ into the spinor space $\mathbb{S} = \text{Cl}(V, g)$.

3.4 Dirac equation

We can use the Clifford algebra, and therefore the metric, to change the classical exterior derivative of differential forms given by

$$d := \varepsilon(dx^\mu) \frac{\partial}{\partial x^\mu}.$$

We then define the Dirac operator on spinor fields $\mathbb{R}^n \rightarrow \mathbb{S}$ as

$$\begin{aligned} D &:= c(dx^\mu) \frac{\partial}{\partial x^\mu} \\ &= \gamma^\mu \frac{\partial}{\partial x^\mu} \end{aligned}$$

where c is the Clifford multiplication, and D acts on the spinor space $\mathbb{S} = \text{Cl}(V, g)$. As $\{c(v), c(w)\} = -2g(v, w)1$, the γ^μ satisfy standard Dirac relations of anticommutation $\{\gamma^\mu, \gamma^\nu\} = -2\delta^{\mu\nu}$ in the Euclidean case.⁹ One can check that $D^2 = \Delta$ is the Laplacian.

3.5 Dirac operator

More generally, if M is a Riemannian manifold, the previous construction can be done for every tangent space $T_x M$ endowed with the quadratic form g_x . In this way we get a *bundle* of Clifford algebras $\text{Cl}(TM, g)$. If S is a spinor bundle, that is a bundle of $\text{Cl}(TM)$ -modules s.t. $\text{Cl}(TM) \simeq \text{End}(S)$, endowed with a covariant derivative ∇ , we associate to it the Dirac operator

$$D : \mathcal{S} = \Gamma(S) = C^\infty(M, S) \rightarrow \Gamma(S)$$

which is a first order elliptic operator interpretable as the “square root” of the Laplacian Δ , Δ interpreting itself the metric in operatorial terms.

⁹The classical Dirac matrices are the $-i\gamma^\mu$ for $\mu = 0, 1, 2, 3$.

The Dirac operator D establishes a coupling between the covariant derivation on S and the Clifford multiplication of 1-forms. It can be extended from the $C^\infty(M)$ -module $\mathcal{S} = \Gamma(S)$ to the Hilbert space $\mathcal{H} = L^2(M, S)$.

In general, because of chirality, S will be the direct sum of an even and an odd part, $S = S^+ \oplus S^-$ and D will have the characteristic form

$$D = \begin{bmatrix} 0 & D^- \\ D^+ & 0 \end{bmatrix}$$

$$D^+ : \Gamma(S^+) \rightarrow \Gamma(S^+)$$

$$D^- : \Gamma(S^-) \rightarrow \Gamma(S^-)$$

D^+ and D^- being adjoint operators.

3.6 NC distance and Dirac operator

In this classical framework, it is easy to compute the bracket $[D, f]$ for $f \in C^\infty(M)$. First, there exists on M the *Levi-Civita connection*:

$$\nabla^g : \Omega^1(M) \rightarrow \Omega^1(M) \otimes_{C^\infty(M)} \Omega^1(M)$$

satisfying the Leibniz rule for $\alpha \in \Omega^1(M)$ and $f \in C^\infty(M)$:

$$\nabla^g(\alpha f) = \nabla^g(\alpha)f + \alpha \otimes df$$

(as $\nabla^g(\alpha) \in \Omega^1(M) \otimes_{C^\infty(M)} \Omega^1(M)$, $\nabla^g(\alpha)f \in \Omega^1(M) \otimes_{C^\infty(M)} \Omega^1(M)$ and as α and $df \in \Omega^1(M)$, $\alpha \otimes df \in \Omega^1(M) \otimes_{C^\infty(M)} \Omega^1(M)$). There exists also the *spin connection* on the spinor bundle S

$$\nabla^S : \Gamma(S) \rightarrow \Omega^1(M) \otimes_{C^\infty(M)} \Gamma(S)$$

satisfying the Leibniz rule for $\psi \in \Gamma(S)$ and $f \in C^\infty(M)$:

$$\nabla^S(\psi f) = \nabla^S(\psi)f + \psi \otimes df$$

$$\nabla^S(\gamma(\alpha)\psi) = \gamma(\nabla^g(\alpha))\psi + \gamma(\alpha)\nabla^S(\psi)$$

where γ is the spin representation. The Dirac operator on $\mathcal{H} = L^2(M, S)$ is then defined as

$$D := \gamma \circ \nabla^S$$

If $\psi \in \Gamma(S)$, we have (making the f acting on the left in \mathcal{H})

$$\begin{aligned} D(f\psi) &= \gamma(\nabla^S(\psi f)) \\ &= \gamma(\nabla^S(\psi)f + \psi \otimes df) \\ &= \gamma(\nabla^S(\psi))f + \gamma(\psi \otimes df) \\ &= fD(\psi) + \gamma(df)\psi \end{aligned}$$

and therefore $[D, f](\psi) = fD(\psi) + \gamma(df)\psi - fD(\psi) = \gamma(df)\psi$, that is

$$[D, f] = \gamma(df).$$

In the standard case where $M = \mathbb{R}^n$ and $S = \mathbb{R}^n \times V$, V being a Cl_n -module of spinors ($\text{Cl}_n = \text{Cl}(\mathbb{R}^n, g_{\text{Euclid}})$), we have seen that D is a differential operator with constant coefficients taking its values in V .

$$D = \sum_{k=1}^{k=n} \gamma^\mu \frac{\partial}{\partial x^\mu}$$

with the constant matrices $\gamma^\mu \in \mathcal{L}(V)$ satisfying the anti-commutation relations

$$\{\gamma^\mu, \gamma^\nu\} = -2\delta^{\mu\nu}$$

The fundamental point is that the γ^μ are associated with the basic 1-forms dx^μ through the isomorphism

$$c : \mathcal{C} = \Lambda^*(M) \rightarrow \text{gr}(\text{Cl}(TM))$$

$$[D, f] = \gamma(df) = c(df)$$

and $\|[D, f]\|$ is the norm of the Clifford action of df on the space of spinors $L^2(M, S)$. But

$$\begin{aligned} \|c(df)\|^2 &= \text{Sup}_{x \in M} g_x^{-1}(d\bar{f}(x), df(x)) \\ &= \text{Sup}_{x \in M} g_x(\text{grad}_x \bar{f}, \text{grad}_x f) \\ &= \|\text{grad}(f)\|_\infty^2 \end{aligned}$$

Whence the definition:

$$d(p, q) = \text{Sup} \{|f(p) - f(q)| : f \in \mathcal{A}, \|[D, f]\| \leq 1\}$$

In this reinterpretation, ds corresponds to *the propagator of the Dirac operator* D . As an operator acting on the Hilbert space \mathcal{H} , D is an unbounded self-adjoint operator such that $[D, f]$ is bounded $\forall f \in \mathcal{A}$ and such that its resolvent $(D - \lambda I)^{-1}$ is compact $\forall \lambda \notin \text{Sp}(D)$ (which corresponds to the fact that ds is infinitesimal) and the trace $\text{Trace}(e^{-D^2})$ is *finite*. In terms of the operator $G = [F, x^\mu]^* g_{\mu\nu} [F, x^\nu]$, we have $G = D^{-2}$.

4 Noncommutative spectral geometry

Basing himself on several examples, Alain Connes arrived at the following concept of NC geometry.

In the classical commutative case, $\mathcal{A} = C^\infty(M)$ is the commutative algebra of “coordinates” on M represented in the Hilbert space $\mathcal{H} = L^2(M, S)$ by pointwise multiplication¹⁰ and ds is a symbol non commuting with the $f \in \mathcal{A}$ and satisfying the commutation relations $[[f, ds^{-1}], g] = 0, \forall f, g \in \mathcal{A}$.

Any specific geometry is defined through the representation $ds = D^{-1}$ of ds by means of a Dirac operator $D = \gamma^\mu \nabla_\mu$. The differential $df = [D, f]$ is then the Clifford multiplication by the gradient ∇f and its norm in \mathcal{H} is the Lipschitz norm of f : $\|[D, f]\| = \text{Sup}_{x \in M} \|\nabla f\|$.

These results can be taken as a definition in the general case. The geometry is defined by a *spectral triple* $(\mathcal{A}, \mathcal{H}, D)$ where \mathcal{A} is a NC C^* -algebra with a representation in an Hilbert \mathcal{H} and D is an unbounded self-adjoint operator on \mathcal{H} such that $ds = D^{-1}$ and more generally the resolvent $(D - \lambda I)^{-1}$, $\lambda \notin \mathbb{R}$, is compact, and at the same time all $[D, a]$ are bounded $\forall a \in \mathcal{A}$ (there is a tension between these two last conditions).¹¹ As Connes [2000] emphasizes

“It is precisely this lack of commutativity between the line element and the coordinates on a space [between ds and the $a \in \mathcal{A}$] that will provide the measurement of distance.”

The new definition of differentials are then $da = [D, a] \forall a \in \mathcal{A}$.

5 Yang-Mills theory of a NC coupling between 2 points and Higgs mechanism

An striking example of pure NC physics is given by Connes’ interpretation of the Higgs phenomenon. In the Standard Model, the Higgs mechanism was an *ad hoc* device used for conferring a mass to gauge bosons. It lacked any geometrical interpretation. One of the deepest achievement of the NC framework has been to show that Higgs fields correspond effectively to gauge bosons, but for a *discrete* NC geometry.

¹⁰If $f \in \mathcal{A}$ and $\xi \in \mathcal{H}$, $(f\xi)(x) = f(x)\xi(x)$.

¹¹Let λ_n be the eigenvalues of D ($\lambda_n \in \mathbb{R}$ since D is self-adjoint). $|\lambda_n| = \mu_n(D)$ and as $(D - \lambda I)^{-1}$ is compact, $|\lambda_n| \xrightarrow{n \rightarrow \infty} \infty$.

5.1 Symmetry breaking and classical Higgs mechanism

Let us first recall the classical Higgs mechanism. Consider e.g. a φ^4 theory for 2 scalar real fields φ_1 and φ_2 . The Lagrangian is

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \varphi_1 \partial^\mu \varphi_1 + \partial_\mu \varphi_2 \partial^\mu \varphi_2) - V(\varphi_1^2 + \varphi_2^2)$$

with the quartic potential

$$V(\varphi_1^2 + \varphi_2^2) = \frac{1}{2} \mu^2 (\varphi_1^2 + \varphi_2^2) + \frac{1}{4} |\lambda| (\varphi_1^2 + \varphi_2^2)^2$$

It is by construction $SO(2)$ -invariant.

For $\mu^2 > 0$ the minimum of V (the quantum vacuum) is non degenerate: $\varphi_0 = (0, 0)$ and the Lagrangian \mathcal{L}_{os} of small oscillations in the neighborhood of φ_0 is the sum of 2 Lagrangians of the form:

$$\mathcal{L}_{os} = \frac{1}{2} (\partial_\mu \psi \partial^\mu \psi) - \frac{1}{2} \mu^2 \psi^2$$

describing particles of mass μ .

But for $\mu^2 < 0$ the situation becomes completely different. Indeed the potential V has a full circle (an $SO(2)$ -orbit) of minima

$$\varphi_0^2 = -\frac{\mu^2}{|\lambda|} = v^2$$

and the vacuum state is highly *degenerate*.

One must therefore *break the symmetry* to choose a vacuum state. Let us take for instance $\varphi_0 = \begin{bmatrix} v \\ 0 \end{bmatrix}$ and translate the situation to φ_0 :

$$\varphi = \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix} = \begin{bmatrix} v \\ 0 \end{bmatrix} + \begin{bmatrix} \xi \\ \eta \end{bmatrix}$$

The oscillation Lagrangian at φ_0 becomes

$$\mathcal{L}_{os} = \frac{1}{2} (\partial_\mu \eta \partial^\mu \eta + 2\mu^2 \eta^2) + \frac{1}{2} (\partial_\mu \xi \partial^\mu \xi)$$

and describes 2 particles:

1. a particle η of mass $m = \sqrt{2} |\mu|$, which corresponds to radial oscillations,

2. a particle ξ of mass $m = 0$, which connects vacuum states. ξ is the *Goldstone boson*.

As is well known, the Higgs mechanism consists in using a cooperation between gauge bosons and Goldstone bosons to confer a mass to gauge bosons. Let $\varphi = \frac{1}{\sqrt{2}}(\varphi_1 + i\varphi_2)$ be the scalar complex field associated to φ_1 and φ_2 . Its Lagrangian is

$$\mathcal{L} = \partial_\mu \bar{\varphi} \partial^\mu \varphi - \mu^2 |\varphi|^2 - |\lambda| |\varphi|^4.$$

It is trivially invariant by the global internal symmetry $\varphi \rightarrow e^{i\theta} \varphi$. If we *localize* the global symmetry using transformations $\varphi(x) \rightarrow e^{iq\alpha(x)} \varphi(x)$ and take into account the coupling with an electro-magnetic field deriving from the vector potential A_μ , we get

$$\mathcal{L} = \nabla_\mu \bar{\varphi} \nabla^\mu \varphi - \mu^2 |\varphi|^2 - |\lambda| |\varphi|^4 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

where ∇ is the covariant derivative

$$\nabla_\mu = \partial_\mu + iqA_\mu$$

and F the force field

$$F_{\mu\nu} = \partial_\nu A_\mu - \partial_\mu A_\nu.$$

The Lagrangian remains invariant if we balance the localization of the global internal symmetry with a change of gauge

$$A_\mu \rightarrow A'_\mu = A_\mu - \partial_\mu \alpha(x).$$

For $\mu^2 > 0$, $\varphi_0 = 0$ is a minimum of $V(\varphi)$, the vacuum is non degenerate, and we get 2 scalar particles φ and $\bar{\varphi}$ and a photon A_μ .

For $\mu^2 < 0$, the vacuum is degenerate and there is a spontaneous symmetry breaking. We have $|\varphi_0|^2 = -\frac{\mu^2}{2|\lambda|} = \frac{v^2}{2}$. If we take $\varphi_0 = \frac{v}{\sqrt{2}}$ and write

$$\varphi = \varphi' + \varphi_0 = \frac{1}{\sqrt{2}}(v + \eta + i\xi) \approx \frac{1}{\sqrt{2}} e^{i\frac{\xi}{v}} (v + \eta) \text{ for } \xi \text{ and } \eta \text{ small,}$$

we get for the Lagrangian of oscillations:

$$\mathcal{L}_{os} = \frac{1}{2} (\partial_\mu \eta \partial^\mu \eta + 2\mu^2 \eta^2) + \frac{1}{2} (\partial_\mu \xi \partial^\mu \xi) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + qv A_\mu (\partial_\mu \xi) + \frac{q^2 v^2}{2} A_\mu A^\mu.$$

1. The field η (radial oscillations) has mass $m = \sqrt{2} |\mu|$.
2. The boson A_μ acquires a mass due to the term $A_\mu A^\mu$ and interacts with the Goldstone boson ξ .

The terms containing the gauge boson A_μ and the Goldstone boson ξ write

$$\frac{q^2 v^2}{2} \left(A_\mu + \frac{1}{qv} \partial_\mu \xi \right) \left(A^\mu + \frac{1}{qv} \partial^\mu \xi \right)$$

and are therefore generated by the gauge change

$$\begin{aligned} \alpha &= \frac{\xi}{qv} \\ A_\mu &\rightarrow A_\mu + \partial_\mu \alpha \end{aligned}$$

We see that we can use the gauge transformations

$$A_\mu \rightarrow A'_\mu = A_\mu + \frac{1}{qv} \partial^\mu \xi$$

for *fixing* the vacuum state. The transformation corresponds to the phase rotation of the scalar field

$$\varphi \rightarrow \varphi' = e^{-i\frac{\xi}{v}} \varphi = \frac{v + \eta}{\sqrt{2}}$$

In this new gauge where the Goldstone boson ξ disappears, the vector particule A'_μ acquires a mass qv . The Lagrangian writes now

$$\mathcal{L}_{os} = \frac{1}{2} (\partial_\mu \eta \partial^\mu \eta + 2\mu^2 \eta^2) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{q^2 v^2}{2} A'_\mu A'^\mu$$

The Goldstone boson connecting the degenerate vacuum states is in some sense “captured” by the gauge boson and transformed into mass.

5.2 NC Yang-Mills theory of 2 points and Higgs phenomenon

The NC equivalent of this description is the following. It shows that Higgs mechanism is actually the standard Yang-Mills formalism applied to a purely discrete NC geometry.

Let $\mathcal{A} = \mathcal{C}(Y) = \mathbb{C} \oplus \mathbb{C}$ be the C^* -algebra of the space Y composed of 2 points a and b . Its elements $f = \begin{bmatrix} f(a) & 0 \\ 0 & f(b) \end{bmatrix}$ act through multiplication on the Hilbert space $\mathcal{H} = \mathcal{H}_a \oplus \mathcal{H}_b$. We take for Dirac operator an operator of the form

$$D = \begin{bmatrix} 0 & M^* = D^- \\ M = D^+ & 0 \end{bmatrix}$$

and introduce the “chirality” $\gamma = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ (the γ_5 of the standard Dirac theory). In this discrete situation we define df as

$$df = [D, f] = \Delta f \begin{bmatrix} 0 & M^* \\ -M & 0 \end{bmatrix}$$

with $\Delta f = f(b) - f(a)$. Therefore

$$\|[D, f]\| = |\Delta f| \lambda$$

where $\lambda = \|M\|$ is the greatest eigenvalue of M .

If we apply now the formula for the distance, we find:

$$\begin{aligned} d(a, b) &= \text{Sup} \{ |f(a) - f(b)| : f \in \mathcal{A}, \|[D, f]\| \leq 1 \} \\ &= \text{Sup} \{ |f(a) - f(b)| : f \in \mathcal{A}, |f(a) - f(b)| \lambda \leq 1 \} \\ &= \frac{1}{\lambda} \end{aligned}$$

and we see that the distance $\frac{1}{\lambda}$ between the two points a and b has a *spectral* content and is measured by the Dirac operator.

To interpret differential calculus in this context, we consider the 2 idempotents (projectors) e and $1 - e$ defined by

$$\begin{aligned} e(a) &= 1, e(b) = 0 \\ (1 - e)(a) &= 0, (1 - e)(b) = 1 \end{aligned}$$

Every $f \in \mathcal{A}$ writes $f = f(a)e + f(b)(1 - e)$, and therefore

$$\begin{aligned} df &= f(a)de + f(b)d(1 - e) \\ &= (f(a) - f(b)) de \\ &= -(\Delta f)de \\ &= -(\Delta f)ede + (\Delta f)(1 - e)d(1 - e) \end{aligned}$$

This shows that ede and $(1 - e)d(1 - e) = -(1 - e)de$ provide a natural basis of the space of 1-forms $\Omega^1 \mathcal{A}$. Let

$$\begin{aligned} \omega &= \lambda ede + \mu(1 - e)d(1 - e) \\ &= \lambda ede - \mu(1 - e)de \end{aligned}$$

a 1-form. ω is represented by

$$\omega = (\lambda e - \mu(1 - e)) [D, e]$$

But on \mathcal{H} $[D, e] = - \begin{bmatrix} 0 & M^* \\ -M & 0 \end{bmatrix}$ and therefore

$$\omega = \begin{bmatrix} 0 & -\lambda M^* \\ -\mu M & 0 \end{bmatrix}$$

Let us now construct the *Yang-Mills theory* corresponding to this situation. A vector potential V — a connection in the sense of gauge theories — is a self-adjoint 1-form and has the form

$$\begin{aligned} V &= -\bar{\varphi}ede + \varphi(1 - e)de \\ &= \begin{bmatrix} 0 & \bar{\varphi}M^* \\ \varphi M & 0 \end{bmatrix}. \end{aligned}$$

Its curvature is the 2-form

$$\theta = dV + V \wedge V$$

and an easy computation gives

$$\theta = -(\varphi + \bar{\varphi} + \varphi\bar{\varphi}) \begin{bmatrix} -M^*M & 0 \\ 0 & -MM^* \end{bmatrix}.$$

The Yang-Mills *action* is the integral of the curvature 2-form, that is the *trace* of θ :

$$YM(V) = \text{Trace}(\theta^2)$$

But as $\varphi + \bar{\varphi} + \varphi\bar{\varphi} = |\varphi + 1|^2 - 1$ and

$$\text{Trace} \left(\begin{bmatrix} -M^*M & 0 \\ 0 & -MM^* \end{bmatrix}^2 \right) = 2 \text{Trace}((M^*M)^2)$$

we get

$$YM(V) = 2(|\varphi + 1|^2 - 1)^2 \text{Trace}((M^*M)^2).$$

5.3 Higgs mechanism

This Yang-Mills action manifests a *pure Higgs phenomenon of symmetry breaking*. The minimum of $YM(V)$ is reached everywhere on the circle $|\varphi + 1|^2 = 1$ (degeneracy) and the gauge group $\mathcal{U} = U(1) \times U(1)$ of the unitary elements of \mathcal{A} acts on it by

$$V \rightarrow uVu^* + udu^*$$

where $u = \begin{bmatrix} u_1 & 0 \\ 0 & u_2 \end{bmatrix}$ with $u_1, u_2 \in U(1)$.

The field φ is a Higgs bosonic field corresponding to a gauge connection on a NC space of 2 points. If $\psi \in \mathcal{H}$ represents a fermionic state, the fermionic action is $I_D(V, \psi) = \langle \psi, (D + V) \psi \rangle$ with

$$D + V = \begin{bmatrix} 0 & (1 + \bar{\varphi}) M^* \\ (1 + \varphi) M & 0 \end{bmatrix}.$$

The complete action coupling the fermion ψ with the Higgs boson φ is therefore

$$YM(V) + I_D(V, \psi).$$

6 The NC derivation of the Glashow-Weinberg-Salam Standard Model (Connes-Lott)

A remarkable achievement of this NC approach of Yang-Mills theories is given by Connes-Lott's NC derivation of the Glashow-Weinberg-Salam Standard Model. This derivation was possible because, as was emphasized by Martin et al. [1997] (p. 5), it ties

“the properties of continuous spacetime with the intrinsic discreteness stemming from the chiral structure of the Standard Model”.

6.1 Gauge theory and NCG

It is easy to reinterpret in the NC framework classical gauge theories where M is a spin manifold, $\mathcal{A} = \mathcal{C}^\infty(M)$, D is the Dirac operator and $\mathcal{H} = L^2(M, S)$ is the space of L^2 sections of the spinor bundle S . $\text{Diff}(M) = \text{Aut}(\mathcal{A}) = \text{Aut}(\mathcal{C}^\infty(M))$ is the relativity group (the gauge group) of the theory: a diffeomorphism $\varphi \in \text{Diff}(M)$ is identified with the *-automorphism $\alpha \in \text{Aut}(\mathcal{A})$ s.t. $\alpha(f)(x) = f(\varphi^{-1}(x))$. The main problem is to reconcile QFT with GR, that is non abelian gauge theories which are non commutative at the level of their *internal* space of quantum variables with the geometry of the *external* space-time M with its group of diffeomorphism $\text{Diff}(M)$. The NC solution is an extraordinary principled one since it links the standard “inner” non commutativity of quantum internal degrees of freedom with the new “outer” non commutativity of the external space.

6.1.1 Inner automorphisms and internal symmetries

The key fact is that, in the NC framework, there exists in $\text{Aut}(\mathcal{A})$ the normal subgroup $\text{Inn}(\mathcal{A})$ of *inner automorphisms* acting by conjugation $a \rightarrow uau^{-1}$. $\text{Inn}(\mathcal{A})$ is trivial in the commutative case and constitutes one of the main feature of the NC case. As Alain Connes [1996] emphasized:

“Amazingly, in this description the group of gauge transformation of the matter fields arises spontaneously as a normal subgroup of the generalized diffeomorphism group $\text{Aut}(\mathcal{A})$. It is the *non commutativity* of the algebra \mathcal{A} which gives for free the group of gauge transformations of matter fields as a (normal) subgroup of the group of diffeomorphisms.”

In $\text{Inn}(\mathcal{A})$ there exists in particular the *unitary* group $\mathcal{U}(\mathcal{A})$ of unitary elements $u^* = u^{-1}$ acting by $\alpha_u(a) = uau^*$.

6.1.2 Connections and vector potentials

In the NC framework we can easily reformulate standard Yang-Mills theories. For that we need the concepts of a connection and of a vector potential.

Let \mathcal{E} be a finite projective (right) \mathcal{A} -module. A connection ∇ on \mathcal{E} is a collection of morphisms (for every p)

$$\nabla : \mathcal{E} \otimes_{\mathcal{A}} \Omega^p(\mathcal{A}) \rightarrow \mathcal{E} \otimes_{\mathcal{A}} \Omega^{p+1}(\mathcal{A})$$

satisfying for every $\omega \in \mathcal{E} \otimes_{\mathcal{A}} \Omega^p(\mathcal{A})$ and every $\rho \in \Omega^q(\mathcal{A})$ the Leibniz rule in $\mathcal{E} \otimes_{\mathcal{A}} \Omega^{p+q+1}(\mathcal{A})$

$$\nabla(\omega \otimes \rho) = \nabla(\omega) \otimes \rho + (-1)^p \omega \otimes d\rho$$

where we use $\Omega^{p+1}(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^q(\mathcal{A}) = \Omega^p(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^{q+1}(\mathcal{A})$.

∇ is determined by its restriction to $\Omega^1(\mathcal{A})$

$$\nabla : \mathcal{E} \otimes_{\mathcal{A}} \Omega^0(\mathcal{A}) = \mathcal{E} \rightarrow \mathcal{E} \otimes_{\mathcal{A}} \Omega^1(\mathcal{A})$$

satisfying $\nabla(\xi a) = \nabla(\xi) a + \xi \otimes da$ for $\xi \in \mathcal{E}$ and $a \in \mathcal{A}$.

The curvature θ of ∇ is given by $\nabla^2 : \mathcal{E} \rightarrow \mathcal{E} \otimes_{\mathcal{A}} \Omega^2(\mathcal{A})$. As

$$\begin{aligned} \nabla^2(\xi a) &= \nabla(\nabla(\xi) a + \xi \otimes da) \\ &= \nabla^2(\xi) a - \nabla(\xi) \otimes da + \nabla(\xi) \otimes da + \xi \otimes d^2 a \\ &= \nabla^2(\xi) a, \end{aligned}$$

∇^2 is \mathcal{A} -linear. And as \mathcal{E} is a projective \mathcal{A} -module,

$$\theta = \nabla^2 \in \text{End}_{\mathcal{A}} \mathcal{E} \otimes_{\mathcal{A}} \Omega^2(\mathcal{A}) = M(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^2(\mathcal{A})$$

is a matrix with elements in $\Omega^2(\mathcal{A})$.

Now, ∇ defines a connection $[\nabla, \bullet]$ on $\text{End}_{\mathcal{A}} \mathcal{E}$ by

$$\begin{aligned} [\nabla, \bullet] : \text{End}_{\mathcal{A}} \mathcal{E} \otimes_{\mathcal{A}} \Omega^p(\mathcal{A}) &\rightarrow \text{End}_{\mathcal{A}} \mathcal{E} \otimes_{\mathcal{A}} \Omega^{p+1}(\mathcal{A}) \\ \alpha &\mapsto [\nabla, \alpha] = \nabla \circ \alpha - \alpha \circ \nabla \end{aligned}$$

and the curvature θ satisfies the *Bianchi identity* $[\nabla, \theta] = 0$.

A vector potential A is a self-adjoint operator interpreting a 1-form

$$A = \sum_j a_j [D, b_j]$$

and the force is the curvature 2-form

$$\theta = dA + A^2.$$

The unitary group $\mathcal{U}(\mathcal{A})$ acts by gauge transformations on A and its 2-form curvature θ

$$\begin{aligned} A &\rightarrow uAu^* + udu^* = uAu^* + u[D, u^*] \\ \theta &\rightarrow u\theta u^*. \end{aligned}$$

6.2 Axioms for geometry

There are characteristic properties of classical (commutative) and NC geometries which can be used to axiomatize them.

1. (Classical and NC geometry). $ds = D^{-1}$ is an infinitesimal of order $\frac{1}{n}$ (n is the dimension)¹² and for any $a \in \mathcal{A}$ integration is given by $\text{Tr}_{\text{Dix}}(a |D|^{-n})$ (which is well defined and $\neq 0$ since $|D|^{-n}$ is an infinitesimal of order 1). One can normalize the integral dividing by $V = \text{Tr}_{\text{Dix}}(|D|^{-n})$.

2. (Classical geometry). Universal commutation relations: $[[D, a], b] = 0, \forall a, b \in \mathcal{A}$. So (Jones, Moscovici [1997])

“while ds no longer commutes with the coordinates, the algebra they generate does satisfy non trivial commutation relations.”

3. (Classical and NC geometry). $a \in \mathcal{A}$ is “smooth” in the sense that a and $[D, a]$ belong to the intersection of the domains of the functionals δ^m where $\delta(T) = [|D|, T]$ for every operator T on \mathcal{H} .

4. (Classical geometry). If the dimension n is *even* there exists a $\tilde{\gamma}$ interpreting a n -form $c \in Z_n(\mathcal{A}, \mathcal{A})$ associated to orientation and chirality (the γ^5 of Dirac), $\tilde{\gamma}$ being of the form $a_0 [D, a_1] \dots [D, a_n]$ and s.t. $\tilde{\gamma} = \tilde{\gamma}^*$ (self-adjointness), $\tilde{\gamma}^2 = 1$, $\{\tilde{\gamma}, D\} = 0$ (anti-commutation relation) and $[\tilde{\gamma}, a] = 0, \forall a \in \mathcal{A}$ (commutation relations). $\tilde{\gamma}$ decomposes

¹²In the NC framework, ds and dx are completely different sort of entities. dx is the differential of a coordinate and ds doesn't commute with it. In the classical case, the order of ds as an infinitesimal is not 1 but the dimension of M . As we will see later, the Hilbert-Einstein action is the NC integral of ds^{n-2} .

D into two parts $D = D^+ + D^-$ where $D^+ = (1-p)Dp$ with $p = \frac{1+\tilde{\gamma}}{2}$. If e is a self-adjoint ($e = e^*$) idempotent ($e^2 = e$) of \mathcal{A} (i.e. a projector), eD^+e is a Fredholm operator from the subspace $ep\mathcal{H}$ to the subspace $e(1-p)\mathcal{H}$. This can be extended to the projectors of $e \in M_q(\mathcal{A})$ defining finite projective left \mathcal{A} -modules $\mathcal{E} = \mathcal{A}^N e$ (if $\xi \in \mathcal{E}$ then $\xi e = \xi$) with the \mathcal{A} -valued inner product $(\xi, \eta) = \sum_{i=1}^N \xi_i \eta_i^*$.

4 bis. (Classical geometry). If n is odd we ask only that there exists such an n -form c interpreted by 1: $a_0 [D, a_1] \dots [D, a_n] = 1$.

5. (Classical and NC geometry). $\mathcal{H}_\infty = \bigcap_m \text{Domain}(D^m)$ is finite and projective as \mathcal{A} -module and $\langle a\xi, \eta \rangle = \text{Tr}_{Dix} a(\xi, \eta) ds^n$ ((ξ, η) being the scalar product of \mathcal{H} and Tr_{Dix} the Dixmier trace of infinitesimals of order 1) define an Hermitian structure on \mathcal{H}_∞ .

6. (Classical geometry). One can define an *index pairing* of D with $K_0(\mathcal{A})$ and an *intersection form* on $K_0(\mathcal{A})$ ¹³. If $[\mathcal{E}] \in K_0(\mathcal{A})$ is defined by the projector e , we consider the scalar product $\langle \text{Ind } D, e \rangle$ which is an integer. We define therefore $\langle \text{Ind } D, e \rangle : K_0(\mathcal{A}) \rightarrow \mathbb{Z}$. As \mathcal{A} is commutative, we can take the multiplication $m : \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}$ given by $m(a \otimes b) = ab$ which induces $m_0 : K_0(\mathcal{A}) \otimes K_0(\mathcal{A}) \rightarrow K_0(\mathcal{A})$. Composing with $\text{Ind } D$ we get the intersection form

$$\begin{aligned} \langle \text{Ind } D, m_0 \rangle : K_0(\mathcal{A}) \otimes K_0(\mathcal{A}) &\rightarrow \mathbb{Z} \\ (e, a) &\rightarrow \langle \text{Ind } D, m_0(e \otimes a) \rangle. \end{aligned}$$

Poincaré duality: the intersection form is invertible.

7. *Real structure* (Classical geometry). There exists an anti-linear isometry (charge conjugation) $J : \mathcal{H} \rightarrow \mathcal{H}$ which combines charge conjugation and complex conjugation and gives the $*$ -involution by algebraic conjugation: $JaJ^{-1} = a^* \forall a \in \mathcal{A}$, and s.t. $J^2 = \varepsilon$, $JD = \varepsilon' DJ$, and $J\gamma = \varepsilon'' \gamma J$ with $\varepsilon, \varepsilon', \varepsilon'' = \pm 1$ depending of the dimension $n \bmod 8$:

n	0	1	2	3	4	5	6	7
ε	1	1	-1	-1	-1	-1	1	1
ε'	1	-1	1	1	1	-1	1	1
ε''	1		-1		1		-1	

In the classical case (M smooth compact manifold of dimension n), Connes proved that these axioms define a unique Riemannian spin geometry whose geodesic distance and the spin structure are those defined by D . Moreover, the value of the Dixmier trace $\text{Tr}_{Dix} ds^{n-2}$ is the *Einstein-Hilbert action functional*:

$$\text{Tr}_{Dix} ds^{n-2} = c_n \int_M R \sqrt{g} d^n x = c_n \int_M R dv$$

¹³Remember that $K_0(\mathcal{A}) = \pi_1(GL_\infty(\mathcal{A}))$ classifies the finite projective \mathcal{A} -modules and that $K_1(\mathcal{A}) = \pi_0(GL_\infty(\mathcal{A}))$ is the group of connected components of $GL_\infty(\mathcal{A})$.

with dv the volume form $dv = \sqrt{g}d^n x$ and $c_n = \frac{n-2}{12} (4\pi)^{-\frac{n}{2}} \Gamma(\frac{n}{2} + 1)^{-1} 2^{\lfloor \frac{n}{2} \rfloor}$. $\text{Tr}_{Dix} ds^{n-2}$ is well defined and $\neq 0$ since ds is an infinitesimal of order $\frac{n-2}{n} < 1$. For $n = 4$, $c_4 = \frac{1}{6} (4\pi)^{-2} \Gamma(3)^{-1} 2^2 = \frac{1}{48\pi^2}$.

In the NC case the characteristic properties (2), (6), (7) must be modified to take into account the NC:

7^{NC} . *Real structure* (NC geometry). In the noncommutative case, the axiom $JaJ^{-1} = a^*$ is transformed into the following axiom saying that the conjugation by J of the involution defines the *opposed* multiplication of \mathcal{A} . Let $b^0 = Jb^*J^{-1}$, then $[a, b^0] = 0$, $\forall a, b \in \mathcal{A}$. By means of this real structure, the Hilbert space \mathcal{H} becomes not only a (left) \mathcal{A} -module through the representation of \mathcal{A} into $\mathcal{L}(\mathcal{H})$ but also a $\mathcal{A} \otimes \mathcal{A}^\circ$ -module (where \mathcal{A}° is the opposed algebra of \mathcal{A}) or a (left-right) \mathcal{A} -bimodule through $(a \otimes b^0)\xi = aJb^*J^{-1}\xi$ or $a\xi b = aJb^*J^{-1}\xi$ for every $\xi \in \mathcal{H}$.

2^{NC} . The universal commutation relations $[[D, f], g] = 0$, $\forall f, g \in \mathcal{A}$ become in the NC case $[[D, a], b^\circ] = 0$, $\forall a, b \in \mathcal{A}$ (which is equivalent to $[[D, b^\circ], a] = 0$, $\forall a, b \in \mathcal{A}$ since a and b° commute by 7^{NC}).

6^{NC} . *K*-theory can be easily generalized to the NC case. We consider finite projective \mathcal{A} -modules \mathcal{E} , that is direct factors of free \mathcal{A} -modules \mathcal{A}^N . They are characterized by a projection $\pi : \mathcal{A}^N \rightarrow \mathcal{E}$ admitting a section $s : \mathcal{E} \rightarrow \mathcal{A}^N$ ($\pi \circ s = Id_{\mathcal{E}}$). $K_0(\mathcal{A})$ classifies them. The structure of $\mathcal{A} \otimes \mathcal{A}^\circ$ -module induced by the real structure J allows to define the intersection form by $(e, a) \rightarrow \langle \text{Ind } D, e \otimes a^\circ \rangle$ with $e \otimes a^\circ$ considered as an element of $K_0(\mathcal{A} \otimes \mathcal{A}^\circ)$.

One of the fundamental aspects of the NC case is that inner automorphisms $\alpha_u(a) = uau^*$, $u \in \mathcal{U}(\mathcal{A})$ act upon the Dirac operator D via NC gauge connections (vector potentials) A

$$\begin{aligned}\tilde{D} &= D + A + JAJ^{-1} \text{ with} \\ A &= u[D, u^*]\end{aligned}$$

the equivalence between D and \tilde{D} being given by $\tilde{D} = UDU^{-1}$ with $U = uJuJ^{-1} = u(u^*)^\circ$.

6.3 The crucial discovery of a structural link between “external” metric and “internal” gauge transformations

One can generalize these transformations of metrics to gauge connections A of the form $A = \sum a_i [D, b_i]$ which can be interpreted as *internal perturbations of the metric* or as *internal fluctuations of the spectral geometry* induced by the internal degrees of freedom of gauge transformations. This coupling between metric and gauge transformations is

what is needed *for coupling gravity with QFT*. In the commutative case, this coupling *vanishes* since $U = uu^* = 1$ and therefore $\tilde{D} = D$. The vanishing $A + JAJ^{-1} = 0$ comes from the fact that A is self-adjoint and that, due to its special form $A = a[D, b]$, we have $JAJ^{-1} = -A^*$. Indeed, since $[D, b^*] = -[D, b]^*$

$$\begin{aligned} JAJ^{-1} &= Ja[D, b]J^{-1} = JaJ^{-1}J[D, b]J^{-1} = a^*[D, b^*] \\ &= -a^*[D, b]^* = -(a[D, b])^* = -A^* \end{aligned}$$

So the coupling between the “external” metric afforded by the Dirac operator and the internal quantum degrees of freedom is a purely NC effect which constitutes a breakthrough for the unification of GR and QFT in a “good” theory of Quantum Gravity (QG).

6.4 Generating the Standard Model (Connes-Lott)

Before concluding this paper with some remarks on QG, let us recall that the first main interest of NC geometry in physics was generated by the possibility of coupling classical gauge theories with purely NC such theories. This led to the NC interpretation of Higgs fields. Connes’ main result is:

Connes’ theorem. The Glashow-Weinberg-Salam Standard Model (SM) can be entirely reconstructed from the NC C^* -algebra

$$\mathcal{A} = \mathcal{C}^\infty(M) \otimes (\mathbb{C} \oplus \mathbb{H} \oplus M^3(\mathbb{C}))$$

where the “internal” algebra $\mathbb{C} \oplus \mathbb{H} \oplus M^3(\mathbb{C})$ has for unitary group the symmetry group

$$U(1) \times SU(2) \times SU(3).$$

The first step is to construct the toy model which is the product $\mathcal{C}^\infty(M) \otimes (\mathbb{C} \oplus \mathbb{C})$ of the classical Dirac fermionic model $(\mathcal{A}_1, \mathcal{H}_1, D_1, \gamma_5)$ and the previously explained, purely NC, 2-points model $(\mathcal{A}_2, \mathcal{H}_2, D_2, \gamma)$ with $D_2 = \begin{bmatrix} 0 & M^* \\ M & 0 \end{bmatrix}$:

$$\begin{cases} \mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2 \\ \mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \\ D = D_1 \otimes 1 + \gamma_5 \otimes D_2 . \end{cases}$$

The second step is to complexify the model and to show that it enables to derive the complete GWS Lagrangian.

The key idea is to take the product of a 4-dimensional spin manifold M with a finite NC geometry $(\mathcal{A}_F, \mathcal{H}_F, D_F)$ of dimension 0 where \mathcal{H}_F is the Hilbert space with basis the generations of fermions: quarks, leptons.

The particle/antiparticle duality decomposes \mathcal{H}_F into $\mathcal{H}_F = \mathcal{H}_F^+ \oplus \mathcal{H}_F^-$, each \mathcal{H}_F^\pm decomposes into $\mathcal{H}_F^\pm = \mathcal{H}_l^\pm \oplus \mathcal{H}_q^\pm$ ($l = \text{lepton}$ and $q = \text{quark}$), and chirality decomposes the \mathcal{H}_p^\pm ($p = \text{particle}$) into $\mathcal{H}_{pL}^\pm \oplus \mathcal{H}_{pR}^\pm$ ($L = \text{left}$, $R = \text{right}$). The 4 quarks are u_L, u_R, d_L, d_R ($u = \text{up}$, $d = \text{down}$) with 3 colours (12 quarks for each generation) and the 3 leptons are e_L, ν_L, e_R , the total being of $2(12 + 3) = 30$ fermions for each generation.

The real structure J is given for $\mathcal{H}_F = \mathcal{H}_F^+ \oplus \mathcal{H}_F^-$ by $J \begin{pmatrix} \xi \\ \bar{\eta} \end{pmatrix} = \begin{pmatrix} \eta \\ \bar{\xi} \end{pmatrix}$ that is, if $\xi = \sum_i \lambda_i p_i$ and $\bar{\eta} = \sum_j \mu_j \bar{p}_j$,

$$J \left(\sum_i \lambda_i p_i + \sum_j \mu_j \bar{p}_j \right) = \left(\sum_j \bar{\mu}_j p_j + \sum_i \bar{\lambda}_i \bar{p}_i \right).$$

The action of the internal algebra $\mathcal{A}_F = \mathbb{C} \oplus \mathbb{H} \oplus M^3(\mathbb{C})$ is defined in the following way. Let $a = (\lambda, q, m) \in \mathcal{A}_F$, $\lambda \in \mathbb{C}$ being a complex scalar acting upon \mathbb{C}^2 as the diagonal quaternion $\begin{pmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{pmatrix}$, $q = \alpha + \beta j \in \mathbb{H}$ being a quaternion written as $\begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}$ $j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and $m \in M^3(\mathbb{C})$ being a 3×3 complex matrix. The element $a = (\lambda, q, m)$ acts on quarks, independently of color, via $au_R = \lambda u_R$, $au_L = \alpha u_L - \bar{\beta} d_L$, $ad_R = \bar{\lambda} d_R$, $ad_L = \beta u_L + \bar{\alpha} d_L$, that is as

$$(\lambda, q, m) \begin{pmatrix} u_L \\ d_L \\ u_R \\ d_R \end{pmatrix} = \begin{pmatrix} \alpha - \bar{\beta} & 0 & 0 \\ \beta & \bar{\alpha} & 0 \\ 0 & 0 & \lambda \\ 0 & 0 & 0 \bar{\lambda} \end{pmatrix} \begin{pmatrix} u_L \\ d_L \\ u_R \\ d_R \end{pmatrix} = \begin{pmatrix} \alpha u_L - \bar{\beta} d_L \\ \beta u_L + \bar{\alpha} d_L \\ \lambda u_R \\ \bar{\lambda} d_R \end{pmatrix}$$

(the pair (u_R, d_R) can be considered as an element of $\mathbb{C} \oplus \mathbb{C}$, while (u_L, d_L) can be considered as an element of \mathbb{C}^2). It acts on leptons via $ae_R = \bar{\lambda} e_R$, $ae_L = \beta \nu_L + \bar{\alpha} e_L$, $a\nu_L = \alpha \nu_L - \bar{\beta} e_L$, that is as

$$(\lambda, q, m) \begin{pmatrix} e_R \\ \nu_L \\ e_L \end{pmatrix} = \begin{pmatrix} \bar{\lambda} & 0 & 0 \\ 0 & \alpha - \bar{\beta} \\ 0 & \beta & \bar{\alpha} \end{pmatrix} \begin{pmatrix} e_R \\ \nu_L \\ e_L \end{pmatrix} = \begin{pmatrix} \bar{\lambda} e_R \\ \alpha \nu_L - \bar{\beta} e_L \\ \beta \nu_L + \bar{\alpha} e_L \end{pmatrix}.$$

It acts on anti-particles via $a\bar{l} = \bar{\lambda} \bar{l}$ for antileptons and via $a\bar{q} = m\bar{q}$ for antiquarks where m acts upon color.

The internal Dirac operator D_F is given by the matrix of Yukawa coupling $D_F = \begin{pmatrix} Y & 0 \\ 0 & \bar{Y} \end{pmatrix}$ where $Y = (Y_q \otimes 1_3) \oplus Y_l$ (the $\otimes 1_3$ comes from

the 3 generations of fermions) with

$$Y_q = \begin{matrix} & u_L & d_L & u_R & d_R \\ \begin{matrix} u_L \\ d_L \\ u_R \\ d_R \end{matrix} & \begin{pmatrix} 0 & 0 & M_u & 0 \\ 0 & 0 & 0 & M_d \\ M_u^* & 0 & 0 & 0 \\ 0 & M_d^* & 0 & 0 \end{pmatrix} \end{matrix}$$

and

$$Y_l = \begin{matrix} & e_R & \nu_L & e_L \\ \begin{matrix} e_R \\ \nu_L \\ e_L \end{matrix} & \begin{pmatrix} 0 & 0 & M_l \\ 0 & 0 & 0 \\ M_l^* & 0 & 0 \end{pmatrix} \end{matrix}$$

where (Connes 1996) M_u , M_d , and M_l are matrices “which encode both the masses of the Fermions and their mixing properties”.

Chirality is given by $\gamma_F(p_R) = p_R$ and $\gamma_F(p_L) = -p_L$ (p being any particle or anti-particle).

Connes and Lott then take the product of this internal model of the fermionic sector with a classical gauge model for the bosonic sector:

$$\begin{cases} \mathcal{A} = C^\infty(M) \otimes \mathcal{A}_F = (C^\infty(M) \otimes \mathbb{C}) \oplus (C^\infty(M) \otimes \mathbb{H}) \oplus (C^\infty(M) \otimes M^3(\mathbb{C})) \\ \mathcal{H} = L^2(M, S) \otimes \mathcal{H}_F = L^2(M, S \otimes \mathcal{H}_F) \\ D = (D_M \otimes 1) \oplus (\gamma_5 \otimes D_F). \end{cases}$$

The extraordinary “tour de force” is that this model, which is rather simple at the conceptual level (a product of two models, respectively fermionic and bosonic, which takes into account only the known fundamental properties of these two sectors), is in fact extremely complex and generates SM in a *principled* way. Computations are very intricate (see Kastler papers in the bibliography). One has to compute first vector potentials of the form $A = \sum_i a_i [D, a'_i]$, $a_i, a'_i \in \mathcal{A}$ which induce fluctuations of the metric. As D is a sum of two terms, it is also the case for A . Its discrete part comes from $\gamma_5 \otimes D_F$ and generates the Higgs bosons. Let $a_i(x) = (\lambda_i(x), q_i(x), m_i(x))$. The term $\sum_i a_i [\gamma_5 \otimes D_F, a'_i]$ yields γ_5 tensored by matrices of the form:

- Quark sector:

$$\begin{pmatrix} 0 & 0 & M_u \varphi_1 & M_u \varphi_2 \\ 0 & 0 & -M_d \overline{\varphi_2} & M_d \overline{\varphi_1} \\ M_u^* \varphi'_1 & M_d^* \varphi'_2 & 0 & 0 \\ -M_u^* \overline{\varphi'_2} & M_d^* \overline{\varphi'_1} & 0 & 0 \end{pmatrix}$$

with

$$\begin{cases} \varphi_1 = \sum_i \lambda_i (\alpha'_i - \lambda'_i) \\ \varphi_2 = \sum_i \lambda_i \beta'_i \\ \varphi'_1 = \sum_i \alpha_i (\lambda'_i - \alpha'_i) + \beta_i \overline{\beta'_i} \\ \varphi'_2 = \sum_i \beta_i (\overline{\lambda'_i} - \overline{\alpha'_i}) - \alpha_i \beta'_i. \end{cases}$$

- Lepton sector:

$$\begin{pmatrix} 0 & -M_d \overline{\varphi_2} & M_d \overline{\varphi_1} \\ M_d^* \varphi'_2 & 0 & 0 \\ M_d^* \varphi'_1 & 0 & 0 \end{pmatrix}.$$

Let $q = \varphi_1 + \varphi_2 j$ and $q' = \varphi'_1 + \varphi'_2 j$ be the quaternionic fields so defined. As $A = A^*$, we have $q' = q^*$. The \mathbb{H} -valued field $q(x)$ is the *Higgs doublet*.

The second part of the vector potential A comes from $D_M \otimes 1$ and generates the gauge bosons. The terms $\sum_i a_i [D_M \otimes 1, a'_i]$ yield

- the $U(1)$ gauge field $\Lambda = \sum_i \lambda_i d\lambda'_i$;
- the $SU(2)$ gauge field $Q = \sum_i q_i dq'_i$;
- the $U(3)$ gauge field $V = \sum_i m_i dm'_i$.

The computation of the fluctuations of the metric $A + JAJ^{-1}$ gives:

- Quark sector:

$$\begin{matrix} & u_L & d_L & u_R & d_R \\ \begin{matrix} u_L \\ d_L \\ u_R \\ d_R \end{matrix} & \begin{pmatrix} Q_{11}1_3 + V & Q_{12}1_3 & 0 & 0 \\ Q_{21}1_3 & Q_{22}1_3 + V & 0 & 0 \\ 0 & 0 & \Lambda 1_3 + V & 0 \\ 0 & 0 & 0 & -\Lambda 1_3 + V \end{pmatrix} \end{matrix}$$

which is a 12×12 matrix since V is 3×3 .

- Lepton sector:

$$\begin{matrix} & e_R & \nu_L & e_L \\ \begin{matrix} e_R \\ \nu_L \\ e_L \end{matrix} & \begin{pmatrix} -2\Lambda & 0 & 0 \\ 0 & Q_{11} - \Lambda & Q_{12} \\ 0 & Q_{21} & Q_{22} - \Lambda \end{pmatrix} \end{matrix}$$

One can suppose moreover that $\text{Trace } V = \Lambda$, that is $V = V' + \frac{1}{3}\Lambda$ with V' traceless, which gives the correct hypercharges.

The crowning of the computation is that the total (bosonic + fermionic) action

$$\text{Tr}_{Dix} \theta^2 ds^4 + \langle (D + A + JAJ^{-1}) \psi, \psi \rangle = YM(A) + \langle D_A \psi, \psi \rangle$$

(where $\theta = dA + A^2$ is the curvature of the connection A) enables to derive *the complete GWS Lagrangian*

$$\mathcal{L} = \mathcal{L}_G + \mathcal{L}_f + \mathcal{L}_\varphi + \mathcal{L}_Y + \mathcal{L}_V .$$

1. \mathcal{L}_G is the Lagrangian of the gauge bosons

$$\begin{aligned} \mathcal{L}_G &= \frac{1}{4} (G_{\mu\nu a} G_a^{\mu\nu}) + \frac{1}{4} (F_{\mu\nu} F^{\mu\nu}) \\ G_{\mu\nu a} &= \partial_\mu W_{\nu a} - \partial_\nu W_{\mu a} + g \varepsilon_{abc} W_{\mu b} W_{\nu c}, \\ &\text{with } W_{\mu a} \text{ a } SU(2) \text{ gauge field (weak isospin)} \\ F_{\mu\nu} &= \partial_\mu B_\nu - \partial_\nu B_\mu, \text{ with } B_\mu \text{ a } SU(1) \text{ gauge field.} \end{aligned}$$

2. \mathcal{L}_f is the fermionic kinetic term

$$\begin{aligned} \mathcal{L}_f &= - \sum \bar{f}_L \gamma^\mu \left(\partial_\mu + ig \frac{\tau_a}{2} W_{\mu a} + ig' \frac{Y_L}{2} B_\mu \right) f_L + \\ &\quad \bar{f}_R \gamma^\mu \left(\partial_\mu + ig' \frac{Y_R}{2} B_\mu \right) f_R \end{aligned}$$

with $f_L = \begin{bmatrix} \nu_L \\ e_L \end{bmatrix}$ left fermion fields of hypercharge $Y_L = -1$ and $f_R = (e_R)$ right fermion fields of hypercharge $Y_R = -2$.

3. \mathcal{L}_φ is the Higgs kinetic term

$$\mathcal{L}_\varphi = - \left| \left(\partial_\mu + ig \frac{\tau_a}{2} W_{\mu a} + ig' \frac{Y}{2} B_\mu \right) \varphi \right|^2$$

with $\varphi = \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix}$ a $SU(2)$ pair of scalar complex fields of hypercharge $Y_\varphi = 1$.

4. \mathcal{L}_Y is a Yukawa coupling between the Higgs fields and the fermions

$$\mathcal{L}_Y = - \sum (H_{ff'} (\bar{f}_L \cdot \varphi) f'_R + H_{ff'}^* \bar{f}'_R (\varphi^+ \cdot f_L))$$

where $H_{ff'}$ is a coupling matrix.

5. \mathcal{L}_V is the Lagrangian of the self-interaction of the Higgs fields

$$\mathcal{L}_V = \mu^2 (\varphi^+ \varphi) - \frac{1}{2} \lambda (\varphi^+ \varphi)^2 \text{ with } \lambda > 0.$$

7 Quantum gravity, fluctuating background geometry, and spectral invariance (Connes - Chamseddine)

7.1 Quantum Field Theory and General Relativity

As we have already emphasized, Alain Connes realized a new breakthrough in Quantum Gravity by coupling such models with General Relativity. In NCG, QG can be thought of in a principled way because it becomes possible to introduce in the model QFT the gravitational Einstein-Hilbert action as a direct consequence of the specific invariance of spectral geometry, namely *spectral invariance*. As Alain Connes [1996] explains:

“However this [the previous NC deduction of the SM] requires the definition of the curvature and is still in the spirit of gauge theories. (...) One should consider the internal gauge symmetries as part of the diffeomorphism group of the non commutative geometry, and the gauge bosons as the internal fluctuations of the metric. It follows then that the action functional should be of a purely gravitational nature. We state the principle of spectral invariance, stronger than the invariance under diffeomorphisms, which requires that the action functional only depends on the spectral properties of $D = ds^{-1}$ in \mathcal{H} .”

The general strategy for coupling a Yang-Mills-Higgs gauge theory with the Einstein-Hilbert action is to find a C^* -algebra \mathcal{A} s.t. the normal subgroup $\text{Inn}(\mathcal{A})$ of inner automorphisms is the gauge group and the quotient group $\text{Out}(\mathcal{A}) = \text{Aut}(\mathcal{A})/\text{Inn}(\mathcal{A})$ of “external” automorphisms plays the role of $\text{Diff}(M)$ in a gravitational theory. Indeed, in the classical setting we have principal bundles $P \rightarrow M$ with a structural group G acting upon the fibers and an exact sequence

$$Id \rightarrow \mathcal{G} \rightarrow \text{Aut}(P) \rightarrow \text{Diff}(M) \rightarrow Id$$

where $\mathcal{G} = C^\infty(M, G)$ is the gauge group. The non abelian character of these gauge theories comes solely from the non commutativity of the group of *internal* symmetries G . The total symmetry group $\text{Aut}(P)$ of the theory is the *semidirect* product \mathfrak{G} of $\text{Diff}(M)$ and $\mathcal{G} = C^\infty(M, G)$. If we want to geometrize the theory completely, we would have to find a generalized space X s.t. $\text{Aut}(X) = \mathfrak{G}$.

“If such a space would exist, then we would have some chance to actually geometrize completely the theory, namely to be able to say that it’s pure gravity on the space X .” (Connes [2000])

But this is impossible if X is a manifold since a theorem of John Mather proves that in that case the group $\text{Diff}(X)$ would be simple (without normal subgroup) and could’nt therefore be a semidirect product. *But it is possible with a NC space $(\mathcal{A}, \mathcal{H}, D)$.* For then (Iochum, Kastler, Schücker [1996])

“the metric ‘fluctuates’, that is, it picks up additional degrees of freedom from the internal space, the Yang-Mills connection and the Higgs scalar. (...) In physicist’s language, the spectral triplet is the Dirac action of a multiplet of dynamical fermions in a background field. This background field is a fluctuating metric, consisting of so far adynamical bosons of spin 0,1 and 2”.

If we find a NC geometry \mathcal{A} with $\text{Inn}(\mathcal{A}) \simeq \mathcal{G}$, a correct spectral triple and apply the spectral action, then gravity will correspond to $\text{Out}(\mathcal{A}) = \text{Aut}(\mathcal{A})/\text{Inn}(\mathcal{A})$. As was emphasized by Martin et al. [1997]:

“The strength of Connes’ conception is that gauge theories are thereby deeply connected to the underlying geometry, on the same footing as gravity. The distinction between gravitational and gauge theories boils down to the difference between outer and inner automorphisms.”

Jones and Moscovici [1997] add that this implies that

“Connes’ spectral approach gains the ability to reach below the Planck scale and attempt to decipher the fine structure of space-time”.

So, just as GR extends the Galilean or Minkowskian invariance into diffeomorphism invariance, NCG extends both diffeomorphism invariance and gauge invariance into a larger invariance, the spectral invariance.

The philosophically significant content of the NC point of view must be emphasized. We already saw that in GR the metric of M is no longer a background structure (but the differentiable structure of M remains a background) while in QFT the metric of M is still a background structure. In the NC framework the metric is no longer a background structure, as in GR, but in addition it is a quantum fluctuating structure.

7.2 The spectral action and the eigenvalues of the Dirac operator as dynamical variables for general relativity

The key device is the bosonic spectral action

$$\text{Trace} \left(\phi \left(\frac{D^2}{\Lambda^2} \right) \right) = \text{Trace} \left(\phi \left(\frac{|D|}{\Lambda} \right) \right)$$

where Λ is a cut-off of the order of the inverse of Planck length and ϕ a smooth approximation of the characteristic function $\chi_{[0,1]}$ of the unit interval. $D^2 = (D_M \otimes 1 + \gamma_5 \otimes D_F)^2$ is computed using Lichnerowicz' formula $D^2 = \Delta^S + \frac{1}{4}R$. As this action counts the number $N(\Lambda)$ of eigenvalues of D in the interval $[-\Lambda, \Lambda]$, the key idea is, as formulated by Giovanni Landi and Carlo Rovelli [1997],

“to consider the eigenvalues of the Dirac operator as dynamical variables for general relativity”.

This formulation highlights the physical and transcendental significance of the NC framework: *since the distance is defined through the Dirac operator D , the spectral properties of D can be used in order to modify the metric.* The eigenvalues are spectral invariants and are therefore, in the classical case, automatically $\text{Diff}(M)$ invariant.

“Thus the general idea is to describe spacetime geometry by giving the eigen-frequencies of the spinors that can live on that spacetime. [...] The Dirac operator D encodes the full information about the spacetime geometry in a way usable for describing gravitational dynamics.” (Landi-Rovelli [1997]: the quotation concerns our D_M acting on the Hilbert space of spinor fields on M .)

This crucial point has also been well explained by Steven Carlip (2001, p. 47). As we have seen in the Introduction, in GR points of space-time lose any physical meaning so that GR observables must be radically non-local. This is the case with the eigenvalues of D which

“provide a nice set of non local, diffeomorphism-invariant observables.”

They yield

“the first good candidates for a (nearly) complete set of diffeomorphism-invariant observables”.

Let us look at $N(\Lambda)$ for $\Lambda \rightarrow \infty$. $N(\Lambda)$ is a step function which encodes a lot of information and can be written as a sum of a mean value and a fluctuation (oscillatory) term $N(\Lambda) = \langle N(\Lambda) \rangle + N_{\text{osc}}(\Lambda)$ where the oscillatory part $N_{\text{osc}}(\Lambda)$ is random. The mean part $\langle N(\Lambda) \rangle$ can be computed using a semi-classical approximation and a heat equation expansion. A wonderful computation shows that for $n = 4$ the asymptotic expansion of the spectral action is

$$\text{Trace} \left(\phi \left(\frac{D^2}{\Lambda^2} \right) \right) = \Lambda^4 f_0 a_0(D^2) + \Lambda^2 f_2 a_2(D^2) + f_4 a_4(D^2) + O(\Lambda^{-2})$$

with

- $f_0 = \int_{\mathbb{R}} \phi(u) u du$, $f_2 = \int_{\mathbb{R}} \phi(u) du$, $f_4 = \phi(0)$.
- $a_j(D^2) = \int_M a_j(x, D^2) dv$ ($dv = \sqrt{g} d^4x$).
- $a_0(x, D^2) = \frac{1}{(4\pi)^2} \text{Trace}_x(1)$.
- $a_2(x, D^2) = \frac{1}{(4\pi)^2} \text{Trace}_x \left(\frac{1}{6} s 1 - E \right)$.
- $a_4(x, D^2) = \frac{1}{360(4\pi)^2} \text{Trace}_x \left(5s^2 1 - 2r^2 1 + 2R^2 1 - 60sE + 180E^2 + 30R_{\mu\nu}^{\nabla} R^{\nabla\mu\nu} \right)$.
- R is the curvature tensor of M and $R^2 = R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta}$.
- r is the Ricci tensor of M and $r^2 = r_{\mu\nu} r^{\mu\nu}$.
- s is the scalar curvature of M .
- E and $R_{\mu\nu}^{\nabla}$ come from Lichnerowicz' formula.

Let

$$\mathcal{E} = C^\infty(M, S \otimes \mathcal{H}_F) = C^\infty(M, S) \otimes_{C^\infty(M)} C^\infty(M, \mathcal{H}_F).$$

The connection on \mathcal{E} is

$$\nabla = \nabla^S \otimes Id_{C^\infty(M, \mathcal{H}_F)} + Id_{C^\infty(M, S)} \otimes \nabla^F$$

and $R_{\mu\nu}^{\nabla}$ is the curvature 2-tensor of this total connection ∇ . If $D = ic^\mu \nabla_\mu + \varphi$ with $c^\mu = \gamma^\mu \otimes Id_{C^\infty(M, \mathcal{H}_F)}$, then $D^2 = \Delta + E$, with

$$\begin{cases} \Delta = -g^{\mu\nu} (\nabla_\mu \nabla_\nu - \Gamma_{\mu\nu}^\alpha \nabla_\alpha) \\ E = \frac{1}{4} s 1 - \frac{1}{2} c(R^F) + ic^\mu [\nabla_\mu, \varphi] + \varphi^2 \\ c(R^F) = -\gamma^\mu \gamma^\nu \otimes R_{\mu\nu}^F \quad (R^F = \text{curvature of } \nabla^F). \end{cases}$$

The asymptotic expansion of the spectral action is dominated by the first two terms which identify with the Einstein-Hilbert action with a cosmological term. The later can be eliminated by a change of ϕ .

Addendum. In a forthcoming book, Alain Connes, Ali Chamseddine and Matilde Marcolli show how the previous results can be strongly improved and yield a derivation of the standard model minimally coupled to gravity (Einstein-Hilbert action) with massive neutrinos, neutrino mixing, Weinberg angle, and Higgs mass (of the order of 170 GeV). This new achievement is quite astonishing.

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