Noncommutative Geometry and Transcendental Physics

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Abstract In our neo-transcendental approach, physical theories are built up from a categorial structure that is mathematically interpreted (what Kant called the "mathematical construction of categories"). The interpretation of physical categories provided by noncommutative geometry is presented in this perspective.

1 Introduction

In the early 1980s I began a research program which developed a new transcendental epistemology for modern theoretical physics. A synthetic summary of this approach can be found (in French) in my book *La Philosophie transcendantale et le problème de l'Objectivité* (1991) and (in English) in my paper "Actuality of Transcendental Aesthetics for Modern Physics" (1992) for the international Conference *1830–1930: Un siècle de géométrie, de C.F. Gauss et B. Riemann à H. Poincaré et E. Cartan : épistémologie, histoire, et mathématiques* held at the Institut Henri Poincaré in Paris the 18–23 September 1989. Further developements can be found in other papers cited in the bibliography.

The key idea is that, if physical theories are conceptually construed on the basis of categorial concepts such as "system", "state", "observable", etc. and geometro-dynamical intuitions such as those of space, time or motion, these representations have to be *mathematically* interpreted in a specific way (what Kant called the "mathematical construction of categories") in order to constitute a well-behaved physical objectivity. In this way, physical objectivity cannot be an ontology, and the departure of objectivity from ontology is, I think, the basic justification for transcendentalism.

Even if objective categories remain fairly invariant in the history of physics, their mathematical interpretation has changed tremendously as physical theories have

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M. Bitbol et al. (eds.), *Constituting objectivity*,

evolved, but this is by no means an argument against transcendentalism. On the contrary, the by now classical criticisms of Reichenbach, Carnap and many others are perhaps valid against a rigid, narrow minded, dogmatic reading of Kant but certainly not against a more general and open conception of transcendentalism. Incidentally, logical positivism is in great part a "grammatical" reinterpretation of classical transcendentalism.

In fact, Kant was the first philosopher to discover the *constitutive* nature of objectivity – "discovery of the constitutive element" that Hans Reichenbach himself called an "eminent philosophical result": objective principles are *prescriptive* rather than descriptive, and they are constitutive of physical reality. But in Kant, the constitutive components of objectivity were rooted in a cognitive representational theory. As Schlick pointed out, they were characteristic of our representational consciousness. That is why a form of transcendental subjectivism became the foundational basis for objectivity. Developments in physics (General Relativity and Quantum Mechanics) created a conflict between the objective components and their cognitive basis. However the appropriate response to this situation is not provided by logical positivism, but rather by a renewed transcendentalism where the objective and the cognitive components are methodologically separated. This transcendentalism is no longer founded on cognitive universals but on procedures of mathematical "construction".

In the previously cited papers I have shown how Hamiltonian (symplectic) mechanics, general relativity, non abelian gauge theories, and even superstring theories can be transcendentally interpreted in a very natural way. I aim at presenting in this perspective the deep and technical mathematical interpretation of physical categories provided by *noncommutative geometry*.

2 Generalizing and "Historicizing" Transcendentalism

As we have seen in the *Introduction* of this volume, a generalized and "historicized" transcendental perspective on modern physics can be based on very general principles:

1. Physics deals only with phenomena. Phenomena are relational entities that are inseparable from their conditions of observation: access conditions (observation, measurement, gathering of information, etc.) are constitutive of the very concept of physical object. In that sense, physical objectivity cannot be the ontology of a mind-independent substantial reality and any ontological realism has to be rejected.
2. But even if they lose ontological content, "categorial" concepts still have a theoretical function. In order to be transformed into scientific objects, phenomena must be conceptually lawful, "legalized" according to a categorial structure. The first philosophical thematization of this principle was Kant's *Metaphysische Anfangsgründe der Naturwissenschaft* (MAN). Kant explained how the four groups of categories and principles specialize in physics into Phoronomy (Kinematics),

Dynamics, Mechanics, and Phenomenology, and how they are mathematically interpreted in Newtonian Mechanics.
3. The essential feature of physics is the mathematical interpretation which transforms the categorial concepts into *algorithms* for the mathematical reconstruction of phenomena. This is a critical point. Physics has to solve an *inverse problem*, namely the inverse problem of the abstraction problem. Conceptual analysis must be supplemented by a *computational synthesis* of phenomena. In Kant, computational synthesis is first based on schematization and then on the "construction" of categories.

The main difficulty with a generalized transcendentalism is to understand the general meaning of *Transcendental Aesthetics*. The latter presents two aspects corresponding to what Kant called two "expositions" (*Erörterung* = "clear representation of what belongs to a concept") in the *Kritik der reinen Vernunft* (KRV): the metaphysical and the transcendental. First, phenomena are observable and therefore must appear to an observer. They appear in a specific medium of manifestation (space and time for sensible phenomena) which provides "forms of intuition". Second, these "forms" can be mathematically determined and converted into what Kant called "formal intuitions" (see the celebrated footnote to section 26 of KRV). To determine phenomena objectively, we need therefore a link between mathematically determined forms of observability (what is "gegeben") and categorial forms of lawfulness (what is "gedacht"). In Kant this link is worked out at two levels. At the level of KRV it is provided by transcendental *schematism* which converts the categories into principles ("*Grundsätze*"). At the level of MAN, it is provided by what Kant called the *construction* ("*Konstruktion*") of categories. The construction is a mode of presentation ("*Darstellung*"). It means that it is possible to interpret mathematically the schematized categorial contents by using mathematics stemming from the transcendental exposition of Transcendental Aesthetics. I think that it is in this very special sort of "mathematical hermeneutics" – not only for the intuitive forms of manifestation but also for the categorial forms of lawfulness themselves – that the synthetic a priori finds its true and deep transcendental meaning.

In the *Introduction* of the volume, we also reminded (in modern terms) the categorial moments of classical Mechanics according to the *Metaphysische Anfangsgründe der Naturwissenschaft*.

1. **Phoronomy (Kinematics).** "Mathematical" categories of quantity and "Axioms of Intuition" ("*Axiomen der Anschauung*") governing "extensive" magnitudes: the Euclidean metric of space is a background (a priori) geometrical structure and physical motion complies with Galilean relativity.
2. **Dynamics.** "Mathematical" categories of quality and "Anticipations of Perception" ("*Anticipationen der Wahrnehmung"*) governing "intensive" magnitudes: physical dynamics has to be described in terms of differential entities (velocities, accelerations, etc.) varying covariantly (link with *Phoronomy*). Physics must therefore be a kind of differential geometry (not a "logic" in the traditional Aristotelian sense).

3. **Mechanics.** "Dynamical", i.e. physical, categories of relation (substance = *Inhärenz und Subsistenz*, causality = *Causalität und Dependenz*, community, reciprocity and interaction = *Gemeinschaft*) and "Analogies of Experience" ("*Analogien der Erfahrung"*): the category of substance is reinterpreted as the transcendental principle of conservation laws, the category of causality as that of forces, and the category of community as that of interactions.
4. **Phenomenology.** Categories of modality and "Postulates of empirical thought" ("*Postulate des empirischen Denkens überhaupt*"): because of relativity, motion cannot be a real but only a "possible" predicate of matter (it is a purely relational phenomenon). Position and velocity are not observable properties whose values could individuate dynamical states. The sentence "The body *S* "has" such position and such velocity" (in the sense of "having a property") is not a physical judgment. We find here the root of the transcendental ideality of space and time, which has nothing to do with a subjective idealism à la Berkeley. But forces (causality) are real and are governed by necessary laws. Necessity is not a logical but a transcendental modality. It is conditional, relative to the radical contingency of possible experience.

A striking modern example of such a transcendental structure is provided by the constitutive role of *symmetries*. In general relativity and non abelian gauge theories, the radical enlargment of the symmetry groups enables us to construct mathematically on the basis of relativity principles not only the physical content of the categories of substance, but also the physical content of the categories of force and interaction. As far as I am concerned (a view shared by Daniel Bennequin, a specialist of symplectic geometry and string theory) this is a far-reaching manifestation of the "Galoisian" essence of modern physics: *symmetries that express entities which cannot be physical observables act as principles of determination for the physical obervables themselves*.

The evolution of modern physics displays fairly stable categorial structures, together with many changes in their successive mathematical interpretations. I think that such a variability is by no means an argument against a transcendental approach. For instance, according to Kant, the a priori nature of space and time means essentially that the Euclidean metric of space–time and the Galilean group act as a background structure for Mechanics. This remains perfectly true. In GR, the metric is no longer a background structure and becomes a dynamical feature of the theory. The *Diff*(M)-invariance implies that localization becomes relational so that points lack any physical content. But this background independence is no refutation of transcendentalism. I have developped the thesis that the *differentiable* structure of space–time and the associated cohomology of differential forms remain a background structure in GR.

In Petitot (1992a) I gave a transcendental approach to:

1. Hamiltonian (symplectic) mechanics, in particular Noether's theorem and the formalism of the momentum map worked out by B. Kostant, J.M. Souriau, V. Arnold, A. Weinstein, R. Abraham, and J. Marsden (deep broadening of the construction of the category of substance).

2. General relativity and the a priori determination of Einstein equations proposed by Wheeler, Misner, and Thorne in their *Geometrodynamics* (construction of the category of force).
3. Non abelian gauge theories (construction of the category of interaction).

As it turns out, this perspective shares many theses with Friedman's works (*Dynamics of Reason*, 1999):

1. The developement of modern physics does not destroy the transcendental constitutive perspective:

> We still need superordinate and highly mathematical first principles in physics – principles that must be injected into our experience of nature before such experience can teach us anything at all. (p. 14)

2. The conditions of possibility of physical theories (a priori synthetic principles of coordination) are not logico-analytic judgements.
3. Kant's a priori principles can be generalized, relativized and historicisized:

> What we end up with (...) is thus a relativized and dynamical conception of a priori mathematical-physical principles, which change and develop along with the development of the mathematical and physical sciences themselves, but which nevertheless retain the characteristically Kantian constitutive function of making the empirical natural knowledge thereby structured and framed by such principles first possible. (p. 31)

4. The central role of constitutive principles:

> What characterizes the distinguished elements of our theories is rather their special constitutive function: the function of making the precise mathematical formulation and empirical application of the theories in question first possible. (p. 40)

3 Noncommutative Geometry as a New Framework

Let me now comment on a new technical example of mathematical reinterpretation of the categorial structures of physics. This reinterpretation is achieved by using John Baez' requisite of *background independence* (less radical than Lee Smolin's). The problem is rather difficult, especially in Quantum Gravity. In GR general covariance implies that the metric is no longer a background structure and points of space–time M lose any physical meaning: GR observables must be $\mathit{Diff}(M)$ invariant and are therefore *non-local*. On the contrary (Carlip, 2001), in Quantum Field Theory there exists a fixed background space–time M and points have a physical meaning: the value $\varphi(x)$ of a field φ at a point $x \in M$ is in principle obervable. How are we to eliminate the background geometry in QFT while maintaining at the same time the computational efficiency of geometry? How are we to reconcile mathematically theories such as GR and QFT which are so heterogeneous to one another? Remarkable suggestions exist – in particular *loop quantum gravity* developped by Abhay Ashtekar, Lee Smolin, Carlo Rovelli, John Baez, etc. – for enlarging the formal framework of Riemann and Cartan geometry and quantize some of their components, but it seems that the problem is not a technical problem to be reckoned with only at the boundary of physical theories but a basic foundational difficulty. This means that we need a change of paradigm, much like GR in the case of Riemannian geometry.

It seems that the most interesting answer to this problem comes from *Noncommutative Geometry* (NCG) which introduces from the outset quantum concepts in the definition of the most fondamental geometrical concepts. I will present here how Connes and Lott achieved the deduction in NCG of the coupling of gravity (Einstein-Hilbert action[1]) with the Standard Model of Quantum Field Theory (QFT), how metric can be reinterpreted in purely spectral terms using the formalism of Clifford algebras and Dirac operators, and how a purely noncommutative generalization yields a natural interpretation of the Higgs phenomenon.

Philosophically speaking, NCG is a new paradigm – or framework – in as much as it includes *both* GR and the standard model of QFT as commutative approximations and provides the first deep theoretical meaning to the Higgs phenomenon. The breakthrough of NCG consists in starting from QM and "quantizing" all classical geometrical concepts. The conflict between geometry and QM disappears from the outset since quantum concepts are no longer subordinated to any prior background geometrical structure.

4 Gelfand Theory

To understand Alain Connes' NC Geometry we must first come back to Gelfand theory for commutative C^*-algebras.

4.1 C^*-algebras

Recall that a C^*-algebra $\mathcal{A}$ is a (unital) Banach algebra on $\mathbb{C}$ (i.e. a $\mathbb{C}$-algebra which is normed and complete for its norm) endowed with an involution $x \to x^*$ s.t. $\|x\|^2 = \|x^*x\|$. The norm (the metric structure) is then deducible from the algebraic structure. Indeed, $\|x\|^2$ is the spectral radius of the positive element x^*x, that is, the *Sup* of the modulus of the spectral values of x^*x:[2]

$$\|x\|^2 = Sup\{|\lambda| : x^*x - \lambda I \text{ is not invertible}\}$$

[1] It would be better to call this action the Hilbert-Einstein action since there is a priority of Hilbert (1915). See e.g. Majer-Sauer (2004).

[2] In the infinite dimensional case, the spectral values ($x - \lambda I$ is not invertible) are not identical with the eigenvalues ($x - \lambda I$ has a non trivial kernel). Indeed non invertibility no longer implies non injectivity (a linear operator can be injective and non surjective). For instance, if e_n, $n \in \mathbb{N}$, is a countable basis, the shift $\sum_n \lambda_n e_n \to \sum_n \lambda_n e_{n+1}$ is injective but not surjective and is not invertible.

(where I is the unit of $\mathcal{A}$). In a C^*-algebra the norm becomes therefore a purely *spectral* concept.

An element $x \in \mathcal{A}$ is called self-adjoint if $x = x^*$, normal if $xx^* = x^*x$, and unitary if $x^{-1} = x^*$ ($\|x\| = 1$).

In this classical setting, the mathematical interpretation of the fundamental (categorial) concepts of

1. Space of states
2. Observable
3. Measure

is the following:

1. The space of states is a smooth manifold: the phase space M (in Hamiltonian mechanics, $M = T^*N$ is the cotangent bundle of the space of configurations N endowed with its canonical symplectic structure).
2. The observables are functions $f : M \to \mathbb{R}$ (interpreted as $f : M \to \mathbb{C}$ with $f = \bar{f}$) which measure some property of states and output a real number.
3. The measure of f in the state $x \in M$ is the evaluation $f(x)$ of f at x; but as $f(x) = \delta_x(f)$ (where δ_x is the Dirac distribution at x) a state can be dually interpreted as a continuous linear operator on observables.

The observables constitute a commutative C^*-algebra $\mathcal{A}$ and Gelfand theory explains that the *geometry* of the manifold M can be completely recovered from the *algebraic* structure of $\mathcal{A}$.

4.2 Gelfand's Theorem

Let M be a topological space and let $\mathcal{A} := \mathcal{C}(M)$ be the $\mathbb{C}$-algebra of continuous functions $f : M \to \mathbb{C}$ (the $\mathbb{C}$-algebra structure being inherited from that of $\mathbb{C}$ itself via pointwise addition and multiplication). Under very general conditions (e.g. if M is compact[3]), it is a C^*-algebra for complex conjugation $f^* = \overline{f}$.

The possible values of f – that is the possible results of a measure of f – can be defined in a purely algebraic way as the *spectrum* of f that is

$$sp_{\mathcal{A}}(f) := \{c : f - cI \text{ is not invertible in } \mathcal{A}\}.$$

Indeed, if $f(x) = c$ then $f - cI$ is not invertible in $\mathcal{A}$. $sp_{\mathcal{A}}(f)$ is the complementary set of what is called the *resolvent* of f,

$$r(f) := \{c : f - cI \text{ is invertible in } \mathcal{A}\}.$$

[3] If M is non compact but only locally compact, then one take $\mathcal{A} = \mathcal{C}_0(M)$ the algebra of continuous functions vanishing at infinity but $\mathcal{A}$ is no longer unital since the constant function 1 doesn't vanish at infinity.

The main point is that the evaluation process $f(x)$ – that is measure – can be interpreted as a *duality* $\langle f,x\rangle$ between the space M and the algebra $\mathcal{A}$. Indeed, to a point x of M we can associate the *maximal ideal* of the $f\in\mathcal{A}$ vanishing at x:

$$\mathfrak{M}_x := \{f\in\mathcal{A} : f(x)=0\}\,.$$

But the maximal ideals $\mathfrak{M}$ of $\mathcal{A}$ constitute themselves a space – called the *spectrum* of the algebra $\mathcal{A}$. They can be considered as the kernels of the *characters* of $\mathcal{A}$, that is of the morphisms (multiplicative linear forms) $\chi:\mathcal{A}\to\mathbb{C}$,

$$\mathfrak{M} = \chi^{-1}(0).$$

A character is by definition a coherent procedure for evaluating the elements $f\in\mathcal{A}$. The evaluation $\chi(f)$ is also a *duality* $\langle\chi,f\rangle$ and its results $\chi(f)$ belong to $sp_{\mathcal{A}}(f)$. Indeed, as *distributions* (continuous linear forms), the characters correspond to the Dirac distributions δ_x and if $\chi=\delta_x$, then $\chi(f)=f(x)=c$ and $c\in sp_{\mathcal{A}}(f)$.

The *spectrum* of the C^*-algebra $\mathcal{A}$ (not to be confused with the spectra $sp_{\mathcal{A}}(f)$ of the single elements f of $\mathcal{A}$) is by definition the space of characters $Sp(\mathcal{A}) := \{\chi\}$ endowed with the topology of simple convergence: $\chi_n\to\chi$ iff $\chi_n(f)\to\chi(f)\ \forall f\in\mathcal{A}$. It is defined uniquely from $\mathcal{A}$ without any reference to the fact that $\mathcal{A}$ is of the form $\mathcal{A}:=\mathcal{C}(M)$. It is also the space of irreducible representations of $\mathcal{A}$ (since $\mathcal{A}$ is commutative, they are 1-dimensional).

Now, if $f\in\mathcal{A}$ is an element of $\mathcal{A}$, using duality, we can associate to it canonically a *function* $\tilde{f}$ on the space $Sp(\mathcal{A})$

$$\begin{aligned}\tilde{f} : Sp(\mathcal{A}) &\to \mathbb{C}\\ \chi &\mapsto \tilde{f}(\chi)=\chi(f)=\langle\chi,f\rangle\,.\end{aligned}$$

We get that way a map

$$\begin{aligned}\tilde{\ } : \mathcal{A} &\to \mathcal{C}(Sp(\mathcal{A}))\\ f &\mapsto \tilde{f}\end{aligned}$$

which is called the *Gelfand transform*. For every f we have

$$\tilde{f}\,(Sp(\mathcal{A})) = sp_{\mathcal{A}}(f).$$

The key result is then:

Gelfand-Neimark theorem. If $\mathcal{A}$ is a *commutative* C^*-algebra, the Gelfand transform $\tilde{\ }$ is an *isometry* between $\mathcal{A}$ and $\mathcal{C}(Sp(\mathcal{A}))$.

Indeed, the norm of $\tilde{f}$ is the spectral radius of f, $\rho(f) := \lim_{n\to\infty}\left(\|f^n\|^{\frac{1}{n}}\right)$ and we have $\left\|\tilde{f}\right\| = \rho(f) = \|f\|$. To see this, suppose first that f is self-adjoint ($f=f^*=\bar{f}$). We have $\|f\|^2 = \|f.f^*\| = \left\|f^2\right\|$. So, $\|f\| = \left\|f^{2^n}\right\|^{2^{-n}}$ and as

$\left\|f^{2^n}\right\|^{2^{-n}} \to \rho(f)$ by definition we have $\|f\| = \rho(f)$. Suppose now that f is any element of $\mathcal{A}$. Since $f.f^*$ is self-adjoint, we have $\|f\|^2 = \|f.f^*\| = \rho(f.f^*) = \left\|\widetilde{f.f^*}\right\|$. But $\left\|\widetilde{f.f^*}\right\| = \left\|\widetilde{f}.\widetilde{f^*}\right\| = \left\|\widetilde{f}\right\|^2$ and therefore $\|f\|^2 = \left\|\widetilde{f}\right\|^2$ and $\|f\| = \left\|\widetilde{f}\right\|$.

Gelfand theory shows that, in the classical case of commutative C^*-algebras $\mathcal{A} := \mathcal{C}(M)$ (M compact), there exists a complete *equivalence* between the geometric and the algebraic perspectives.

4.3 Towards a New (Functional) "Phoronomy"

We think that Gelfand theorem has a deep philosophical meaning. In classical mechanics "phoronomy" (kinematics) concerns the structure of the configuration space N and the phase space $M := T^*N$. Observables and measurements are defined in terms of functions on these basic spaces directly construed from the geometry of space–time (transcendental aesthetics). Gelfand theorem shows than we can *exchange* the primary geometrical background and the secondary algebraic moment of measure, take measure as a primitive fact and reconstruct the geometric background from it as a secondary moment. In one word, *we can substitute a "functional" transcendental aesthetics to a purely geometrical one*.

4.4 Towards Noncommutative Geometry

In Quantum Mechanics, the basic structure is that of the *noncommutative* C^*-algebras $\mathcal{A}$ of observables. In Petitot (1992a) I suggested that "phoronomy" operates at this level. It is challenging and natural to wonder if there could exist a *geometric correlate* of this noncommutative algebraic setting. The deepest answer is Connes' *Noncommutative Geometry* (NCG) also called *Spectral Geometry* or *Quantum Geometry*. In NCG the basic structure is the NC C^*-algebra $\mathcal{A}$ of obervables: any phenomenon is primarily something which is observable in the quantum sense, and not an event in space–time. But observables must be defined for states and are therefore represented in the space of states of the system, which is an Hilbert space and not the classical space. The associated NC space is then the space of *irreducible representations*.

NCG is a fundamentally new step toward a *geometrization* of physics. Instead of beginning with classical differential geometry and try to develop Quantum Mechanics on this backgrond, it begins with Quantum Mechanics and construct a new *quantum geometrical framework*. In that sense, Connes is the Einstein of Quantum Mechanics. The most fascinating aspect of his research program is how he succeeded in reinterpreting all the basic structures of classical geometry inside the framework of NC C^*-algebras operating on Hilbert spaces. *The basic concepts (with their categorial content) remain almost the same but their mathematical interpretation is significantly complexified*, since their classical meaning becomes a *commutative limit*. We meet here a new very deep example of the conceptual transformation of physical theories through mathematical enlargements, as it is the case in GR or QM. As explained by Daniel Kastler [NCG]:

> Alain Connes' noncommutative geometry (...) is a systematic quantization of mathematics parallel to the quantization of physics effected in the twenties. (...) This theory widens the scope of mathematics in a manner congenial to physics.

5 NCG and Differential Forms

Connes reinterpreted (in an extremely deep and technical way) the six classical levels:

1. Measure theory
2. Algebraic topology and topology (K-theory)
3. Differentiable structure
4. Differential forms and De Rham cohomology
5. Fiber bundles, connections, covariant derivations, Yang-Mills theories
6. Riemannian manifolds and metric structures.

Let us take as a first example the reinterpretation of the differential calculus.

5.1 A Universal and Formal Differential Calculus

How can we interpret differential calculus in the new NC paradigm? Connes wanted first to define *derivations* $D : \mathcal{A} \to \mathcal{E}$, that is $\mathbb{C}$-linear maps satisfying the *Leibniz rule* (which is the universal formal rule for derivations):

$$D(ab) = (Da)b + a(Db)$$

For that, $\mathcal{E}$ must be endowed with a structure of $\mathcal{A}$-bimodule (right and left products of elements of $\mathcal{E}$ by elements of $\mathcal{A}$). It is evident that $D(c) = 0$ for every scalar $c \in \mathbb{C}$ since $D(1.a) = D(1)a + 1D(a) = D(a)$ and therefore $D(1) = 0$.

Let $Der(\mathcal{A}, \mathcal{E})$ be the $\mathbb{C}$-vector space of such derivations. In $Der(\mathcal{A}, \mathcal{E})$ there exist very particular elements, the inner derivatives, associated with the elements m of $\mathcal{E}$, which express the difference between the right and left $\mathcal{A}$-module structures of $\mathcal{E}$:

$$D(a) := ad(m)(a) = ma - am.$$

Indeed,

$$\begin{aligned} ad(m)(a).b + a.ad(m)(b) &= (ma - am)b + a(mb - bm) \\ &= mab - abm \\ &= ad(m)(ab). \end{aligned}$$

In the case where $\mathcal{E} = \mathcal{A}$, $ad(b)(a) = [b,a]$ expresses the non commutativity of $\mathcal{A}$. By the way, $Der(\mathcal{A},\mathcal{A})$ is a Lie algebra since $[D_1,D_2]$ is a derivation if D_1,D_2 are derivations.

Now, the fact must be stressed that there exists *a universal derivation* depending only upon the algebraic structure of $\mathcal{A}$, and having therefore *absolutely nothing to do* with the classical "infinitesimal" intuitions underlying the classical concepts of differential and derivation. It is given by

$$\begin{aligned} d : \mathcal{A} &\to \mathcal{A}\otimes_{\mathbb{C}}\mathcal{A} \\ a &\mapsto da := 1\otimes a - a\otimes 1. \end{aligned}$$

Let $\Omega^1\mathcal{A}$ be the sub-bimodule of $\mathcal{A}\otimes_{\mathbb{C}}\mathcal{A}$ generated by the elements $adb := a\otimes b - ab\otimes 1$, i.e. the kernel of the multiplication $a\otimes b \mapsto ab$.[4] $\Omega^1\mathcal{A}$ is isomorphic to the tensorial product $\mathcal{A}\otimes_{\mathcal{A}}\overline{\mathcal{A}}$, where $\overline{\mathcal{A}}$ is the quotient $\mathcal{A}/\mathbb{C}$, with $adb = a\otimes\overline{b}$. It is called the bimodule of *universal* 1*-forms* on $\mathcal{A}$ where "universality" means that

$$Der(\mathcal{A},\mathcal{E}) \simeq Hom_{\mathcal{A}}\left(\Omega^1\mathcal{A},\mathcal{E}\right)$$

i.e. that a derivation $D : \mathcal{A} \to \mathcal{E}$ is the same thing as a morphism *of algebras* between $\Omega^1\mathcal{A}$ and $\mathcal{E}$. If $D : \mathcal{A} \to \mathcal{E}$ is an element of $Der(\mathcal{A},\mathcal{E})$, the associated morphism $\tilde{D} : \Omega^1\mathcal{A} \to \mathcal{E}$ is defined by

$$a\otimes b \mapsto aD(b).$$

So $da = 1\otimes a - a\otimes 1 \mapsto 1.D(a) - a.D(1) = D(a)$ (since $D(1) = 0$).

We can generalize this construction to universal n-forms, which have the symbolic form[5]

$$a_0da_1...da_n.$$

If $\Omega^n\mathcal{A} := \left(\Omega^1\mathcal{A}\right)^{\otimes n} = \mathcal{A}\otimes_{\mathcal{A}}\left(\overline{\mathcal{A}}\right)^{\otimes n}$ with $a_0da_1...da_n = a_0\otimes\overline{a_1}\otimes...\otimes\overline{a_n}$, the differential is then

$$\begin{aligned} d : \Omega^n\mathcal{A} &\to \Omega^{n+1}\mathcal{A} \\ a_0da_1...da_n &\mapsto da_0da_1...da_n \\ a_0\otimes\overline{a_1}\otimes...\otimes\overline{a_n} &\mapsto 1\otimes\overline{a_0}\otimes\overline{a_1}\otimes...\otimes\overline{a_n}\ . \end{aligned}$$

Since $d1 = 0$, it is easy to verify the fundamental cohomological property $d^2 = 0$ of the graduate differential algebra $\Omega\mathcal{A} := \bigoplus_{n\in\mathbb{N}}\Omega^n\mathcal{A}$. Some technical difficulties must be overcome (existence of "junk" forms) to transform this framework into a "good" formal differential calculus.

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[4] For $a\otimes b - ab\otimes 1$ the multiplication gives $ab - ab = 0$. Reciprocally if $ab = 0$ then $a\otimes b = a\otimes b - ab\otimes 1$ and $a\otimes b$ belongs to $\Omega^1\mathcal{A}$.

[5] $da_1...da_n$ is the exterior product of 1-forms, classically denoted $da_1\wedge...\wedge da_n$.

5.2 Noncommutative Differential Calculus or "Quantized" Calculus

Connes wanted to *represent* this universal differential algebra in spaces of physical states. Let us suppose therefore that the C^*-algebra $\mathcal{A}$ acts upon an Hilbert space $\mathcal{H}$ and we want to interpret in this representation the universal, formal, and purely symbolic differential calculus of the previous section. For that, we must interpret the differential df of the elements $f \in \mathcal{A}$, these f being now represented as *operators* on $\mathcal{H}$. Connes' main idea was to use the well-known formula of QM

$$\frac{df}{dt} = \frac{2i\pi}{h}[F, f]$$

where F is the Hamiltonian of the system and f any obervable.

Consequently, he interpreted the symbol df as

$$df := [F, f]$$

for an appropriate self-adjoint operator F. We want of course $d^2 f = 0$. But $d^2 f = \left[F^2, f\right]$ and therefore F^2 must commute with all observables.

The main constraint is that, once interpreted in $\mathcal{H}$, the symbol df must correspond to an *infinitesimal*. *The classical concept of infinitesimal ought to be reinterpreted in the NC framework*. Connes' definition is that an operator T is infinitesimal if it is *compact*, that is if the eigenvalues $\mu_n(T)$ of its absolute value $|T| = (T^*T)^{1/2}$ – called the *characteristic values* of T – converge to 0, that is if for every $\varepsilon > 0$ the norm $\|T\|$ of T is $< \varepsilon$ outside a subspace of *finite* dimension. If $\mu_n(T) \underset{n\to\infty}{\longrightarrow} 0$ as $\frac{1}{n^\alpha}$ then T is an infinitesimal of order α (α not necessarily an integer).

If T is compact, let ξ_n be a complete orthonormal basis of $\mathcal{H}$ associated to $|T|$, $T = U|T|$ the polar decomposition of T[6] and $\eta_n = U\xi_n$. Then T is the sum

$$T = \sum_{n\geq 0} \mu_n(T) |\eta_n\rangle \langle \xi_n| \ .$$

If T is a positive infinitesimal of order 1, its trace $Trace\,(T) = \sum_n \mu_n(T)$ has a logarithmic divergence. If T is of order >1, its trace is finite > 0. It is the basis for *NC integration* which uses the *Dixmier trace*, a technical tool for constructing a new trace extracting the logarithmic divergence of the classical trace. Dixmier trace is a technical way for giving a sense to the formula $\lim_{N\to\infty} \frac{1}{\ln N} \sum_{n=0}^{n=N-1} \mu_n(T)$. It vanishes for infinitesimals of order >1.

Therefore, we interpret the differential calculus in the NC framework through triples $(\mathcal{A}, \mathcal{H}, F)$ where $[F, f]$ is compact for every $f \in \mathcal{A}$. Such a structure is called *a Fredholm module*.

[6] The polar decomposition $T = U|T|$ is the equivalent for operators of the decomposition $z = |z|e^{i\theta}$ for a complex number. In general U cannot be unitary but only a partial isometry.

The differential forms $a_0da_1...da_n$ can now be interpreted as operators on $\mathcal{H}$

$$a_0da_1...da_n := a_0[F,a_1]...[F,a_n]$$

and we see how the second transcendental moment of physical objectivity, namely that of "dynamics", becomes interpreted in the NC framework.

It must be emphasized that the NC generalization of differential calculus is a wide and wild generalization since it enables us to extend differential calculus to fractals!

6 NC Riemannian Geometry, Clifford Algebras, and Dirac Operator

Another great achievement of Alain Connes was the complete and deep reinterpretation of the ds^2 in Riemannian geometry. Classically, $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$. In the NC framework, dx must be interpreted as $dx = [F,x]$ (where $(\mathcal{A},\mathcal{H},F)$ is a Fredholm module), and the matrix $(g_{\mu\nu})$ as an element of the $n \times n$ matrix algebra $M_n(\mathcal{A})$. The ds^2 must therefore become a *compact* and *positive* operator of the form

$$G = [F,x^\mu]^* g_{\mu\nu}[F,x^\nu].$$

6.1 A Redefinition of Distance

Connes' idea is to reinterpret the classical definition of distance $d(p,q)$ between two points p,q of a Riemannian manifold M as the *Inf* of the length $L(\gamma)$ of the paths $\gamma : p \to q$

$$d(p,q) = \underset{\gamma:p\to q}{Inf}\ L(\gamma)$$

$$L(\gamma) = \int_p^q ds = \int_p^q \left(g_{\mu\nu}dx^\mu dx^\nu\right)^{1/2}.$$

Using the equivalence between a point x of M and the pure state δ_x on the commutative C^*-algebra $\mathcal{A} := C^\infty(M)$, an elementary computation shows that this definition of the distance is equivalent to the dual algebraic definition using only concepts concerning the C^*-algebra $\mathcal{A}$

$$d(p,q) = Sup\{|f(q) - f(p)| : \|grad(f)\|_\infty \leq 1\}$$

where $\|...\|_\infty$ is the L^∞ norm, that is the *Sup* on $x \in M$ of the norms on the tangent spaces T_xM.[7]

[7] Let $\gamma : I = [0,1] \to M$ be a C^∞ curve in M from p to q. $L(\gamma) = \int_p^q |\dot{\gamma}(t)|\,dt = \int_0^1 g(\dot{\gamma}(t), \dot{\gamma}(t))^{1/2}\,dt$. If $f \in C^\infty(M)$, using the duality between df and $gradf$ induced by the metric, we find

6.2 Clifford Algebras

Now the core of the NC definition of distance uses the *Dirac operator*. In order to explain this key point, *which makes distance a quantum concept*, the so called *Clifford algebra* of a Riemannian manifold must be introduced.

Recall that the formalism of Clifford algebras relates *the differential forms and the metric* on Riemannian manifolds. In the classical case of the Euclidean space $\mathbb{R}^n$, the main idea is to encode the isometries $O(n)$ in an algebra structure. Since every isometry is a product of reflections (Cartan), we can associate to any vector $v \in \mathbb{R}^n$ the reflection $\overline{v}$ relative to the orthogonal hyperplane $v^\perp$ and introduce a multiplication $v.w$ which is nothing else than the composition $\overline{v} \circ \overline{w}$. We are then naturally led to the anti-commutation relations

$$\{v,w\} := v.w + w.v = -2(v,w)$$

where (v,w) is the Euclidean scalar product.

More generally, let V be a $\mathbb{R}$-vector space endowed with a quadratic form g. Its Clifford algebra $Cl(V,g)$ is its tensor algebra $\mathcal{T}(V) = \oplus_{k=0}^{k=\infty} V^{\otimes k}$ quotiented by the relations

$$v \otimes v = -g(v)1, \ \forall v \in V$$

(where $g(v) = g(v,v) = \|v\|^2$). In $Cl(V,g)$ the tensorial product $v \otimes v$ becomes a product $v.v = v^2$. It must be stressed that there exists always in $Cl(V,g)$ the constants $\mathbb{R}$ which correspond to the 0th tensorial power of V.

Using the scalar product

$$2g(v,w) = g(v+w) - g(v) - g(w)$$

one gets the *anti-commutation* relations

$$\{v,w\} = -2g(v,w)$$

Elementary examples are given by the $Cl_n = Cl(\mathbb{R}^n, g_{Euclid})$.

- $Cl_0 = \mathbb{R}$
- $Cl_1 = \mathbb{C}$ ($V = i\mathbb{R}$, $i^2 = -1$, $Cl_1 = \mathbb{R} \oplus i\mathbb{R}$)
- $Cl_2 = \mathbb{H}$ ($V = i\mathbb{R} + j\mathbb{R}$, $ij = k$, $Cl_2 = \mathbb{R} \oplus i\mathbb{R} \oplus j\mathbb{R} \oplus k\mathbb{R}$)
- $Cl_3 = \mathbb{H} \oplus \mathbb{H}$
- $Cl_4 = \mathbb{H}[2]$ (2×2 matrices with entries in $\mathbb{H}$)
- $Cl_5 = \mathbb{C}[4]$

$f(q) - f(p) = \int_0^1 df_{\gamma(t)}(\dot{\gamma}(t))\,dt = \int_0^1 g_{\gamma(t)}\left(grad_{\gamma(t)} f, \dot{\gamma}(t)\right)dt$. This shows that $|f(q) - f(p)| \leq \int_0^1 \left|grad_{\gamma(t)} f\right| |\dot{\gamma}(t)|\,dt \leq \|grad f\|_\infty L(\gamma)$. Therefore, if $\|grad(f)\|_\infty \leq 1$ we have $|f(q) - f(p)| \leq d(p,q)$. When we take the *Sup* we retreive $d(p,q)$ using the special function $f_p(x) = d(p,x)$ since $|f_p(q) - f_p(p)| = d(p,q)$.

- $Cl_6 = \mathbb{R}[8]$
- $Cl_7 = \mathbb{R}[8] \oplus \mathbb{R}[8]$
- $Cl_{n+8} = Cl_n \otimes \mathbb{R}[16]$ (Bott periodicity theorem)

If $g(v) \neq 0$ (which would always be the case for $v \neq 0$ if g is non degenerate) v is invertible in this algebra structure and

$$v^{-1} = -\frac{v}{g(v)}.$$

The multiplicative Lie group $Cl^{\times}(V,g)$ of the invertible elements of $Cl(V,g)$ act through *inner automorphisms* on $Cl(V,g)$. This yields the *adjoint representation*

$$\begin{aligned} Ad : Cl^{\times}(V,g) &\to Aut\left(Cl(V,g)\right) \\ v &\mapsto Ad_v : w \mapsto v.w.v^{-1}. \end{aligned}$$

But[8]

$$v.w.v^{-1} = -w + \frac{2g(v,w)v}{g(v)} = Ad_v(w).$$

As $-Ad_v$ is the reflection relative to $v^{\perp}$, this means that reflections act through the adjoint representation of the Clifford algebra. The derivative *ad* of the adjoint representation enables to recover the Lie bracket of the Lie algebra $cl^{\times}(V,g) = Cl(V,g)$ of the Lie group $Cl^{\times}(V,g)$

$$\begin{aligned} ad : cl^{\times}(V,g) = Cl(V,g) &\to Der\left(Cl(V,g)\right) \\ v &\mapsto ad_v : w \mapsto [v,w] \end{aligned}$$

Now there exists a fundamental relation between the Clifford algebra $Cl(V,g)$ of V and its exterior algebra Λ^*V. If $g = 0$ and if we interpret $v.w$ as $v \wedge w$, the anti-commutation relations become simply $\{v,w\} = 0$, that is the classical antisymmetry $w \wedge v = -v \wedge w$ of differential 1-forms. Therefore

$$\Lambda^*V = Cl(V,0).$$

In fact, $Cl(V,g)$ can be considered as a way of *quantizing* Λ^*V using the metric g in order to get *non trivial* anti-commutation relations.

Due to the relations $v^2 = -g(v)1$ which decrease the degree of a product by 2, $Cl(V,g)$ is no longer a $\mathbb{Z}$-graded algebra but only a $\mathbb{Z}/2$-graded algebra, the $\mathbb{Z}/2$-gradation corresponding to the even/odd elements. But we can reconstruct a $\mathbb{Z}$-graded algebra $\mathcal{C} = \bigoplus_{k=0}^{k=\infty} C^k$ associated to $Cl(V,g)$, the C^k being the homogeneous terms of degree k: $v_1.\cdots.v_k$.

[8] $v.w.v^{-1} = -v.w.\frac{v}{g(v)} = -(-w.v - 2g(v,w))\frac{v}{g(v)} = w.\frac{v^2}{g(v)} + \frac{2g(v,w)v}{g(v)} = -w + \frac{2g(v,w)v}{g(v)}.$

Theorem. The map of graded algebras $\mathcal{C} = \bigoplus_{k=0}^{k=\infty} C^k \rightarrow \Lambda^* V = \bigoplus_{k=0}^{k=\infty} \Lambda^k$ given by $v_1.\cdots.v_k \rightarrow v_1 \wedge \cdots \wedge v_k$ is a *linear isomorphism* (but not an *algebra* isomorphism).

We consider now 2 operations on the *exterior* algebra $\Lambda^* V$:

1. The outer multiplication $\varepsilon(v)$ by $v \in V$:

$$\varepsilon(v)\left(\underset{i}{\wedge} u_i\right) = v \wedge \left(\underset{i}{\wedge} u_i\right).$$

   We have $\varepsilon(v)^2 = 0$ since $v \wedge v = 0$.
2. The contraction (inner multiplication) $\iota(v)$ induced by the metric g:[9]

$$\iota(v)\left(\underset{i}{\wedge} u_i\right) = \sum_{j=1}^{j=k} (-1)^j g(v, u_j)\, u_1 \wedge \cdots \wedge \widehat{u_j} \wedge \cdots u_k.$$

   We have also $\iota(v)^2 = 0$. The inner multiplication $\iota(v)$ is a supplementary structure involving the metric structure.

   One shows that the following anti-commutations relations obtain:

$$\{\varepsilon(v), \iota(w)\} = -g(v, w)1\ .$$

Let now $c(v) = \varepsilon(v) + \iota(v)$. We get the anti-commutation relations of the Clifford algebra

$$\{c(v), c(w)\} = -2g(v, w)1$$

and $Cl(V, g)$ is therefore generated in $End_{\mathbb{R}}(\Lambda^* V)$ by the $c(v)$ (identified with v).

6.3 Spin Groups

The isometry group $O(n)$ is canonically embedded in $Cl(V, g)$ since every isometry is a product of reflections. In fact $Cl(V, g)$ contains also the *pin group* $Pin(n)$ which is a twofold covering of $O(n)$. If we take into account the orientation and restrict to $SO(n)$, the twofold covering becomes the *spin group* $Spin(n)$. $Spin(n)$ is generated by the even products of v s.t. $g(v) = \pm 1$, $SO(n)$ is generated by even products of $-Ad_v$ and the covering $Spin(n) \longrightarrow SO(n)$ is given by $v \mapsto -Ad_v$. By restriction of the Clifford multiplication and of the adjoint representation $w \mapsto v.w.v^{-1}$ to $Spin(n)$, we get therefore a representation γ of $Spin(n)$ into the spinor space $\mathbb{S} = Cl(V, g)$.

[9] In the following formula $\widehat{u_j}$ means that the term u_j is deleted.

6.4 Dirac Equation

We can use the Clifford algebra, and therefore the metric, to change the classical exterior derivative of differential forms given by

$$d := \varepsilon\,(dx^{\mu})\,\frac{\partial}{\partial x^{\mu}}.$$

We then define the Dirac operator on spinor fields $\mathbb{R}^n \to \mathbb{S}$ as

$$\begin{aligned} D\ :\ &= c\,(dx^{\mu})\,\frac{\partial}{\partial x^{\mu}} \\ &= \gamma^{\mu}\,\frac{\partial}{\partial x^{\mu}} \end{aligned}$$

where c is the Clifford multiplication, and D acts on the spinor space $\mathbb{S} = Cl(V,g)$. As $\{c(v),c(w)\} = -2g(v,w)1$, the γ^{μ} satisfy standard Dirac relations of anticommutation $\{\gamma^{\mu},\gamma^{\nu}\} = -2\delta^{\mu\nu}$ in the Euclidean case.[10] On can check that $D^2 = \Delta$ is the Laplacian.

6.5 Dirac Operator

More generally, if M is a Riemannian manifold, the previous construction can be done for every tangent space T_xM endowed with the quadratic form g_x. In this way we get a *bundle* of Clifford algebras $Cl(TM,g)$. If S is a spinor bundle, that is a bundle of $Cl(TM)$ -modules s.t. $Cl(TM) \simeq End(S)$, endowed with a covariant derivative ∇, we associate to it the Dirac operator

$$D : \mathcal{S} = \Gamma(S) = C^{\infty}(M,S) \to \Gamma(S)$$

which is a first order elliptic operator interpretable as the "square root" of the Laplacian Δ, Δ interpreting itself the metric in operatorial terms. The Dirac operator D establishes a coupling between the covariant derivation on S and the Clifford multiplication of 1-forms. It can be extended from the $C^{\infty}(M)$-module $\mathcal{S} = \Gamma(S)$ to the Hilbert space $\mathcal{H} = L^2(M,S)$.

In general, because of chirality, S will be the direct sum of an even and an odd part, $S = S^+ \oplus S^-$ and D will have the characteristic form

[10] The classical Dirac matrices are the $-i\gamma^{\mu}$ for $\mu = 0,1,2,3$.

$$D = \begin{bmatrix} 0 & D^- \\ D^+ & 0 \end{bmatrix}$$
$$D^+ \; : \; \Gamma(S^+) \to \Gamma(S^+)$$
$$D^- \; : \; \Gamma(S^-) \to \Gamma(S^-)$$

D^+ and D^- being adjoint operators.

6.6 NC Distance and Dirac Operator

In this classical framework, it easy to compute the bracket $[D, f]$ for $f \in C^\infty(M)$. First, there exists on M the *Levi-Civita connection*:

$$\nabla^g : \Omega^1(M) \to \Omega^1(M) \underset{C^\infty(M)}{\otimes} \Omega^1(M)$$

satisfying the Leibniz rule for $\alpha \in \Omega^1(M)$ and $f \in C^\infty(M)$:

$$\nabla^g(\alpha f) = \nabla^g(\alpha) f + \alpha \otimes df$$

(as $\nabla^g(\alpha) \in \Omega^1(M) \underset{C^\infty(M)}{\otimes} \Omega^1(M)$, $\nabla^g(\alpha) f \in \Omega^1(M) \underset{C^\infty(M)}{\otimes} \Omega^1(M)$ and as α and $df \in \Omega^1(M)$, $\alpha \otimes df \in \Omega^1(M) \underset{C^\infty(M)}{\otimes} \Omega^1(M)$). There exists also the *spin connection* on the spinor bundle S

$$\nabla^S : \Gamma(S) \to \Omega^1(M) \underset{C^\infty(M)}{\otimes} \Gamma(S)$$

satisfying the Leibniz rule for $\psi \in \Gamma(S)$ and $f \in C^\infty(M)$:

$$\nabla^S(\psi f) = \nabla^S(\psi) f + \psi \otimes df$$
$$\nabla^S(\gamma(\alpha)\psi) = \gamma(\nabla^g(\alpha))\,\psi + \gamma(\alpha)\nabla^S(\psi)$$

where γ is the spin representation. The Dirac operator on $\mathcal{H} = L^2(M,S)$ is then defined as

$$D := \gamma \circ \nabla^S .$$

If $\psi \in \Gamma(S)$, we have (making the f acting on the left in $\mathcal{H}$)

$$\begin{aligned} D(f\psi) &= \gamma\left(\nabla^S(\psi f)\right) \\ &= \gamma\left(\nabla^S(\psi) f + \psi \otimes df\right) \\ &= \gamma\left(\nabla^S(\psi)\right) f + \gamma(\psi \otimes df) \\ &= fD(\psi) + \gamma(df)\,\psi \end{aligned}$$

and therefore $[D,f](\psi) = fD(\psi) + \gamma(df)\,\psi - fD(\psi) = \gamma(df)\,\psi$, that is

$$[D,f] = \gamma(df)\,.$$

In the standard case where $M = \mathbb{R}^n$ and $S = \mathbb{R}^n \times V$, V being a Cl_n-module of spinors ($Cl_n = Cl(\mathbb{R}^n, g_{Euclid})$), we have seen that D is a differential operator with constant coefficients taking its values in V.

$$D = \sum_{k=1}^{k=n} \gamma^\mu \frac{\partial}{\partial x^\mu}$$

with the constant matrices $\gamma^\mu \in \mathcal{L}(V)$ satisfying the anti-commutation relations

$$\{\gamma^\mu, \gamma^\nu\} = -2\delta^{\mu\nu}$$

The fundamental point is that the γ^μ are associated with the basic 1-forms dx^μ through the isomorphism

$$c : \mathcal{C} = \Lambda^*(M) \to gr(Cl(TM))$$

$$[D,f] = \gamma(df) = c(df)$$

and $\|[D,f]\|$ is the norm of the Clifford action of *df* on the space of spinors $L^2(M,S)$. But

$$\begin{aligned}\|c(df)\|^2 &= \underset{x\in M}{Sup}\, g_x^{-1}\left(d\overline{f}(x), df(x)\right) \\ &= \underset{x\in M}{Sup}\, g_x\left(grad_x\overline{f}, grad_x f\right) \\ &= \|grad(f)\|_\infty^2\;.\end{aligned}$$

Whence the definition:

$$d(p,q) = Sup\{|f(p) - f(q)| : f \in \mathcal{A}, \|[D,f]\| \leq 1\}\;.$$

In this reinterpretation, *ds* corresponds to *the propagator of the Dirac operator D*. As an operator acting on the Hilbert space $\mathcal{H}$, D is an unbounded self-adjoint operator such that $[D,f]$ is bounded $\forall f \in \mathcal{A}$ and such that its resolvent $(D - \lambda I)^{-1}$ is compact $\forall \lambda \notin Sp(D)$ (which corresponds to the fact that *ds* is infinitesimal) and the trace $Trace\left(e^{-D^2}\right)$ is *finite*. In terms of the operator $G = [F, x^\mu]^* g_{\mu\nu}[F, x^\nu]$, we have $G = D^{-2}$.

7 Noncommutative Spectral Geometry

Basing himself on several examples, Alain Connes arrived at the following concept of NC geometry.

In the classical commutative case, $\mathcal{A} = C^\infty(M)$ is the commutative algebra of "coordinates" on M represented in the Hilbert space $\mathcal{H} = L^2(M,S)$ by pointwise multiplication[11] and ds is a symbol non commuting with the $f \in \mathcal{A}$ and satisfying the commutation relations $\left[\left[f, ds^{-1}\right], g\right] = 0, \forall f, g \in \mathcal{A}$.

Any specific geometry is defined through the representation $ds = D^{-1}$ of ds by means of a Dirac operator $D = \gamma^\mu \nabla_\mu$. The differential $df = [D, f]$ is then the Clifford multiplication by the gradient ∇f and its norm in $\mathcal{H}$ is the Lipschitz norm of f: $\|[D, f]\| = \underset{x \in M}{Sup} \|\nabla f\|$.

These results can be taken as a definition in the general case. The geometry is defined by a *spectral triple* $(\mathcal{A}, \mathcal{H}, D)$ where $\mathcal{A}$ is a NC C^*-algebra with a representation in an Hilbert $\mathcal{H}$ and D is an unbounded self-adjoint operator on $\mathcal{H}$ such that $ds = D^{-1}$ and more generally the resolvent $(D - \lambda I)^{-1}$, $\lambda \notin \mathbb{R}$, is compact, and at the same time all $[D, a]$ are bounded $\forall a \in \mathcal{A}$ (there is a tension between these two last conditions).[12] As Connes (2000) emphasizes

> It is precisely this lack of commutativity between the line element and the coordinates on a space [between ds and the $a \in \mathcal{A}$] that will provide the measurement of distance.

The new definition of differentials are then $da = [D, a]$ $\forall a \in \mathcal{A}$.

8 Yang-Mills Theory of a NC Coupling Between Two Points and Higgs Mechanism

A striking example of pure NC physics is given by Connes' interpretation of the Higgs phenomenon. In the Standard Model, the Higgs mechanism was an *ad hoc* device used for confering a mass to gauge bosons. It lacked any geometrical interpretation. One of the deepest achievement of the NC framework has been to show that Higgs fields correspond effectively to gauge bosons, but for a *discrete* NC geometry.

8.1 Symmetry Breaking and Classical Higgs Mechanism

Let us first recall the classical Higgs mechanism. Consider e.g. a φ^4 theory for two scalar real fields φ_1 and φ_2. The Lagrangian is

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[11] If $f \in \mathcal{A}$ and $\xi \in \mathcal{H}$, $(f\xi)(x) = f(x)\xi(x)$.

[12] Let λ_n be the eigenvalues of D ($\lambda_n \in \mathbb{R}$ since D is self-adjoint). $|\lambda_n| = \mu_n(D)$ and as $(D - \lambda I)^{-1}$ is compact, $|\lambda_n| \underset{n \to \infty}{\longrightarrow} \infty$.

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \varphi_1 \partial^\mu \varphi_1 + \partial_\mu \varphi_2 \partial^\mu \varphi_2\right) - V\left(\varphi_1^2 + \varphi_2^2\right)$$

with the quartic potential

$$V\left(\varphi_1^2 + \varphi_2^2\right) = \frac{1}{2}\mu^2\left(\varphi_1^2 + \varphi_2^2\right) + \frac{1}{4}|\lambda|\left(\varphi_1^2 + \varphi_2^2\right)^2$$

It is by construction $SO(2)$-invariant.

For $\mu^2 > 0$ the minimum of V (the quantum vacuum) is non degenerate: $\varphi_0 = (0,0)$ and the Lagrangian $\mathcal{L}_{os}$ of small oscillations in the neighborhood of φ_0 is the sum of 2 Lagrangians of the form:

$$\mathcal{L}_{os} = \frac{1}{2}\left(\partial_\mu \psi \partial^\mu \psi\right) - \frac{1}{2}\mu^2\psi^2$$

describing particles of mass μ^2.

But for $\mu^2 < 0$ the situation becomes completely different. Indeed the potential V has a full circle (an $SO(2)$-orbit) of minima

$$\varphi_0^2 = -\frac{\mu^2}{|\lambda|} = v^2$$

and the vacuum state is highly *degenerate*.

One must therefore *break the symmetry* to choose a vacuum state. Let us take for instance $\varphi_0 = \begin{bmatrix} v \\ 0 \end{bmatrix}$ and translate the situation to φ_0:

$$\varphi = \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix} = \begin{bmatrix} v \\ 0 \end{bmatrix} + \begin{bmatrix} \xi \\ \eta \end{bmatrix} .$$

The oscillation Lagrangian at φ_0 becomes

$$\mathcal{L}_{os} = \frac{1}{2}\left(\partial_\mu \eta \partial^\mu \eta + 2\mu^2\eta^2\right) + \frac{1}{2}\left(\partial_\mu \xi \partial^\mu \xi\right)$$

and describes two particles:

1. A particle η of mass $m = \sqrt{2}|\mu|$, which corresponds to radial oscillations.
2. A particule ξ of mass $m = 0$, which connects vacuum states. ξ is the *Goldstone boson*.

As is well known, the Higgs mechanism consists in using a cooperation between gauge bosons and Goldstone bosons to confer a mass to gauge bosons. Let $\varphi = \frac{1}{\sqrt{2}}(\varphi_1 + i\varphi_2)$ be the scalar complex field associated to φ_1 and φ_2. Its Lagrangian is

$$\mathcal{L} = \partial_\mu \overline{\varphi} \partial^\mu \varphi - \mu^2 |\varphi|^2 - |\lambda| |\varphi|^4 .$$

It is trivially invariant by the global internal symmetry $\varphi \to e^{i\theta}\varphi$. If we *localize* the global symmetry using transformations $\varphi(x) \to e^{iq\alpha(x)}\varphi(x)$ and take into account the coupling with an electro-magnetic field deriving from the vector potential A_μ, we get

$$\mathcal{L} = \nabla_\mu \overline{\varphi} \nabla^\mu \varphi - \mu^2 |\varphi|^2 - |\lambda| |\varphi|^4 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

where ∇ is the covariant derivative

$$\nabla_\mu = \partial_\mu + iqA_\mu$$

and F the force field

$$F_{\mu\nu} = \partial_\nu A_\mu - \partial_\mu A_\nu.$$

The Lagrangian remains invariant if we balance the localization of the global internal symmetry with a change of gauge

$$A_\mu \to A'_\mu = A_\mu - \partial_\mu \alpha(x).$$

For $\mu^2 > 0$, $\varphi_0 = 0$ is a minimum of $V(\varphi)$, the vacuum is non degenerate, and we get 2 scalar particles φ and $\overline{\varphi}$ and a photon A_μ.

For $\mu^2 < 0$, the vacuum is degenerate and there is a spontaneous symmetry breaking. We have $|\varphi_0|^2 = -\frac{\mu^2}{2|\lambda|} = \frac{v^2}{2}$. If we take $\varphi_0 = \frac{v}{\sqrt{2}}$ and write

$$\varphi = \varphi' + \varphi_0 = \frac{1}{\sqrt{2}}(v + \eta + i\xi) \approx \frac{1}{\sqrt{2}} e^{i\frac{\xi}{v}} (v + \eta) \text{ for } \xi \text{ and } \eta \text{ small,}$$

we get for the Lagrangian of oscillations:

$$\mathcal{L}_{os} = \frac{1}{2}\left(\partial_\mu \eta \partial^\mu \eta + 2\mu^2 \eta^2\right) + \frac{1}{2}\left(\partial_\mu \xi \partial^\mu \xi\right) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + qvA_\mu \left(\partial_\mu \xi\right) + \frac{q^2 v^2}{2} A_\mu A^\mu.$$

1. The field η (radial oscillations) has mass $m = \sqrt{2}\,|\mu|$.
2. The boson A_μ acquires a mass due to the term $A_\mu A^\mu$ and interacts with the Goldstone boson ξ.

The terms containing the gauge boson A_μ and the Goldstone boson ξ write

$$\frac{q^2 v^2}{2} \left(A_\mu + \frac{1}{qv} \partial_\mu \xi\right) \left(A^\mu + \frac{1}{qv} \partial^\mu \xi\right)$$

and are therefore generated by the gauge change

$$\alpha = \frac{\xi}{qv}$$
$$A_\mu \to A_\mu + \partial_\mu \alpha\ .$$

We see that we can use the gauge transformations

$$A_\mu \to A'_\mu = A_\mu + \frac{1}{qv}\partial^\mu \xi$$

for *fixing* the vacuum state. The transformation corresponds to the phase rotation of the scalar field

$$\varphi \to \varphi' = e^{-i\frac{\xi}{v}}\varphi = \frac{v+\eta}{\sqrt{2}} \ .$$

In this new gauge where the Goldstone boson ξ disappears, the vector particule A'_μ acquires a mass qv. The Lagrangian writes now

$$\mathcal{L}_{os} = \frac{1}{2}\left(\partial_\mu \eta \partial^\mu \eta + 2\mu^2\eta^2\right) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{q^2v^2}{2}A'_\mu A'^\mu \ .$$

The Goldstone boson connecting the degenerate vacuum states is in some sense "captured" by the gauge boson and transformed into mass.

8.2 NC Yang-Mills Theory of Two Points and Higgs Phenomenon

The NC equivalent of this description is the following. It shows that Higgs mechanism is actually the standard Yang-Mills formalism applied to a purely discrete NC geometry.

Let $\mathcal{A} = \mathcal{C}(Y) = \mathbb{C} \oplus \mathbb{C}$ be the C^*-algebra of the space Y composed of two points a and b. Its elements $f = \begin{bmatrix} f(a) & 0 \\ 0 & f(b) \end{bmatrix}$ act through multiplication on the Hilbert space $\mathcal{H} = \mathcal{H}_a \oplus \mathcal{H}_b$. We take for Dirac operator an operator of the form

$$D = \begin{bmatrix} 0 & M^* = D^- \\ M = D^+ & 0 \end{bmatrix}$$

and introduce the "chirality" $\gamma = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ (the γ_5 of the standard Dirac theory). In this discrete situation we define *df* as

$$df = [D, f] = \Delta f \begin{bmatrix} 0 & M^* \\ -M & 0 \end{bmatrix}$$

with $\Delta f = f(b) - f(a)$. Therefore

$$\|[D, f]\| = |\Delta f|\,\lambda$$

where $\lambda = \|M\|$ is the greatest eigenvalue of M.

If we apply now the formula for the distance, we find:

$$\begin{aligned} d(a,b) &= Sup\{|f(a)-f(b)| : f \in \mathcal{A}, \|[D,f]\| \leq 1\} \\ &= Sup\{|f(a)-f(b)| : f \in \mathcal{A}, |f(a)-f(b)|\lambda \leq 1\} \\ &= \frac{1}{\lambda} \end{aligned}$$

and we see that the distance $\frac{1}{\lambda}$ between the two points a and b has a *spectral* content and is measured by the Dirac operator.

To interpret differential calculus in this context, we consider the two idempotents (projectors) e and $1-e$ defined by

$$\begin{aligned} e(a) &= 1, e(b) = 0 \\ (1-e)(a) &= 0, (1-e)(b) = 1\ . \end{aligned}$$

Every $f \in \mathcal{A}$ writes $f = f(a)e + f(b)(1-e)$, and therefore

$$\begin{aligned} df &= f(a)de + f(b)d(1-e) \\ &= (f(a)-f(b))\,de \\ &= -(\Delta f)de \\ &= -(\Delta f)ede + (\Delta f)(1-e)d(1-e) \end{aligned}$$

This shows that ede and $(1-e)d(1-e) = -(1-e)de$ provide a natural basis of the space of 1-forms $\Omega^1\mathcal{A}$. Let

$$\begin{aligned} \omega &= \lambda ede + \mu(1-e)d(1-e) \\ &= \lambda ede - \mu(1-e)de \end{aligned}$$

be a 1-form. ω is represented by

$$\omega = (\lambda e - \mu(1-e))\,[D,e]\ .$$

But on $\mathcal{H}$ $[D,e] = -\begin{bmatrix} 0 & M^* \\ -M & 0 \end{bmatrix}$ and therefore

$$\omega = \begin{bmatrix} 0 & -\lambda M^* \\ -\mu M & 0 \end{bmatrix}\ .$$

Let us now construct the *Yang-Mills theory* corresponding to this situation. A vector potential V – a connection in the sense of gauge theories – is a self-adjoint 1-form and has the form

$$\begin{aligned} V &= -\overline{\varphi} e d e + \varphi(1-e)de \\ &= \begin{bmatrix} 0 & \overline{\varphi} M^* \\ \varphi M & 0 \end{bmatrix}. \end{aligned}$$

Its curvature is the 2-form

$$\theta = dV + V \wedge V$$

and an easy computation gives

$$\theta = -\left(\varphi + \overline{\varphi} + \varphi\overline{\varphi}\right) \begin{bmatrix} -M^*M & 0 \\ 0 & -MM^* \end{bmatrix}.$$

The Yang-Mills *action* is the integral of the curvature 2-form, that is the *trace* of θ:

$$YM(V) = Trace\left(\theta^2\right) .$$

But as $\varphi + \overline{\varphi} + \varphi\overline{\varphi} = |\varphi + 1|^2 - 1$ and

$$Trace\left(\begin{bmatrix} -M^*M & 0 \\ 0 & -MM^* \end{bmatrix}^2\right) = 2Trace\left((M^*M)^2\right)$$

we get

$$YM(V) = 2\left(|\varphi + 1|^2 - 1\right)^2 Trace\left((M^*M)^2\right).$$

8.3 Higgs Mechanism

This Yang-Mills action manifests *a pure Higgs phenomenon of symmetry breaking*. The minimum of $YM(V)$ is reached everywhere on the circle $|\varphi + 1|^2 = 1$ (degeneracy) and the gauge group $\mathcal{U} = U(1) \times U(1)$ of the unitary elements of $\mathcal{A}$ acts on it by

$$V \to uVu^* + udu^*$$

where $u = \begin{bmatrix} u_1 & 0 \\ 0 & u_2 \end{bmatrix}$ with $u_1, u_2 \in U(1)$.

The field φ is a Higgs bosonic field corresponding to a gauge connection on a NC space of two points. If $\psi \in \mathcal{H}$ represents a fermionic state, the fermionic action is $I_D(V,\psi) = \langle \psi, (D+V)\psi \rangle$ with

$$D + V = \begin{bmatrix} 0 & (1+\overline{\varphi})M^* \\ (1+\varphi)M & 0 \end{bmatrix}.$$

The complete action coupling the fermion ψ with the Higgs boson φ is therefore

$$YM(V) + I_D(V,\psi).$$

9 The NC Derivation of the Glashow-Weinberg-Salam Standard Model (Connes-Lott)

A remarkable achievement of this NC approach of Yang-Mills theories is given by Connes-Lott's NC derivation of the Glashow-Weinberg-Salam Standard Model. This derivation was possible because, as was emphasized by Martin et al. (1997, p. 5), it ties

> the properties of continuous spacetime with the intrinsic discreteness stemming from the chiral structure of the Standard Model.

9.1 Gauge Theory and NCG

It is easy to reinterpret in the NC framework classical gauge theories where M is a spin manifold, $\mathcal{A} = \mathcal{C}^{\infty}(M)$, D is the Dirac operator and $\mathcal{H} = L^2(M,S)$ is the space of L^2 sections of the spinor bundle S. $Diff(M) = Aut(\mathcal{A}) = Aut(\mathcal{C}^{\infty}(M))$ is the relativity group (the gauge group) of the theory: a diffeomorphism $\varphi \in Diff(M)$ is identified with the $*$-automorphism $\alpha \in Aut(\mathcal{A})$ s.t. $\alpha(f)(x) = f\left(\varphi^{-1}(x)\right)$. The main problem is to reconcile QFT with GR, that is non abelian gauge theories which are non commutative at the level of their *internal* space of quantum variables with the geometry of the *external* space–time M with its group of diffeomorphism $Diff(M)$. The NC solution is an extraordinary principled one since it links the standard "inner" non commutativity of quantum internal degrees of freedom with the new "outer" non commutativity of the external space.

9.1.1 Inner Automorphisms and Internal Symmetries

The key fact is that, in the NC framework, there exists in $Aut(\mathcal{A})$ the normal subgroup $Inn(\mathcal{A})$ of *inner automorphisms* acting by conjugation $a \to uau^{-1}$. $Inn(\mathcal{A})$ is trivial in the commutative case and constitutes one of the main feature of the NC case. As Alain Connes (1996) emphasized:

> Amazingly, in this description the group of gauge transformation of the matter fields arises spontaneously as a normal subgroup of the generalized diffeomorphism group $Aut(\mathcal{A})$. It is the *non commutativity* of the algebra $\mathcal{A}$ which gives for free the group of gauge transformations of matter fields as a (normal) subgroup of the group of diffeomorphisms.

In $Inn(\mathcal{A})$ there exists in particular the *unitary* group $\mathcal{U}(\mathcal{A})$ of unitary elements $u^* = u^{-1}$ acting by $\alpha_u(a) = uau^*$.

9.1.2 Connections and Vector Potentials

In the NC framework we can easily reformulate standard Yang-Mills theories. For that we need the concepts of a connection and of a vector potential.

Let $\mathcal{E}$ be a finite projective (right) $\mathcal{A}$-module. A connection ∇ on $\mathcal{E}$ is a collection of morphisms (for every p)

$$\nabla : \mathcal{E} \otimes_{\mathcal{A}} \Omega^p(\mathcal{A}) \to \mathcal{E} \otimes_{\mathcal{A}} \Omega^{p+1}(\mathcal{A})$$

satisfying for every $\omega \in \mathcal{E} \otimes_{\mathcal{A}} \Omega^p(\mathcal{A})$ and every $\rho \in \Omega^q(\mathcal{A})$ the Leibniz rule in $\mathcal{E} \otimes_{\mathcal{A}} \Omega^{p+q+1}(\mathcal{A})$

$$\nabla(\omega \otimes \rho) = \nabla(\omega) \otimes \rho + (-1)^p \omega \otimes d\rho$$

where we use $\Omega^{p+1}(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^q(\mathcal{A}) = \Omega^p(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^{q+1}(\mathcal{A})$.

∇ is determined by its restriction to $\Omega^1(\mathcal{A})$

$$\nabla : \mathcal{E} \otimes_{\mathcal{A}} \Omega^0(\mathcal{A}) = \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \Omega^1(\mathcal{A})$$

satisfying $\nabla(\xi a) = \nabla(\xi) a + \xi \otimes da$ for $\xi \in \mathcal{E}$ and $a \in \mathcal{A}$.

The curvature θ of ∇ is given by $\nabla^2 : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \Omega^2(\mathcal{A})$. As

$$\begin{aligned}
\nabla^2(\xi a) &= \nabla(\nabla(\xi) a + \xi \otimes da) \\
&= \nabla^2(\xi) a - \nabla(\xi) \otimes da + \nabla(\xi) \otimes da + \xi \otimes d^2 a \\
&= \nabla^2(\xi) a \,,
\end{aligned}$$

∇^2 is $\mathcal{A}$-linear. And as $\mathcal{E}$ is a projective $\mathcal{A}$-module,

$$\theta = \nabla^2 \in End_{\mathcal{A}}\mathcal{E} \otimes_{\mathcal{A}} \Omega^2(\mathcal{A}) = M(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^2(\mathcal{A})$$

is a matrix with elements in $\Omega^2(\mathcal{A})$.

Now, ∇ defines a connection $[\nabla, \bullet]$ on $End_{\mathcal{A}}\mathcal{E}$ by

$$\begin{aligned}
[\nabla, \bullet] : End_{\mathcal{A}}\mathcal{E} \otimes_{\mathcal{A}} \Omega^p(\mathcal{A}) &\to End_{\mathcal{A}}\mathcal{E} \otimes_{\mathcal{A}} \Omega^{p+1}(\mathcal{A}) \\
\alpha &\mapsto [\nabla, \alpha] = \nabla \circ \alpha - \alpha \circ \nabla
\end{aligned}$$

and the curvature θ satisfies the *Bianchi identity* $[\nabla, \theta] = 0$.

A vector potential A is a self-adjoint operator interpreting a 1-form

$$A = \sum_j a_j [D, b_j]$$

and the force is the curvature 2-form

$$\theta = dA + A^2.$$

The unitary group $\mathcal{U}(\mathcal{A})$ acts by gauge transformations on A and its 2-form curvature θ

$$A \to uAu^* + udu^* = uAu^* + u[D,u^*]$$
$$\theta \to u\theta u^*.$$

9.2 Axioms for Geometry

There are characteristic properties of classical (commutative) and NC geometries which can be used to axiomatize them.

1. (Classical and NC geometry). $ds = D^{-1}$ is an infinitesimal of order $\frac{1}{n}$ (n is the dimension)[13] and for any $a \in \mathcal{A}$ integration is given by $Tr_{Dix}\left(a\,|D|^{-n}\right)$ (which is well defined and $\neq 0$ since $|D|^{-n}$ is an infinitesimal of order 1). One can normalize the integral dividing by $V = Tr_{Dix}\left(|D|^{-n}\right)$.
2. (Classical geometry). Universal commutation relations: $[[D,a],b] = 0, \forall a,b \in \mathcal{A}$. So (Jones and Moscovici, 1997)

   > while ds no longer commutes with the coordinates, the algebra they generate does satisfy non trivial commutation relations.

3. (Classical and NC geometry). $a \in \mathcal{A}$ is "smooth" in the sense that a and $[D,a]$ belong to the intersection of the domains of the functionals δ^m where $\delta(T) = [|D|,T]$ for every operator T on $\mathcal{H}$.
4. (Classical geometry). If the dimension n is *even* there exists a $\widetilde{\gamma}$ interpreting a n-form $c \in Z_n(\mathcal{A},\mathcal{A})$ associated to orientation and chirality (the γ^5 of Dirac), $\widetilde{\gamma}$ being of the form $a_0[D,a_1]\dots[D,a_n]$ and s.t. $\widetilde{\gamma} = \widetilde{\gamma}^*$ (self-adjointness), $\widetilde{\gamma}^2 = 1$, $\{\widetilde{\gamma},D\} = 0$ (anti-commutation relation) and $[\widetilde{\gamma},a] = 0, \forall a \in \mathcal{A}$ (commutation relations). $\widetilde{\gamma}$ decomposes D into two parts $D = D^+ + D^-$ where $D^+ = (1-p)Dp$ with $p = \frac{1+\widetilde{\gamma}}{2}$. If e is a self-adjoint ($e = e^*$) idempotent ($e^2 = e$) of $\mathcal{A}$ (i.e. a projector), eD^+e is a Fredholm operator from the subspace $ep\mathcal{H}$ to the subspace $e(1-p)\mathcal{H}$. This can be extended to the projectors of $e \in M_q(\mathcal{A})$ defining finite projective left $\mathcal{A}$-modules $\mathcal{E} = \mathcal{A}^N e$ (if $\xi \in \mathcal{E}$ then $\xi e = \xi$) with the $\mathcal{A}$-valued inner product $(\xi,\eta) = \sum_{i=1}^{i=N} \xi_i \eta_i^*$. ~~bis. (Classical geometry).~~ If n is odd we ask only that there exists such an n-form c interpreted by 1: $a_0[D,a_1]\dots[D,a_n] = 1$.
5. (Classical and NC geometry). $\mathcal{H}_\infty = \bigcap_m Domain(D^m)$ is finite and projective as $\mathcal{A}$ -module and $\langle a\xi,\eta\rangle = Tr_{Dix}a(\xi,\eta)\,ds^n$ ((ξ,η) being the scalar product of

[13] In the NC framework, ds and dx are completely different sort of entities. dx is the differential of a coordinate and ds doesn't commute with it. In the classical case, the order of ds as an infinitesimal is not 1 but the dimension of M. As we will see later, the Hilbert-Einstein action is the NC integral of ds^{n-2}.

$\mathcal{H}$ and Tr_{Dix} the Dixmier trace of infinitesimals of order 1) define an Hermitian structure on $\mathcal{H}_\infty$.

6. (Classical geometry). One can define an *index pairing* of D with $K_0(\mathcal{A})$ and an *intersection form* on $K_0(\mathcal{A})$[14]. If $[\mathcal{E}] \in K_0(\mathcal{A})$ is defined by the projector e, we consider the scalar product $\langle IndD, e\rangle$ which is an integer. We define therefore $\langle IndD, e\rangle : K_0(\mathcal{A}) \to \mathbb{Z}$. As $\mathcal{A}$ is commutative, we can take the multiplication $m : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ given by $m(a \otimes b) = ab$ which induces $m_0 : K_0(\mathcal{A}) \otimes K_0(\mathcal{A}) \to K_0(\mathcal{A})$. Composing with $IndD$ we get the intersection form

$$\begin{aligned} \langle IndD, m_0\rangle \ &: \ K_0(\mathcal{A}) \otimes K_0(\mathcal{A}) \to \mathbb{Z} \\ (e,a) &\to \langle IndD, m_0(e \otimes a)\rangle \,. \end{aligned}$$

Poincaré duality: the intersection form is invertible.

7. *Real structure* (Classical geometry). There exists an anti-linear isometry (charge conjugation) $J : \mathcal{H} \to \mathcal{H}$ which combines charge conjugation and complex conjugation and gives the $*$-involution by algebraic conjugation: $JaJ^{-1} = a^* \ \forall a \in \mathcal{A}$, and s.t. $J^2 = \varepsilon$, $JD = \varepsilon' DJ$, and $J\gamma = \varepsilon'' \gamma J$ with ε, ε', $\varepsilon'' = \pm 1$ depending of the dimension n $mod 8$:

n	0	1	2	3	4	5	6	7
ε	1	1	−1	−1	−1	−1	1	1
ε'	1	−1	1	1	1	−1	1	1
ε''	1		−1		1		−1	

In the classical case (M smooth compact manifold of dimension n), Connes proved that these axioms define a unique Riemannian spin geometry whose geodesic distance and the spin structure are those defined by D. Moreover, the value of the Dixmier trace $Tr_{Dix} ds^{n-2}$ is the *Einstein-Hilbert action functional*:

$$Tr_{Dix} ds^{n-2} = c_n \int_M R\sqrt{g}\, d^n x = c_n \int_M R dv$$

with dv the volume form $dv = \sqrt{g}\, d^n x$ and $c_n = \frac{n-2}{12}(4\pi)^{-\frac{n}{2}}\Gamma\left(\frac{n}{2}+1\right)^{-1} 2^{\left[\frac{n}{2}\right]}$. $Tr_{Dix} ds^{n-2}$ is well defined and $\neq 0$ since ds is an infinitesimal of order $\frac{n-2}{n} < 1$). For $n = 4$, $c_4 = \frac{1}{6}(4\pi)^{-2}\Gamma(3)^{-1} 2^2 = \frac{1}{48\pi^2}$.

In the NC case the characteristic properties (2), (6), (7) must be modified to take into account the NC:

7^{NC}. *Real structure* (NC geometry). In the noncommutative case, the axiom $JaJ^{-1} = a^*$ is transformed into the following axiom saying that the conjugation by J of the involution defines the *opposed* multiplication of $\mathcal{A}$. Let $b^0 = Jb^* J^{-1}$, then $\left[a, b^0\right] = 0$, $\forall a, b \in \mathcal{A}$. By means of this real structure, the Hilbert space $\mathcal{H}$

[14] Remember that $K_0(\mathcal{A}) = \pi_1(GL_\infty(\mathcal{A}))$ classifies the finite projective $\mathcal{A}$-modules and that $K_1(\mathcal{A}) = \pi_0(GL_\infty(\mathcal{A}))$ is the group of connected components of $GL_\infty(\mathcal{A})$.

becomes not only a (left) $\mathcal{A}$-module through the representation of $\mathcal{A}$ into $\mathcal{L}(\mathcal{H})$ but also a $\mathcal{A}\otimes\mathcal{A}^\circ$-module (where $\mathcal{A}^\circ$ is the opposed algebra of $\mathcal{A}$) or a (left-right) $\mathcal{A}$-bimodule through $\left(a\otimes b^0\right)\xi = aJb^*J^{-1}\xi$ or $a\xi b = aJb^*J^{-1}\xi$ for every $\xi\in\mathcal{H}$.

2^{NC}. The universal commutation relations $[[D,f],g]=0, \forall f,g\in\mathcal{A}$ become in the NC case $[[D,a],b^\circ]=0, \forall a,b\in\mathcal{A}$ (which is equivalent to $[[D,b^\circ],a]=0, \forall a,b\in\mathcal{A}$ since a and b° commute by 7^{NC}).

6^{NC}. K-theory can be easily generalized to the NC case. We consider finite projective $\mathcal{A}$-modules $\mathcal{E}$, that is direct factors of free $\mathcal{A}$-modules $\mathcal{A}^N$. They are characterized by a projection $\pi:\mathcal{A}^N\rightarrow\mathcal{E}$ admitting a section $s:\mathcal{E}\rightarrow\mathcal{A}^N$ ($\pi\circ s = Id_{\mathcal{E}}$). $K_0(\mathcal{A})$ classifies them. The structure of $\mathcal{A}\otimes\mathcal{A}^\circ$-module induced by the real structure J allows to define the intersection form by $(e,a)\rightarrow\langle IndD, e\otimes a^\circ\rangle$ with $e\otimes a^\circ$ considered as an element of $K_0(\mathcal{A}\otimes\mathcal{A}^\circ)$.

One of the fundamental aspects of the NC case is that inner automorphisms $\alpha_u(a)=uau^*$, $u\in\mathcal{U}(\mathcal{A})$ act upon the Dirac operator D via NC gauge connections (vector potentials) A

$$\begin{aligned}&\widetilde{D} = D + A + JAJ^{-1} \text{ with}\\ &A = u[D,u^*]\ .\end{aligned}$$

the equivalence between D and $\widetilde{D}$ being given by $\widetilde{D} = UDU^{-1}$ with $U = uJuJ^{-1} = u(u^*)^\circ$.

9.3 The Crucial Discovery of a Structural Link Between "External" Metric and "Internal" Gauge Transformations

One can generalize these transformations of metrics to gauge connections A of the form $A=\sum a_i[D,b_i]$ which can be interpreted as *internal perturbations of the metric* or as *internal fluctuations of the spectral geometry* induced by the internal degrees of freedom of gauge transformations. This coupling between metric and gauge transformations is what is needed *for coupling gravity with QFT*. In the commutative case, this coupling *vanishes* since $U=uu^*=1$ and therefore $\widetilde{D}=D$. The vanishing $A+JAJ^{-1}=0$ comes from the fact that A is self-adjoint and that, due to its special form $A=a[D,b]$, we have $JAJ^{-1}=-A^*$. Indeed, since $[D,b^*]=-[D,b]^*$

$$\begin{aligned}JAJ^{-1} &= Ja[D,b]J^{-1} = JaJ^{-1}J[D,b]J^{-1} = a^*[D,b^*]\\ &= -a^*[D,b]^* = -(a[D,b])^* = -A^*\end{aligned}$$

So the coupling between the "external" metric afforded by the Dirac operator and the internal quantum degrees of freedom is a purely NC effect which constitutes a breakthrough for the unification of GR and QFT in a "good" theory of Quantum Gravity (QG).

9.4 Generating the Standard Model (Connes-Lott)

Before concluding this paper with some remarks on QG, let us recall that the first main interest of NC geometry in physics was generated by the possibility of coupling classical gauge theories with purely NC such theories. This led to the NC interpretation of Higgs fields. Connes' main result is:

Connes' theorem. The Glashow-Weinberg-Salam Standard Model (SM) can be entirely reconstructed from the NC C^*-algebra

$$\mathcal{A} = \mathcal{C}^\infty(M) \otimes (\mathbb{C} \oplus \mathbb{H} \oplus M^3(\mathbb{C}))$$

where the "internal" algebra $\mathbb{C} \oplus \mathbb{H} \oplus M^3(\mathbb{C})$ has for unitary group the symmetry group

$$U(1) \times SU(2) \times SU(3).$$

The first step is to construct the toy model which is the product $\mathcal{C}^\infty(M) \otimes (\mathbb{C} \oplus \mathbb{C})$ of the classical Dirac fermionic model $(\mathcal{A}_1, \mathcal{H}_1, D_1, \gamma_5)$ and the previously explained, purely NC, 2-points model $(\mathcal{A}_2, \mathcal{H}_2, D_2, \gamma)$ with $D_2 = \begin{bmatrix} 0 & M^* \\ M & 0 \end{bmatrix}$:

$$\begin{cases} \mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2 \\ \mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \\ D = D_1 \otimes 1 + \gamma_5 \otimes D_2 \ . \end{cases}$$

The second step is to complexify the model and to show that it enables to derive the complete GWS Lagrangian.

The key idea is to take the product of a 4-dimensional spin manifold M with a finite NC geometry $(\mathcal{A}_F, \mathcal{H}_F, D_F)$ of dimension 0 where $\mathcal{H}_F$ is the Hilbert space with basis the generations of fermions: quarks, leptons. The particule/antiparticule duality decomposes $\mathcal{H}_F$ into $\mathcal{H}_F = \mathcal{H}_F^+ \oplus \mathcal{H}_F^-$, each $\mathcal{H}_F^\pm$ decomposes into $\mathcal{H}_F^\pm = \mathcal{H}_l^\pm \oplus \mathcal{H}_q^\pm$ ($l =$ lepton and $q =$ quark), and chirality decomposes the $\mathcal{H}_p^\pm$ ($p =$ particule) into $\mathcal{H}_{pL}^\pm \oplus \mathcal{H}_{pR}^\pm$ ($L =$ left, $R =$ right). The four quarks are u_L, u_R, d_L, d_R ($u =$ up, $d =$ down) with three colours (12 quarks for each generation) and the three leptons are e_L, ν_L, e_R, the total being of $2(12+3) = 30$ fermions for each generation.

The real structure J is given for $\mathcal{H}_F = \mathcal{H}_F^+ \oplus \mathcal{H}_F^-$ by $J \begin{pmatrix} \xi \\ \overline{\eta} \end{pmatrix} = \begin{pmatrix} \eta \\ \overline{\xi} \end{pmatrix}$ that is, if $\xi = \sum_i \lambda_i p_i$ and $\overline{\eta} = \sum_j \mu_j \overline{p_j}$,

$$J\left(\sum_i \lambda_i p_i + \sum_j \mu_j \overline{p_j}\right) = \left(\sum_j \overline{\mu_j} p_j + \sum_i \overline{\lambda_i} \overline{p_i}\right).$$

The action of the internal algebra $\mathcal{A}_F = \mathbb{C} \oplus \mathbb{H} \oplus M^3(\mathbb{C})$ is defined in the following way. Let $a = (\lambda, q, m) \in \mathcal{A}_F$, $\lambda \in \mathbb{C}$ being a complex scalar acting upon $\mathbb{C}^2$ as the diagonal quaternion $\begin{pmatrix} \lambda & 0 \\ 0 & \overline{\lambda} \end{pmatrix}$, $q = \alpha + \beta j \in \mathbb{H}$ being a quaternion written as $\begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix}$, $j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and $m \in M^3(\mathbb{C})$ being a 3×3 complex matrix. The element $a = (\lambda, q, m)$ acts on quarks, independently of color, via $au_R = \lambda u_R$, $au_L = \alpha u_L - \overline{\beta} d_L$, $ad_R = \overline{\lambda} d_R$, $ad_L = \beta u_L + \overline{\alpha} d_L$, that is as

$$(\lambda, q, m) \begin{pmatrix} u_L \\ d_L \\ u_R \\ d_R \end{pmatrix} = \begin{pmatrix} \alpha & -\overline{\beta} & 0 & 0 \\ \beta & \overline{\alpha} & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \overline{\lambda} \end{pmatrix} \begin{pmatrix} u_L \\ d_L \\ u_R \\ d_R \end{pmatrix} = \begin{pmatrix} \alpha u_L - \overline{\beta} d_L \\ \beta u_L + \overline{\alpha} d_L \\ \lambda u_R \\ \overline{\lambda} d_R \end{pmatrix}$$

(the pair (u_R, d_R) can be considered as an element of $\mathbb{C} \oplus \mathbb{C}$, while (u_L, d_L) can be considered as an element of $\mathbb{C}^2$). It acts on leptons via $ae_R = \overline{\lambda} e_R$, $ae_L = \beta \nu_L + \overline{\alpha} e_L$, $a\nu_L = \alpha \nu_L - \overline{\beta} e_L$, that is as

$$(\lambda, q, m) \begin{pmatrix} e_R \\ \nu_L \\ e_L \end{pmatrix} = \begin{pmatrix} \overline{\lambda} & 0 & 0 \\ 0 & \alpha & -\overline{\beta} \\ 0 & \beta & \overline{\alpha} \end{pmatrix} \begin{pmatrix} e_R \\ \nu_L \\ e_L \end{pmatrix} = \begin{pmatrix} \overline{\lambda} e_R \\ \alpha \nu_L - \overline{\beta} e_L \\ \beta \nu_L + \overline{\alpha} e_L \end{pmatrix}.$$

It acts on anti-particules via $a\overline{l} = \lambda \overline{l}$ for antileptons and via $a\overline{q} = m\overline{q}$ for antiquarks where m acts upon color.

The internal Dirac operator D_F is given by the matrix of Yukawa coupling $D_F = \begin{pmatrix} Y & 0 \\ 0 & \overline{Y} \end{pmatrix}$ where $Y = (Y_q \otimes 1_3) \oplus Y_l$ (the $\otimes 1_3$ comes from the three generations of fermions) with

$$Y_q = \begin{array}{c} \\ u_L \\ d_L \\ u_R \\ d_R \end{array} \begin{array}{c} \begin{array}{cccc} u_L & d_L & u_R & d_R \end{array} \\ \begin{pmatrix} 0 & 0 & M_u & 0 \\ 0 & 0 & 0 & M_d \\ M_u^* & 0 & 0 & 0 \\ 0 & M_d^* & 0 & 0 \end{pmatrix} \end{array}$$

and

$$Y_l = \begin{array}{c} \\ e_R \\ \nu_L \\ e_L \end{array} \begin{array}{c} \begin{array}{ccc} e_R & \nu_L & e_L \end{array} \\ \begin{pmatrix} 0 & 0 & M_l \\ 0 & 0 & 0 \\ M_l^* & 0 & 0 \end{pmatrix} \end{array}$$

where (Connes, 1996) M_u, M_d, and M_l are matrices "which encode both the masses of the Fermions and their mixing properties".

Chirality is given by $\gamma_F(p_R) = p_R$ and $\gamma_F(p_L) = -p_L$ (p being any particule or anti-particule).

Connes and Lott then take the product of this internal model of the fermionic sector with a classical gauge model for the bosonic sector:

$$\begin{cases} \mathcal{A} = C^\infty(M) \otimes \mathcal{A}_F = (C^\infty(M) \otimes \mathbb{C}) \oplus (C^\infty(M) \otimes \mathbb{H}) \oplus \left(C^\infty(M) \otimes M^3(\mathbb{C})\right) \\ \mathcal{H} = L^2(M,S) \otimes \mathcal{H}_F = L^2(M, S \otimes \mathcal{H}_F) \\ D = (D_M \otimes 1) \oplus (\gamma_5 \otimes D_F). \end{cases}$$

The extraordinary "tour de force" is that this model, which is rather simple at the conceptual level (a product of two models, respectively fermionic and bosonic, which takes into account only the known fundamental properties of these two sectors), is in fact extremely complex and generates SM in a *principled* way. Computations are very intricate (see Kastler papers in the bibliography). One has to compute first vector potentials of the form $A = \sum_i a_i [D, a'_i]$, $a_i, a'_i \in \mathcal{A}$ which induce fluctuations of the metric. As D is a sum of two terms, it is also the case for A. Its discrete part comes from $\gamma_5 \otimes D_F$ and generates the Higgs bosons. Let $a_i(x) = (\lambda_i(x), q_i(x), m_i(x))$. The term $\sum_i a_i [\gamma_5 \otimes D_F, a'_i]$ yields γ_5 tensored by matrices of the form:

- Quark sector:

$$\begin{pmatrix} 0 & 0 & M_u\varphi_1 & M_u\varphi_2 \\ 0 & 0 & -M_d\overline{\varphi_2} & M_d\overline{\varphi_1} \\ M_u^*\varphi'_1 & M_d^*\varphi'_2 & 0 & 0 \\ -M_u^*\overline{\varphi'_2} & M_d^*\overline{\varphi'_1} & 0 & 0 \end{pmatrix}$$

  with

$$\begin{cases} \varphi_1 = \sum_i \lambda_i (\alpha'_i - \lambda'_i) \\ \varphi_2 = \sum_i \lambda_i \beta'_i \\ \varphi'_1 = \sum_i \alpha_i (\lambda'_i - \alpha'_i) + \beta_i \overline{\beta'_i} \\ \varphi'_2 = \sum_i \beta_i \left(\overline{\lambda'_i} - \overline{\alpha'_i}\right) - \alpha_i \beta'_i \,. \end{cases}$$

- Lepton sector:

$$\begin{pmatrix} 0 & -M_d\overline{\varphi_2} & M_d\overline{\varphi_1} \\ M_d^*\varphi'_2 & 0 & 0 \\ M_d^*\varphi'_1 & 0 & 0 \end{pmatrix}.$$

Let $q = \varphi_1 + \varphi_2 j$ and $q' = \varphi'_1 + \varphi'_2 j$ be the quaternionic fields so defined. As $A = A^*$, we have $q' = q^*$. The $\mathbb{H}$-valued field $q(x)$ is the *Higgs doublet*.

The second part of the vector potential A comes from $D_M \otimes 1$ and generates the gauge bosons. The terms $\sum_i a_i [D_M \otimes 1, a'_i]$ yield

- The $U(1)$ gauge field $\Lambda = \sum_i \lambda_i d\lambda'_i$.
- The $SU(2)$ gauge field $Q = \sum_i q_i dq'_i$.
- The $U(3)$ gauge field $V = \sum_i m_i dm'_i$.

The computation of the fluctuations of the metric $A+JAJ^{-1}$ gives:

- Quark sector:

$$\begin{array}{c} \\ u_L \\ d_L \\ u_R \\ d_R \end{array} \begin{array}{c} \begin{array}{cccc} u_L & d_L & u_R & d_R \end{array} \\ \begin{pmatrix} Q_{11}1_3+V & Q_{12}1_3 & 0 & 0 \\ Q_{21}1_3 & Q_{22}1_3+V & 0 & 0 \\ 0 & 0 & \Lambda 1_3+V & 0 \\ 0 & 0 & 0 & -\Lambda 1_3+V \end{pmatrix} \end{array}$$

which is a 12×12 matrix since V is 3×3.

- Lepton sector:

$$\begin{array}{c} \\ e_R \\ \nu_L \\ e_L \end{array} \begin{array}{c} \begin{array}{ccc} e_R & \nu_L & e_L \end{array} \\ \begin{pmatrix} -2\Lambda & 0 & 0 \\ 0 & Q_{11}-\Lambda & Q_{12} \\ 0 & Q_{21} & Q_{22}-\Lambda \end{pmatrix} \end{array}$$

One can suppose moreover that $TraceV=\Lambda$, that is $V=V'+\frac{1}{3}\Lambda$ with V' traceless, which gives the correct hypercharges.

The crowning of the computation is that the total (bosonic + fermionic) action

$$Tr_{Dix}\theta^2 ds^4+\left\langle \left(D+A+JAJ^{-1}\right)\psi,\psi\right\rangle = YM(A)+\langle D_A\psi,\psi\rangle$$

(where $\theta=dA+A^2$ is the curvature of the connection A) enables to derive *the complete GWS Lagrangian*

$$\mathcal{L}=\mathcal{L}_G+\mathcal{L}_f+\mathcal{L}_\varphi+\mathcal{L}_Y+\mathcal{L}_V\ .$$

1. $\mathcal{L}_G$ is the Lagrangian of the gauge bosons

$$\begin{aligned} \mathcal{L}_G &= \frac{1}{4}\left(G_{\mu\nu a}G_a^{\mu\nu}\right)+\frac{1}{4}\left(F_{\mu\nu}F^{\mu\nu}\right) \\ G_{\mu\nu a} &= \partial_\mu W_{\nu a}-\partial_\nu W_{\mu a}+g\varepsilon_{abc}W_{\mu b}W_{\nu c}, \\ &\quad \text{with } W_{\mu a} \text{ a } SU(2) \text{ gauge field (weakisospin)} \\ F_{\mu\nu} &= \partial_\mu B_\nu-\partial_\nu B_\mu,\ \text{with } B_\mu \text{ a } SU(1) \text{ gauge field.} \end{aligned}$$

2. $\mathcal{L}_f$ is the fermionic kinetic term

$$\begin{aligned} \mathcal{L}_f &= -\sum \overline{f_L}\gamma^\mu\left(\partial_\mu+ig\frac{\tau_a}{2}W_{\mu a}+ig'\frac{Y_L}{2}B_\mu\right)f_L+ \\ &\quad \overline{f_R}\gamma^\mu\left(\partial_\mu+ig'\frac{Y_R}{2}B_\mu\right)f_R \end{aligned}$$

with $f_L=\begin{bmatrix}\nu_L\\ e_L\end{bmatrix}$ left fermion fields of hypercharge $Y_L=-1$ and $f_R=(e_R)$ right fermion fields of hypercharge $Y_R=-2$.

3. $\mathcal{L}_\varphi$ is the Higgs kinetic term

$$\mathcal{L}_\varphi = -\left|\left(\partial_\mu + ig\frac{\tau_a}{2}W_{\mu a} + i\frac{g'}{2}B_\mu\right)\varphi\right|^2$$

with $\varphi = \begin{bmatrix}\varphi_1 \\ \varphi_2\end{bmatrix}$ a $SU(2)$ pair of scalar complex fields of hypercharge $Y_\varphi = 1$.

4. $\mathcal{L}_Y$ is a Yukawa coupling between the Higgs fields and the fermions

$$\mathcal{L}_Y = -\sum\left(H_{ff'}\left(\overline{f_L}.\varphi\right)f'_R + H^*_{ff'}\overline{f'_R}\left(\varphi^+.f_L\right)\right)$$

where $H_{ff'}$ is a coupling matrix.

5. $\mathcal{L}_V$ is the Lagrangian of the self-interaction of the Higgs fields

$$\mathcal{L}_V = \mu^2\left(\varphi^+\varphi\right) - \frac{1}{2}\lambda\left(\varphi^+\varphi\right)^2 \ with\ \lambda > 0.$$

10 Quantum Gravity, Fluctuating Background Geometry, and Spectral Invariance (Connes-Chamseddine)

10.1 Quantum Field Theory and General Relativity

As we have already emphasized, Alain Connes realized a new breakthrough in Quantum Gravity by coupling such models with General Relativity. In NCG, QG can be thought of in a principled way because it becomes possible to introduce in the model of QFT the gravitational Einstein-Hilbert action as a direct consequence of the specific invariance of spectral geometry, namely *spectral invariance*. As Alain Connes (1996) explains:

> However this [the previous NC deduction of the SM] requires the definition of the curvature and is still in the spirit of gauge theories. (...) One should consider the internal gauge symmetries as part of the diffeomorphism group of the non commutative geometry, and the gauge bosons as the internal fluctuations of the metric. It follows then that the action functional should be of a purely gravitational nature. We state the principle of spectral invariance, stronger than the invariance under diffeomorphisms, which requires that the action functional only depends on the spectral properties of $D = ds^{-1}$ in $\mathcal{H}$.

The general strategy for coupling a Yang-Mills-Higgs gauge theory with the Einstein-Hilbert action is to find a C^*-algebra $\mathcal{A}$ s.t. the normal subgroup $Inn(\mathcal{A})$ of inner automorphisms is the gauge group and the quotient group $Out(\mathcal{A}) = Aut(\mathcal{A})/Inn(\mathcal{A})$ of "external" automorphisms plays the role of $Diff(M)$ in a gravitational theory. Indeed, in the classical setting we have principal bundles $P \to M$ with a structural group G acting upon the fibers and an exact sequence

$$Id \to \mathcal{G} \to Aut(P) \to Diff(M) \to Id$$

where $\mathcal{G} = C^{\infty}(M,G)$ is the gauge group. The non abelian character of these gauge theories comes solely from the non commutativity of the group of *internal* symmetries G. The total symmetry group $Aut(P)$ of the theory is the *semidirect* product $\mathfrak{G}$ of $Diff(M)$ and $\mathcal{G} = C^{\infty}(M,G)$. If we want to geometrize the theory completely, we would have to find a generalized space X s.t. $Aut(X) = \mathfrak{G}$.

> If such a space would exist, then we would have some chance to actually geometrize completely the theory, namely to be able to say that it's pure gravity on the space X. (Connes, 2000)

But this is impossible if X is a manifold since a theorem of John Mather proves that in that case the group $Diff(X)$ would be simple (without normal subgroup) and could'nt therefore be a semidirect product. *But it is possible with a NC space* $(\mathcal{A},\mathcal{H},D)$. For then (Iochum et al., 1996)

> the metric 'fluctuates', that is, it picks up additional degrees of freedom from the internal space, the Yang-Mills connection and the Higgs scalar. (...) In physicist's language, the spectral triplet is the Dirac action of a multiplet of dynamical fermions in a background field. This background field is a fluctuating metric, consisting of so far adynamical bosons of spin 0,1 and 2.

If we find a NC geometry $\mathcal{A}$ with $Inn(\mathcal{A}) \simeq \mathcal{G}$, a correct spectral triple and apply the spectral action, then gravity will correspond to $Out(\mathcal{A}) = Aut(\mathcal{A})/Inn(\mathcal{A})$. As was emphasized by Martin et al. (1997):

> The strength of Connes' conception is that gauge theories are thereby deeply connected to the underlying geometry, on the same footing as gravity. The distinction between gravitational and gauge theories boils down to the difference between outer and inner automorphisms.

Jones and Moscovici (1997) add that this implies that

> Connes' spectral approach gains the ability to reach below the Planck scale and attempt to decipher the fine structure of space–time.

So, just as GR extends the Galilean or Minkowskian invariance into diffeomorphism invariance, NCG extends both diffeomorphism invariance and gauge invariance into a larger invariance, the spectral invariance.

The philosophically significant content of the NC point of view must be emphasized. We already saw that in GR the metric of M is no longer a background structure (but the differentiable structure of M remains a background) while in QFT the metric of M is still a background structure. In the NC framework the metric is no longer a background structure, as in GR, but in addition it is a quantum fluctuating structure.

10.2 The Spectral Action and the Eigenvalues of the Dirac Operator as Dynamical Variables for General Relativity

The key device is the bosonic spectral action

$$Trace\left(\phi\left(\frac{D^2}{\Lambda^2}\right)\right) = Trace\left(\phi\left(\frac{|D|}{\Lambda}\right)\right)$$

where Λ is a cut-off of the order of the inverse of Planck length and ϕ a smooth approximation of the characteristic function $\chi_{[0,1]}$ of the unit interval. $D^2 = (D_M \otimes 1 + \gamma_5 \otimes D_F)^2$ is computed using Lichnerowicz' formula $D^2 = \Delta^S + \frac{1}{4}R$. As this action counts the number $N(\Lambda)$ of eigenvalues of D in the interval $[-\Lambda, \Lambda]$, the key idea is, as formulated by Giovanni Landi and Carlo Rovelli (1997),

> to consider the eigenvalues of the Dirac operator as dynamical variables for general relativity.

This formulation highlights the physical and transcendental significance of the NC framework: *since the distance is defined through the Dirac operator D, the spectral properties of D can be used in order to modify the metric*. The eigenvalues are spectral invariants and are therefore, in the classical case, automatically $Diff(M)$ invariant.

> Thus the general idea is to describe spacetime geometry by giving the eigen-frequencies of the spinors that can live on that spacetime. [...] The Dirac operator D encodes the full information about the spacetime geometry in a way usable for describing gravitational dynamics. (Landi-Rovelli (1997): the quotation concerns D_M acting on the Hilbert space of spinor fields on M.)

This crucial point has also been well explained by Steven Carlip (2001, p. 47). As we have seen in the Introduction, in GR points of space–time lose any physical meaning so that GR obervables must be radically non-local. This is the case with the eingenvalues of D which

> provide a nice set of non local, diffeomorphism-invariant obervables.

They yield

> the first good candidates for a (nearly) complete set of diffeomorphism-invariant observables.

Let us look at $N(\Lambda)$ for $\Lambda \to \infty$. $N(\Lambda)$ is a step function which encodes a lot of information and can be written as a sum of a mean value and a fluctuation (oscillatory) term $N(\Lambda) = \langle N(\Lambda)\rangle + N_{osc}(\Lambda)$ where the oscillatory part $N_{osc}(\Lambda)$ is random. The mean part $\langle N(\Lambda)\rangle$ can be computed using a semi-classical approximation and a heat equation expansion. A wonderful computation shows that for $n = 4$ the asymptotic expansion of the spectral action is

$$Trace\left(\phi\left(\frac{D^2}{\Lambda^2}\right)\right) = \Lambda^4 f_0 a_0\left(D^2\right) + \Lambda^2 f_2 a_2\left(D^2\right) + f_4 a_4\left(D^2\right) + O\left(\Lambda^{-2}\right)$$

where

- $f_0 = \int_{\mathbb{R}} \phi(u)\, u du,\ f_2 = \int_{\mathbb{R}} \phi(u)\, du,\ f_4 = \phi(0)$
- $a_j\left(D^2\right) = \int_M a_j\left(x, D^2\right) dv\ (\ dv = \sqrt{g} d^4 x)$
- $a_0\left(x, D^2\right) = \frac{1}{(4\pi)^2} Trace_x(1)$
- $a_2\left(x, D^2\right) = \frac{1}{(4\pi)^2} Trace_x\left(\frac{1}{6}s1 - E\right)$
- $a_4\left(x, D^2\right) = \frac{1}{360(4\pi)^2} Trace_x\left(5s^2 1 - 2r^2 1 + 2R^2 1 - 60sE + 180E^2 + 30R^{\nabla}_{\mu\nu} R^{\nabla\mu\nu}\right)$
- R is the curvature tensor of M and $R^2 = R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta}$
- r is the Ricci tensor of M and $r^2 = r_{\mu\nu} r^{\mu\nu}$
- s is the scalar curvature of M
- E and $R^{\nabla}_{\mu\nu}$ come from Lichnerowicz' formula.

Let

$$\mathcal{E} = C^\infty(M, S \otimes \mathcal{H}_F) = C^\infty(M, S) \otimes_{C^\infty(M)} C^\infty(M, \mathcal{H}_F).$$

The connection on $\mathcal{E}$ is

$$\nabla = \nabla^S \otimes Id_{C^\infty(M,\mathcal{H}_F)} + Id_{C^\infty(M,S)} \otimes \nabla^F$$

and $R^{\nabla}_{\mu\nu}$ is the curvature 2-tensor of this total connection ∇. If $D = ic^\mu \nabla_\mu + \varphi$ with $c^\mu = \gamma^\mu \otimes Id_{C^\infty(M,\mathcal{H}_F)}$, then $D^2 = \Delta + E$, with

$$\begin{cases} \Delta = -g^{\mu\nu}\left(\nabla_\mu \nabla_\nu - \Gamma^\alpha_{\mu\nu} \nabla_\alpha\right) \\ E = \frac{1}{4}s1 - \frac{1}{2}c\left(R^F\right) + ic^\mu\left[\nabla_\mu, \varphi\right] + \varphi^2 \\ c\left(R^F\right) = -\gamma^\mu \gamma^\nu \otimes R^F_{\mu\nu}\ (R^F = \text{curvature of } \nabla^F). \end{cases}$$

The asymptotic expansion of the spectral action is dominated by the first two terms which identify with the Einstein-Hilbert action with a cosmological term. The later can be eliminated by a change of ϕ.

11 Conclusion

We have seen how NCG reformulated on a new basis the mathematical interpretation of the categories of physical objectivity. Let us summarize its main steps.

1. The primitive fact, namely how phenomena are given, is constituted by the NC C^*-algebra $\mathcal{A}$ of observables. What is physically observable and measurable are the *spectral* properties of the observables of $\mathcal{A}$ interpreted as operators on an Hilbert space $\mathcal{H}$. Spectral data are physically more primitive than geometrical ones and physical geometry must be reconstructed from the outset as a spectral geometry. Classical geometrical transcendental aesthetics determines the first transcendental

moment of "Phoronomy". As was already shown in Petitot (1991a), this was already converted into a spectral moment in Quantum Mechanics. Now in NCG this moment becomes a *geometrical-spectral* moment. We can speak of a "spectral phoronomy".

2. Differential calculus and infinitesimals, which determine the second transcendental moment, namely that of "Dynamics", are entirely interpreted anew from the formula $da = [D, a]$.

3. As in GR, metric is "promoted" from the "Phoronomy" moment (where it acts as a background structure) to the "Mechanics" moment (where it becomes a physical field) while, conversely, the "Mechanics" moment is "demoted" to the "Phoronomy" moment (forces are absorbed in a larger relativity principle). This transcendental chiasm provides the philosophical interpretation for the elimination of metric as background structure. In NCG this is expressed by the constitutive role of the Dirac operator D in the definition of metric. D is a physical operator and in that sense metric is "physicalized". But at the same time, differentials are defined by $da = [D, a]$ and in the classical case the eigenvalues of D are $Diff(M)$ invariant, that is, the metric still belongs to the moments of "Phoronmy" and "Dynamics".

4. This deep recasting of the mathematical "construction" of transcendental moments of physical objectivity has many important consequences. Let me focus here on two of them.

1. The possibility of deriving the whole complexity of the Standard Model from an empirical nucleus via the product of a classical spin geometry with a NC discrete geometry generating Higgs fields.
2. The possibility of defining a spectral action unifying a QFT à la Yang-Mills with GR via the eigenvalues of the Dirac operator used as dynamical variables for the metric.

We see that, after having been applied to symplectic mechanics, general relativity, non-abelian gauge theories and string theories, a correctly generalized and "historicized" transcendentalism is able to support the conceptual breakthrough brought about by Noncommutative Geometry.

Addendum. In a forthcoming book, Alain Connes, Ali Chamseddine and Matilde Marcolli show how their previous results can be strongly improved and yield a derivation of the standard model minimally coupled to gravity (Einstein-Hilbert action) with massive neutrinos, neutrino mixing, Weinberg angle, and Higgs mass (of the order of 170 GeV). This new achievement is quite astonishing.

Acknowledgements Of course, these reflexions would not have been possible without the opportunity of attending during many years Alain Connes' lectures at the Collège de France in Paris. I want to thank Pierre Cartier for many enlighting comments on these difficult topics. In the last years, my research program on a transcendentalist philosophy of modern physics gained a lot from discussions with my colleagues Michel Bitbol and Pierre Kerszberg, and master physicists such as Bernard d'Espagnat, Roland Omnès, Gilles Cohen-Tannoudji, and Bas Van Fraasssen. Some discussions with the "Structuralism in Physics" group: Katherine Brading, Elena Castellani, Brigitte Falkenburg, Steven French, and Tom Ryckman, were also constructive. It was a great pleasure to meet in Paris, at the Fondation des Treilles[15] and elsewhere in other occasions Jeffrey Bub, John Baez, Michael Friedmann, Marc Lachièze-Rey, Herman Nicolai, Paolo Parrini, and Carlo Rovelli.

I want also to mention some Ph.D. and post-doc students in philosophy of physics at CREA: Manuel Bächtold, Gabriel Catren, Alexei Grinbaum, Patricia Kauark, Stefano Osnaghi, and Isabelle Peschard.

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[15] A great thank to Bernard Teissier, one of the pilars of the Foundation.

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