Schématisme : logique des jugements VS géométrie des phénomènes

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3 décembre 2020

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- Fil directeur : *le schématisme*.
  - Un pôle logico-linguistique (logique des jugements au sens traditionnel de prédication),
  - Un pôle de *format* d'objet (souvent intuitif, géométrique, morphologique).
- Lien entre les deux.

- Relier une logique à une structure objectale et à des modèles d'objets (ce que Husserl appelait une “ontologie régionale”) est un problème “transcendantal”.

- Physique mathématique.
- Phénoménologie de la perception.
- Structuralisme : phonologie, linguistique cognitive, sémiotique (Thèse: *Pour un schématisme de la structure*).
- Esthétique.

- Deux problématiques :
  - physique au niveau fondationnel le plus élémentaire,
  - phénoménologie de la perception.
- “Logique” est pris au sens le plus élémentaire et classique (aristotélicien) de “prédication” : attribution de propriétés à des entités sans tenir compte de la façon dont les entités sont *constituées* en objets d'une science particulière.

Physique

- L'exemple de la phoronomie (cinématique) chez Kant dans les *Metaphysische Anfangsgründe der Naturwissenschaft* (1786).
- Cf. Jules Vuillemin : *Physique et métaphysique kantiennes* (1955).
- Michael Friedman : *Kant's Construction of Nature* (2013).

- Logique générale “pure” analytique et formelle.
- Elle ne concerne que la “forme de la pensée” et fait abstraction de la spécificité des domaines de connaissance.
- C'est un “canon” de l'entendement et pas un “organon”.
- C'est une simple “logique de la vérité” (le vrai/faux des jugements).
- La logique générale “appliquée” tient compte de la psychologie associée.

- Une logique contrainte par les strutures synthétiques a priori d'un domaine d'objet (“ontologie régionale” chez Husserl) est une logique *transcendantale*.
- Elle seule peut être un “organon” pour la science considérée.

- Les MAN traitent de l'exemple de la mécanique newtonienne.
- L'objet régional est le *mouvement* (Bewegung).
- Les catégories *ne sont pas* directement applicables, en tant que “concepts purs de l'entendement”, à un objet régional spécifique.
- Elles nécessitent un *schématisme* les transformant en *méthodes de construction d'objets*.

- **Remarque.** C'est évident pour les concepts *empiriques* : transformer le *concept* de chat en des méthodes de construction *d'images* de chats. Mais c'est un énorme problème neurocognitif (“art caché dans les profondeurs de l'âme” disait Kant).
- Ici, le schème *transcendantal* de la quantité est le nombre et, pour une ontologie régionale donnée, concerne ce qui permet de *calculer* sur les phénomènes de la région avec des *algèbres* de nombres.

- Pourquoi Kant a-t-il dû introduire un nouveau type de jugements à cause de la relativité galiléenne ?
- C'est aujourd'hui totalement trivial, mais à l'époque cela ne l'était pas du tout.
- C'est la première mise en crise de la logique élémentaire des jugements par un problème fondationnel (“Anfangsgründe”) venant de la physique (révolution galileo-newtonienne).
- Pour Kant : objectivité physique (“factum rationis”) **>** logique.

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Petitot_Phoronomy.pdf

Phoronomy

- *Phenomenality* of motion is characterized by transformations of frames.
- Phoronomy concerns the motion of matter (Bewegung) relative to an inertial frame (relative Raum).
- Inertial frames are governed by Galilean relativity.

Kinematics Background structure: space and time. Observable states: position and velocity. Measuring: Euclidean metric and arithmetic. Algebraic structure: composing motions. Galilean relativity and inertial frames.	**Phoronomy** Categories of quantity. Extensive magnitudes. Axioms of intuition.
Dynamics Primitive internal forces inside matter (repulsion, attraction). Reduction of matter to mass.	**Dynamics** Categories of quality. Intensive magnitudes. Anticipations of perception.
Mechanics Conservation laws. Invariance. Newton law: Galilean invariance. Action-reaction equality	**Mechanics** Categories of relation. Analogies of experience: permanence of substance, causality, coexistence.
	Phenomenology Categories of modality. Postulates of empirical thought in general.

- The fundamental problem of *Phoronomy* : inertial motions as “magnitudes” (Grösse) must be *algebraically* constructed as “quantities” (quantitas).
- *Axioms of Intuition* : a magnitude is “composition of the homogeneous” (*Zusammen-setzung des Gleichartigen*, B202).
- There must exist *mathematical operations of composition* for magnitudes (an algebra).

- The *algebraic* composition (*Zusammensetzung*) of velocities of inertial motions must be deduced from the *phenomenal* composition of changes of frame and from their Galilean relativity.
- Key problem : relativity implies that the algebraic composition *is not physical* and cannot be a composition *inherent* to matter (as it is the case for accelerations).

- For Newton, there exists an *absolute space* and therefore Galilean relativity is not metaphysically disruptive.
- For Kant, absolute space is antinomic and cannot be introduced in physics.
  - "Absolute space is (...) in itself nothing, and no object at all." (481/16).
  - "An absolute motion (...) cannot be experienced at all" (487/23).
  - "Absolute space is nothing for all possible experience" (488/23).

- Absolute space is only “an Idea of Reason”.
- It is the same for an absolute position in space, an absolute direction in space, an absolute moment in time.
- Physics had to wait until its Lagrangian formulation to correlate these relativities to fondamental conservation laws via Noether's theorem : momentum, angular momentum, energy, center of mass.

- Nevertheless, Kant admits that if, far beyond Phoronomy, we look at all the gravitational inter-actions between all bodies in the universe, we can choose a special *cosmological* inertial frame attached to the *center of mass* of the universe and acting as a surrogate of the absolute space. But this has no sense at the phoronomical level.

- The consequence of this radical conception of Galilean relativity, is that velocities *cannot be attached to bodies* and added as vectors attached to a single and same body (the parallelogram rule).
- Accelerations can be attached to bodies because they are Galilean invariants, but not velocities.

- Lehrsatz :

"The composition of two motions of one and the same point can only be thought in such a way that one of them is represented in absolute space, and, instead of the other, a motion of the relative space with the same speed occurring in the opposite direction is represented as the same as the latter." (490/26).

- The problem is that v_K is a relative $v_{K/F}$ and the composition + for the v must respect the composition of the K/F.
- But K/F and $(K/F)'$ cannot be composed (obstruction). Only K/F and F/F' can be composed (this gives K/F'), or F'/K and K/F (this givesF'/F).

- “<u>Axiom</u>” : if K is a body with linear uniform (LU) motion of velocity v, to add w to v means
  - to look at the inertial frame F_0 where K has velocity v,
  - to take the inertial frame F_1 with velocity $-w$ w.r.t. F_0,
  - and to take the velocity of K w.r.t. F_1.
- Kant was very proud of this axiom concerning the "observer" ("Zuschauer") side and the background structure.

- The equality of motions is called “Kongruenz”.

“[The] congruence of two combined motions with a third (as with the *motus compositus* itself) can never take place if this two combined motions are represented in one and the same space.” (493/29, Remark 1 of the Proposition).

- The phoronomic “*construction*” of motion is this *conversion* of relativity into an algebraic calculus (vector space).

- A magnitude (*Grösse*) becomes a *quantitas* (vector with module, *Geschwindigkeit*, and direction *Richtung*):

$$\begin{cases} v+w:(\mathbb{R}^{+},+)\,, \text{ same line, same direction} \\ v+w:(\mathbb{R},+)\,, \text{ same line} \\ v+w:\left(\mathbb{R}^{2},+\right), \text{ parallelogram rule} \end{cases}$$

- LU motions of inertial frames : composition $*$ (as in categories)

$$(F_2/F_0) * (F_0/F_1) = (F_2/F_1)$$

$$(F_0/F_1) * (F_1/F_0) = Id_{F_0}$$

- Velocity vectors : algebraic operation +
- Relation between the two :

$$v_{F_2/F_0} + v_{F_0/F_1} = v_{(F_2/F_0)*(F_0/F_1)}$$

- So, moving bodies K have “kinematical states” ρ = K/L but the result of a measure of v in a state does not measure a true property of K.

- In fact the problem is quite subtle because Kant works in the tangent spaces $T_x E$ of the space E at x.
- He considers the LU motions “tangent” to motions.

Relativity and judgment

- With Phoronomy and relativity, Kant ran into a deep philosophical problem.
- He tackled it in the *Phenomenology*, which develops the categories of <u>modality</u> and the <u>Postulates of the empirical thought in general</u>.
- How a logic of judgments can be compatible with the object format "background structure + Galilean relativity" ?

- Due to relativity, a judgment such as

  “the body K has velocity v_K”

  has *no* physical content, *no* ascribable truth value.

- "In Phoronomy (…) it is completely undetermined and arbitrary how much speed, if any, I wish to ascribe to the one [the body] or to the other [the frame]."

- Phoronomy concerns motions as phenomena and not objects of experience.
- But logic concerns the relation of knowledge to *objects* :

  disjunctive judgments, predicates, and *truth values*.

"of two *objectively* opposed predicates, one is assumed to the exclusion of the other for the determination of the object." (556/95).

- Law of the excluded middle.

- But Phoronomy as *Erscheinungslehre* concerns the relation of knowledge to *the subject* (Zuschauer, observer) :

  <u>*alternative*</u> judgments and relativity.

"of two judgments objectively equivalent, yet subjectively opposed to one another, one is assumed for the determination of the object without excluding its opposite – and thus by mere choice." (Ibidem)

"For here the proposition that the body is either moved and the space at rest, or conversely, is not a disjunctive proposition in an objective relation, but only in a subjective one, and the two judgments contained therein are valid alternatively." (560/99).

- The difference “phenomenon VS object” is expressed on the side of judgments by the difference between
  - disjunctive judgments,
  - alternative judgments.
- Disjunctive judgments can have truth values, that is $v_K = v$ *or* $v_K \neq v$ is true.
- Alternative judgments cannot have truth values : since two predicates objectively contradictory can be ascribed: $v_K = v$ *and* $v_K \neq v$ can be true at the same time.

- Kant says that they are only “possible” judgments ascribing a predicate.
- There is a problem of “*value definiteness*”: “to have a value” <u>does not mean</u> “to possess a property”.
- Relativity means that the concept of velocity is <u>contextual</u>, relative to an observer (*Zuschauer*).
- Only the (K, F, $v_{K/F}$) have a physical content.

- For inertial motions there exists no “element of physical reality” associated to velocity (no physical content).
- Ontological realism is excluded by the theory itself.

- This problem had deep consequences after Kant for EM.
- The motion of a charged particule in an EM field is driven by the force

$$F = q(E + v \wedge B) .$$

- But F is Gal-invariant while v is not.
- Therefore the formula seems to be not well composed. It is a problem.
- Solution : the decomposition (E, B) of the EM field depends on the inertiel frame. E and B do not exist as independent fields.

- E is generated by static charges and B by moving charges and, in another inertial frame with velocity V, E becomes $E + V \wedge B$ and B remains B.
- Furthermore, the group of relativity for the Maxwell equations of the EM field (E, B) is not the Galilean group but the Lorentz group.

- Physicists introduced the “aether” converting the empty space into a *physical* (paradoxical) substrate w.r.t. which v becomes “absolute”.
- They needed to introduce contractions of length and dilations of time.

- But the addition of velocities remained paradoxical (c is constant, Michelson-Morley).
- Einstein “dephysicalized” the aether, turned back to a pure kinematics with a new relativity group (Lorentz).
- But a sort of aether regained a physical status with the quantum vacuum.

Quantum mechanics

- Philosophy of quantum mechanics faced the same type of problem, but in a much more acute way.
- Quantum Logic.
- Bohr topos (Chris Isham, Jeremy Butterfield, Andreas Döring, Chris Heunen, Bas Spitters, etc.) .

- In QM the obstruction to ascribe truth values to judgments comes from the non-commutativity of the algebra of observables.

- The problem is well known since Bell and Kochen-Specker no-go theorems.
- Value definiteness and non contextuality.
K-S : For $dim(\mathfrak{H}) \geq 3$ ($\mathfrak{H}$ the Hilbert space of the system) it is impossible to ascribe definite values $V(a)$ to the observables a with the hypotheses :
  - the *measure* of a gives $V(a) \Leftrightarrow$ the system *has* an *intrinsic property* (non contextual) whose value is $V(a)$,
  - $FUNC : V(f(a,b)) = f(V(a), V(b))$

- No-go theorems show that “measure” is not “predication”.
- There exists a gap between:
  - “the measure of the observable a gives the result r when the system is in state ρ ”, and
  - “when the system is in state ρ , the quantity a *has* the value r .”
- There exists <u>no “element of physical reality”</u> associated to a measure.

- Simon Kochen-Ernst Specker (1967) :
- They worked out an exemple with $dim(\mathfrak{H}) = 3$ and 117 observables.
- Adán Cabello (1996) worked out a simpler example with $dim(\mathfrak{H}) = 4$ and 18 observables.
- To each observable is associated an orthonormal frame of eigenvectors and 4 projectors P_i (yes-no questions) with

$$V\left(P_i\right) = 0 \text{ or } 1$$

$$\sum_i V\left(P_i\right) = 1$$

- In each frame one P has $V(P) = 1$ and three P have $V(P) = 0$.
- Cabello found 9 orthonormal frames

$$9 \text{ x } 4 = 36$$

where each axis is counted 2 times

$$36 = 18 \text{ x } 2$$

- Non contextuality implies that the $V(P)$ are *intrinsic* and do not depend upon the frame.

- When we add the 9 $\sum_i V(P_i) = 1$ the RHS is 9 (odd) and the LHS is even since the $V(P)$ are *intrinsic* and each appears two times.

u_1	(0, 0, 0, 1)	(0, 0, 0, 1)	(1, −1, 1, −1)	(1, −1, 1, −1)	(0, 0, 1, 0)	(1, −1, −1, 1)	(1, 1, −1, 1)	(1, 1, −1, 1)	(1, 1, 1, −1)
u_2	(0, 0, 1, 0)	(0, 1, 0, 0)	(1, −1, −1, 1)	(1, 1, 1, 1)	(0, 1, 0, 0)	(1, 1, 1, 1)	(1, 1, 1, −1)	(−1, 1, 1, 1)	(−1, 1, 1, 1)
u_3	(1, 1, 0, 0)	(1, 0, 1, 0)	(1, 1, 0, 0)	(1, 0, −1, 0)	(1, 0, 0, 1)	(1, 0, 0, −1)	(1, −1, 0, 0)	(1, 0, 1, 0)	(1, 0, 0, 1)
u_4	(1, −1, 0, 0)	(1, 0, −1, 0)	(0, 0, 1, 1)	(0, 1, 0, −1)	(1, 0, 0, −1)	(0, 1, −1, 0)	(0, 0, 1, 1)	(0, 1, 0, −1)	(0, 1, −1, 0)

Transcendental logic and topos theory

- On the “judgment” side we find *logic*.
- On the “objective” side we find *geometrical* structures of objects (vector spaces, relativity groups, Hilbert spaces, C*-algebras, etc.).

- To combine both sides, that is logic and geometry, <u>is</u> the transcendental problem since Kant.
- For the specific science of a specific regional object, <u>logic must be constrained by an object structure</u>.
- It is *"modal"* in Kant's sense w.r.t. a *"Phoronomy"* .

- To day, I think that one of the best framework for transcendental logic is topos theory, which
  - expresses the “object structure” by some category $\mathfrak{C}$,
  - formulates the theory by a certain type of functors $\mathfrak{C} \to \underline{\mathbf{Ens}}$.

- Topos
  - pull backs,
  - terminal object,
  - exponentials i.e. internal Hom,
  - subobject classifier.
- Seaves, sites, sheaves.

- Saunders Mac Lane - Ieke Moerdijk (first sentence of *Sheaves in Geometry and Logic*, 1992) :

  “A startling aspect of topos theory is that it unifies two seemingly wholly distinct mathematical subjects : on the one hand topology and algebraic geometry and on the other hand, logic and set theory.”

- The topos <u>**Ens**</u> is purely logic, without object structure.
- As claimed Ferdinand Gonseth,

  “la logique est la physique de l'objet quelconque.”

Bohr topos and K-S

- Interpretation of K-S in the late 90s by Chris Isham, Jeremy Butterfield, Andreas Döring, Chris Heunen, Bas Spitters, etc.
- The approach is mainly "foundational, structural and conceptual".
- How to ascribe truth values to judgments ?
- They use the "interplay between spatial and logical structures inherent to topos theory." (Heunen)

Bohr topos

Born rule

Example of *elementary* propositions ("yes" or "no" questions)

$$a \in \Delta$$

with $a \in \mathcal{B}_{s.a.}(\mathcal{H})$ and $\Delta \subset \mathbb{R}$ (Borel subset).

If $\psi \in \mathcal{H}$, $a \in \mathcal{B}_{s.a.}(\mathcal{H})$, and $[a \in \Delta]$ projector,

Born rule: the probability for $a \in \Delta$ to be true in state ψ is

$$\langle \psi, a \in \Delta \rangle = \| [a \in \Delta] \psi \|^2$$

A definite truth value $V(a \in \Delta)$ (not a probability) exists iff

$$\begin{cases} [a \in \Delta] \psi = \psi \text{ then } V = 1 \\ [a \in \Delta] \psi = 0 \text{ then } V = 0 \end{cases}$$

If we want more general truth values, since hidden variables don't work, we need a non classical truth object (subobject classifier) Ω.

Ω must *reflect* the object structure.

Logic must be "internalized" and become
a transcendental logic.

Classical case

M = phase space,
category (poset, frame i.e.distributive complete lattice) $\mathcal{O}(M)$ of open subsets.
states = points $x \in M$,
observable = real function $a : M \to \mathbb{R}$,

$a \in \Delta$ true $\Leftrightarrow a(x) \in \Delta \Leftrightarrow x \in a^{-1}(\Delta)$.

A point x defines a valuation V_x on propositions $a \in \Delta$ with values in $\Omega = \{0, 1\}$:

$$\begin{cases} V_x\left(a \in \Delta\right) = 1 \text{ if } a\left(x\right) \in \Delta \\ V_x\left(a \in \Delta\right) = 0 \text{ if } a\left(x\right) \notin \Delta \end{cases}$$

Gelfand-Neimark-Segal

(M compactified)
The algebra $\mathfrak{A}_{s.a.}$ of observables $\subset$ the commutative C^*-algebra

$$\mathfrak{A} = C\left(M, \mathbb{C}\right), a : M \to \mathbb{R}.$$

A *state* is a map $\rho : \mathfrak{A} \to \mathbb{C}$ which is $\mathbb{C}$-linear, positive, normalized.
$\rho\left(a\right)$ is the value of the observable a in the state ρ.

Let $\Sigma = \text{Spec}(\mathfrak{A})$, $\Sigma \simeq M$.
Gelfand duality: $\mathfrak{A} = C(\Sigma, \mathbb{C})$, i.e.

$$a \Leftrightarrow \widehat{a} : \Sigma \to \mathbb{C} \Leftrightarrow \widetilde{a} : C(\mathbb{R})_0 \to \mathfrak{A}$$

where $C(\mathbb{R})_0$ is the unitalization of the algebra of functions with compact support on $\mathbb{R}$ (compactification of $\mathbb{R}$).

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The map $\widetilde{a} : C(\mathbb{R})_0 \to \mathfrak{A}$ corresponds to the *functional calculus* FUNC on $\mathfrak{A}$, that is

$$\widetilde{a}(1) = a, \text{and } \widetilde{a}(f(x)) = f(a).$$

The state ρ defines a probability measure μ_ρ on Σ (Riesz theorem) and $\rho(a) = \int_\Sigma \widehat{a}(x) \, d\mu_\rho(x)$.

If the state ρ is not only $\mathbb{C}$-linear but also a morphism of *algebras*, then it is a point x of Σ, $\rho = \delta_x$.

Points $x \in \Sigma \iff$ valuations $V_x : \mathfrak{A} \to \mathbb{C}$ defined by $V_x(a) = \widehat{a}(x)$.

Gleason theorem

If $\mathfrak{A}$ is no longer commutative, the states would a priori be only *quasi*-states $\rho : \mathfrak{A} \rightarrow \mathbb{C}$ which are $\mathbb{C}$-linear only on the commutative sub-C^*-algebra C of $\mathfrak{A}$.

But Gleason theorem (1957):
If $\mathcal{H}$ is a separable Hilbert space of dim ≥ 3 and if $\mathfrak{A} = \mathcal{B}(\mathcal{H})$ then every quasi-state is a state.

- In QM we need $\Omega \neq \{0, 1\}$.
- Quantum logic is not satisfactory.
- Isham's & Co strategy is to “localize” truth to *partial* and *contextual* valuations defines only on the commutative sub-C*-algebras C.
  - Philosophical relations with modal theory (Bas Van Fraassen, Richard Healey, Jeffrey Bub, etc.).
  - Mathematical relations with topos theory.

Let $\mathfrak{A}$ be the C^*-algebra describing the system.
Let $\mathcal{C}(\mathfrak{A})$ be the small category (poset) of *commutative* sub-C^*-algebra of $\mathfrak{A}$.
Let $\mathcal{T}(\mathfrak{A}) = Sh(\mathcal{C}(\mathfrak{A}))$ be the topos of sheaves over $\mathcal{C}(\mathfrak{A})$.

The natural topology of $\mathcal{C}(\mathfrak{A})$ is defined by the upper-sets U over C
(i.e. $U \subseteq C^{\uparrow}$ the maximal upper-set over C and if $D \in U$ and $D \subset E$ then $E \in U$).

If $\overline{F} \in \mathcal{T}(\mathfrak{A})$, it is equivalent to the *covariant* functor $F : \mathcal{C}(\mathfrak{A}) \to \mathbf{Ens}$ defined by $F(C) = \overline{F}(C^{\uparrow})$.

Indeed, $C \subset D \Rightarrow D^{\uparrow} \subset C^{\uparrow}$ since

$$D \subset E \Rightarrow C \subset D \subset E \text{ } i.e. \text{ } C \subset E$$

The terminal object 1 of $\mathcal{T}(\mathfrak{A})$ is the constant functor.

Pullbacks are computed pointwise.

For the exponentials

$$F^G(C) = \mathrm{Nat}\left(G_{C^\uparrow}, F_{C^\uparrow}\right)$$
$$F^G(\mathbb{C}.1) = \mathrm{Nat}\left(G, F\right)$$

As the maximal upper-sets $C^\uparrow$ are sub-categories of $\mathcal{C}(\mathfrak{A})$, if $F \in \mathcal{T}(\mathfrak{A})$, we can consider the restrictions of F to the $C^\uparrow$. We note that $\mathbb{C}.1^\uparrow = \mathcal{C}(\mathfrak{A})$.

The subobject classifier is defined by

$$\Omega(C) \to \mathcal{U}_C = \{\text{upper sets } U \text{ over } C\}$$
$$\Omega(C \subset D) = \mathcal{U}_C \to \mathcal{U}_D, U \in \mathcal{U}_C \mapsto U \cap D^{\uparrow} \in \mathcal{U}_D$$

$$\Omega^F(C) \simeq \mathrm{Sub}(F_{C^{\uparrow}})$$
$$\Omega^F(\mathbb{C}.1) \simeq \mathrm{Sub}(F)$$
$$\Omega^F(C \subset D) = \text{ restriction of the } \mathrm{Sub}(F_{C^{\uparrow}}) \text{ to the } \mathrm{Sub}(F_{D^{\uparrow}})$$

The key point is that *inside* the topos $\mathcal{T}(\mathfrak{A})$, the *identity* functor $\underline{\mathfrak{A}}\colon \mathcal{C}(\mathfrak{A}) \to \mathbf{Ens}$, $C \mapsto C$, defines an *internal commutative C^*-algebra.*

$(\mathcal{T}(\mathfrak{A}), \underline{\mathfrak{A}})$ is called the *Bohr topos.*

As $\underline{\mathfrak{A}}$ is *commutative* we can apply Gelfand theory but *inside* $\mathcal{T}(\mathfrak{A})$.

We get an *internal* Gelfand spectrum $\underline{\Sigma}$ which synthetizes the Gelfand spectra Σ_C.

This NC "phase space" $\underline{\Sigma}$ can be described *externally* by a frame $\mathcal{O}(\underline{\Sigma})$ (distributive complete lattice) of "open sets", but this frame doesn't come from any classical topological space.

The interpretation of K-S in the Bohr topos

K-S means that the *internal* Gelfand spectrum $\underline{\Sigma}$ has *no* points (it is point free).

Indeed, a point is a morphism

$$\rho : 1 \to \underline{\Sigma}$$

(that is a morphism $\rho^{-1} : \mathcal{O}(\underline{\Sigma}) \to \Omega$).

Internally, observables a are elements of

$$\underline{\mathfrak{A}}_{s.a.} : C \to C_{s.a.}$$

If $a \in \underline{\mathfrak{A}}_{s.a.}$, its Gelfand transform is $\widehat{a} : \underline{\Sigma} \to \underline{\mathbb{R}}$ where $\underline{\mathbb{R}}$ is the constant sheaf $\mathbb{R}$.

$\widehat{a} \circ \rho : 1 \to \underline{\Sigma} \to \underline{\mathbb{R}}$ is a point of $\underline{\mathbb{R}}$.

We get a map

$$\underline{V}_\rho : \underline{\mathfrak{A}}_{s.a.} \to \mathrm{Pt}\,(\underline{\mathbb{R}})$$

$$\underline{V}_\rho\,(C) = \underline{\mathfrak{A}}_{s.a.}\,(C) = C_{s.a.} \to \mathrm{Pt}\,(\underline{\mathbb{R}})\,(C) = \mathbb{R}$$

One shows that $\underline{V}_\rho\,(C) = C_{s.a.} \to \mathbb{R}$ is a morphism of algebras and is therefore a *valuation*.
K-S implies it is impossible. Therefore a point ρ of $\underline{\Sigma}$ can't exist.

We can also say that $\underline{\Sigma}$ is a "fibration" $\underline{\pi}$ over $\mathcal{C}(\mathfrak{A})$ with fibers $\Sigma_C \to C$. A point $\rho : 1 \to \underline{\Sigma}$ is in fact a *gobal section* of $\underline{\pi}$ and K-S means that $\underline{\pi}$ has no global sections.