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PHENOMENOLOGY OF PERCEPTION, QUALITATIVE PHYSICS AND SHEAF MEREOLOGY

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INTRODUCTION

One of the main consequences of cognitive sciences for philosophy concerns *the naturalization of eidetic descriptions*, e.g. eidetic phenomenological descriptions or formal logical ones. “Naturalization” means here experimental devices, physical explanations, mathematical modeling (which is in general completely different from logical formalization) and computational simulation.

This raises immediately a question: what can be the link between natural explanations and pure phenomenological, ontological (e.g. in Husserl’s sense of formal ontology) or logical descriptions of the same phenomena? For investigating this point we need some mediating device which can operate at the same time at two completely different levels:

(i) the one of physical explanations and

(ii) the one of eidetic and formal descriptions.

Mathematical modeling provides such a mediation.

My purpose is to give an example of the link between *physics and logic* which can be worked out using mathematical modeling in cognitive sciences. The example will be very simple but also very characteristic. It concerns the concept of *form*. As I want to clarify some difficult philosophical points I will take the *simplest* case — too simple of course —, the one of 2D static forms. For a mathematical investigation of 3D dynamical forms in computational vision I refer to my paper “Le Physique, le Morphologique, le Symbolique. Remarques sur la vision” (Petitot, 1990).

In the first part of my paper I will show briefly that there exists a remarkable convergence between:

(i) the phenomenological description of forms worked out by Husserl in the third *Logical Investigation*;

(ii) the topological-geometrical description proposed by Thom in the late sixties;

(iii) the physical (in the sense of objective, external, in the outside world) explanation of forms in morphodynamics and qualitative physics;
(iv) the physical (in the sense of cognitive, internal, in the brain) explanation of forms in computational vision.

This convergence will lead us to what can be called a *morphological geometry* which shares the *twofold* status of a descriptive eidetics (in Husserl's sense) and of a mathematical modeling of physical (external and internal) explanations.

In the second part of my paper I will show that this morphological eidetics yields a mereology and a logic which can help to solve some traditional philosophical problems. We will see that morphological eidetics is naturally related to the geometrical concept of *sheaf* and we will use the essential and deep link established by Lawvere and Tierney between sheaf theory and (intuitionistic) logic by means of the categorical concept of *topos*.

I. THE EIDETIC NUCLEUS OF THE CONCEPT OF FORM

Let us start with the phenomenological pure eidetic description given by Husserl in the Third Logical Investigation. Husserl begins with the difference between "abstrakten" and "konkreten Inhalten". He identifies it with the other (Stumpfian) opposition between dependent ("unselbständigen") and independent ("selbständigen") contents.

It is only in the second chapter Gedanken zu einer Theorie der reinen formen von Ganzen und Teilen that Husserl develops an axiomatics of whole/parts relations. In the first chapter Der Unterschied der selbständingen und unselbständingen Gegenstände, he develops in fact a "material" analysis of empirical morphologies.

The central problem analyzed by Husserl is that of the *unilateral* dependence between qualitative moments (e.g. color) and spatial extension (Ausdehnung). According to him, qualities are abstract essences (species) and categorized manifolds. He thought of the "quality → extension" dependence as an eidetic law binding generic abstracta or types.

> "The dependence [Abhängigkeit] of the immediate moments [der unmittelbaren Momente] concerns a certain relation conform to a law existing between them, relation which is determined only by the immediately super-ordered abstracta of these moments" (p. 233).

There is a *functional* dependence (funktionelle Abhängigkeit) connecting the *immediate* moments of quality and extension: it associates to every point x of the extension W the value $q(x)$ of the quality q. But it is objectively legalized by a pure law ("objektiv-ideale Notwendigkeit", "reine und objektive Gesetzlichkeit") which acts only at the level of pure essences (reinen Wesen). This “ideal a priori necessity grounded in the material essences” (“in den sachlichen Wesen gründenden idealen oder apriorischen Notwendigkeit”) is, according to Husserl, a typical exemple of *synthetic a priori*.

In the §§8-9, Husserl analyzes the difference between the contents which profile themselves intuitively against a background ("anschaulich sich abhebenden Inhalten") and the contents which are intuitively merged and fused together (verschmolzenen). Perception presupposes a global unity of the intuitive moments and a “phänomenal Abhebung”, that is, a saliency in Thom's sense. It is such a saliency which is expressed by the difference between, on the one hand, contents intuitively separated (gesonderten, sich abhebenden, sich abscheidenden) from the neighboring ones and, on the other hand, contents merged with the neighboring ones (verschmolzenen, überflieβenden, ohne Scheidung).

The concept of fusion or merging — of *Verschmelzung* — is a key one. It expresses the spreading of qualities, that is the topological transition from the local level to the global one. Its complementary concept is that of separation, of disjonction — of *Sonderung*. Sonderung is an obstacle to Verschmelzung. It generates boundaries delimiting parts. At the intuitive synthetic a priori level, the "whole/part" difference grounds itself on the "Verschmelzung/Sonderung" one.

<u>Remark 1.</u>

It must be pointed here that Husserl’s pure description fits very well with contemporary research. For instance Stephen Grossberg, one of the leading specialists of vision, concludes from his numerous works that there are two fundamental systems in visual perception.

(i) The *Boundary Contour System* (BCS) which controls the segmentation of the visual scene. It detects, sharpens, enhances and completes edges, especially boundaries, by means of a “spatially long-range cooperative process”. The boundaries organize the image geometrically (morphologically).

(ii) The *Featural Contour System* (FCS) which performs featural filling-in, that is spreading of qualities. It stabilizes qualities such as color or brightness. These diffusion processes are triggered and limited by the boundaries provided by the BCS.

Therefore, according to Grossberg (1988, 35),

> “Boundary Contours activate a boundary completion process that synthesizes the boundaries that define perceptual domains. Feature Contours activate a diffusion filling-in process that spreads featural qualities, such as brightness or color, across these perceptual domains” (p. 35).

<u>Remark 2.</u>

In fact, the concept of Verschmelzung does not come from Stumpf but from the German psychologist Johann Friedrich Herbart (1776-1841) who developed a *continuous* theory of mental representations. Essentially in the same vein as Peirce after him, Herbart was convinced that mental contents are *vague* and can vary continuously. For him, a “serial form” (Reihenform) was a class of mental representations which undergo a graded fusion (abgestufte Verschmelzung) glueing them together via continuous transitions. He coined the neologism of *synechology* for his metaphysics (Peirce's neologism of *synechism* is clearly parallel). It is not sufficiently known that Herbart's point of view was one of the main interests of Bernhard Riemann when he was elaborating his key concept of Riemannian manifold. Even if Riemann did not agree with Herbart's metaphysics, he strongly claimed that he was “a Herbartian in psychology and epistemology”. Erhard Scholtz (1992) has shown that in Riemann's celebrated Über die Hypothesen, welche der Geometrie zu Grunde liegen (1867) the role of the differentiable manifold underlying a Riemannian manifold “is taken in a vague sense by a Herbartian-type of ‘serial form’, backed by mathematical intuition”.

Still in the §8 of the third *Logische Untersuchung*, Husserl claims “Sonderung beruht (...) auf *Diskontinuität*”. Verschmelzung corresponds to a continuous (stetig) spreading of qualities in an undifferentiated unity (unterschiedslose Einheit) (p. 244) while Sonderung corresponds to qualitative discontinuities in the way in which extension is covered (Deckungszusammenhang) by qualities.

These qualitative discontinuities are salient only if :

(i) they are contiguously unfolded (sie angrenzend ausgebreitet sind) against the background of a moment which varies continuously (ein kontinuierlich variierendes Moment), namely the spatial and temporal moment.

(ii) they present a sufficient gap (threshold of discrimation).

Husserl's morphological description is precise and remarkable.

> “It is from a spatial or temporal limit [einer Raum-oder Zeitgreuze] that one springs from a visual quality to another. In the continuous transition [kontinuierlichen Übergang] from a spatial part to another, one does not progress also continuously in the covering quality [in der überdeckenden Qualität] : in some place of the space, the adjacent neighboring qualities [die angrenzenden Qualitäten] present a finite (and not too small) gap [Abstand]” (p. 246).

This Husserlian pure eidetic description of the unilateral dependence “quality→extension” yields therefore the following homologations.

Totality (Whole)	**Parts**
Verschmelzung	Sonderung

Spreading activation (featural filling-in)	Boundaries
Continuity	Discontinuity

But even if it is precise and remarkable, Husserl's morphological eidetic description raises nevertheless a fundamental problem. As we have seen, qualitative discontinuities

> “concern the minimal specific differences [die niedersten spezifischen Differenzen] in a same immediately super-ordered [übergeordnet] pure genus [Gattung]” (p. 246).

They are discontinuities of the *concrete functional dependences* “quality → extension”. But, according to Husserl, it is impossible to formalize them. Formalization can only operate at a higher level of abstraction, the level of the general eidetic law of dependence.

We meet here a *formalist* thesis which subordinates the regional material ontologies to formal ontology, and therefore the synthetic a priori (synthetisch-a priorischen) laws to the analytic (analytisch-a priorischen) ones. As regards its material content, the eidetic law of dependence “quality → extension”, belongs to the sphere of “der vagen Anschaulichkeiten”. But, according to Husserl, these vague — *anexact* — morphological essences cannot be *geometrically* constructed. As he claims at the beginning of the §9:

> “Kontinuität und Diskontinuität sind natürlich nicht in matematischer Exaktheit zu nehmen“ (p. 245).

It is not possible to clarify here this fundamental point. But nevertheless I want to emphasize the fact that one of the main limits of phenomenology is to divorce any “material descriptive eidetic” of Erlebniss from any form of geometry. Husserl has always rejected the possibility of a morphological geometry. In some outstanding sections of the *Ideen I* (§§ 71-75), he explains the fundamental difference between, on the one hand, the vague anexact descriptive concepts correlated with morphological essences, and, on the other hand, the exact ideal mathematical concepts. According to him, *ideation*, which brings exact essences to ideality, is drastically different from *abstraction*, which brings anexact essences to genericity (categorization and typicality). This opposition is a key one for Husserl. Nevertheless, it is no longer acceptable.

II. FIVE CONVERGENT SCIENTIFIC EXPLANATIONS OF THE PHENOMENOLOGICAL DESCRIPTION

We want now to show that the phenomenological pure eidetic description fits very well with several scientific explanations which are remarkably convergent. We will give 5 examples.

1. The topological-geometrical schematization (Thom)

In this section "internal" means "internal to the material system under consideration".

Phenomenologically, a material system *S* occupying a spatial domain *W* manifests its form through observable and measurable qualities $q^1(w),\dots,q^n(w)$ which are characteristic of its actual internal state A_w at every point $w \in W$, and, as we will see in a moment, are sections of fibrations of typical fibers the quality types $Q^1,\dots,Q^n$ (colour, texture, etc.). When the spatial control *w* varies smoothly in *W*, A_w varies smoothly. If A_w subsists as the actual state, then the q^i vary also smoothly. But if the actual state A_w bifurcates towards another actual state B_w when *w* crosses some *critical value*, then some of the q^i must present a discontinuity. Thom called *regular* the points $w \in W$ where locally all the qualities q^i vary smoothly and *singular* the points $w \in W$ where locally some of the q^i present a qualitative discontinuity. The set R_w of regular points is by definition an open set of *W* and its complementary set K_W, the set of singular points, is therefore a closed set. By definition, K_W is the *morphology* yielded by the internal dynamical behavior of the system *S*.

The singular points $w \in K_W$ are *critical values* of the control parameters and, in the physical cases, the system *S* presents for them *a critical internal behavior*. Thom was one of the first scientists to stress the point that qualitative discontinuities are phenomenologically dominant, that every qualitative discontinuity is a sort of critical phenomenon and that a general mathematical theory of morphologies presented by general systems had to be an enlarged theory of critical phenomena.

Now, it is clear that Thom's description is an exact topological version of Husserl's one. Regular points correspond exactly to Verschmelzung, and singular ones to Sonderung.

2. The morphodynamical explanation and qualitative macroscopic physics (Thom)

(For more details, see Petitot-Smith 1991).

Let *S* be a material substrate. The problem is to explain its obervable morphology. We choose a level of observation, that is a *scale*. The main morphodynamical model is then the following one.

In *S* an internal dynamical mechanism *X* defines the *internal states*. More precisely :

(i) There exists a configuration space (or a phase space) M of S which is a differentiable manifold and whose points x represent the instantaneous transient states of S. M is called the *internal space* of S.

(ii) X is a flow on M, that is, a system of ordinary differential equations $\dot{x} = X(x)$ which shares three properties : it is first complete (its trajectories are integrable from $t = -\infty$ to $t = +\infty$); second deterministic; and third smooth relatively to the initial conditions. The smooth vector field X is called the *internal dynamics* of S.

As a flow, X is identifiable with the one parameter subgroup of diffeomorphisms of M, Γ_t , where Γ_t is the diffeomorphism of M which associates to every point $x \in M$ the point x_t which is the point at time t on the trajectory of X leaving x at time $t = 0$. Clearly, $\Gamma_{t'} \circ \Gamma_t = \Gamma_{t+t'}$ and $\Gamma_{(-t)} = (\Gamma_t)^{-1}$. $\Gamma : \mathbb{R} \rightarrow \mathrm{Diff}\,(M)$ is therefore a morphism of groups from the additive group of $\mathbb{R}$ to the group of diffeomorphisms of M. Γ is the integral version of the vector field X. The internal states of S are then the (asymptotically stable) *attractors* of X.

In fact, for a general dynamical system, it is very difficult to define rigorously the notion of an attractor. The usual definition is the following. Let $\omega^+(a)$ be the positive limit set of $a \in M$: $\omega^+(a)$ is the topological closure "at infinity" of the positive trajectory of a. Let $A \subset M$ be a subset of the internal space M. A is an attractor of the flow X if it is topologically *closed*, *X-invariant* (i.e. if $a \in A$ then $\Gamma_t(a) \in A\ \forall t \in \mathbb{R}$), *minimal* for these properties (i.e. $A = \omega^+(a)\ \forall a \in A$), and if it attracts asymptotically every point x belonging to one of its neighborhoods U (i.e. $\exists U$ s.t. $A = \omega^+(x)\ \forall x \in U$). A is asymptotically stable if in addition it *confines* the trajectories of its sufficiently neighboring points. If A is an attractor of X, its *basin* $B(A)$ is the set of points $x \in M$ which are attracted by A (i.e. s.t. $\omega^+(x) = A$). If U is an attracted neighborhood of A, we have of course $B(A) = \bigcup_{t<0} \Gamma_t(U)\cdot$

Now, as only one internal state A can of course be the *actual* state of S, there exists necessarily some criterion I (for instance a physical principle of minimization of energy) which selects A from among the possible internal states of S. The system S is also *controlled* by control parameters varying in the extension W of S. W is called the *external space* of S. The internal dynamics X is therefore a dynamics X_w which is parametrized by the external points $w \in W$ and varies smoothly relative to them.

With such a morphodynamical model we can physically explain the topological description

We see that we can go from physics to phenomenological description through mathematical modeling using the following steps.

(i) The choice of a scale which allows us to use a differentiable approximation for formatting the physical phenomena.

(ii) The use of a *qualitative* approach to the dynamical models of the internal physics (qualitative dynamics in the Poincaré-Birkhoff-Smale-Thom-Arnold sense).
(iii) The bracketing (the *epoche*) of the internal physics and the coding of the internal attractors by observable external qualities (qualia).
This leads to the morphological (topological-geometrical) description.
(iv) The phenomenological eidetic description of the morphological one.

3. The cognitive explanation I: 2D primal sketch and wavelet analysis (Marr, Mallat)

In this section "internal" means "internal to the observer".

We shift now from "external" physical explanations to "internal" cognitive ones. At the most peripherical level visual processing is a signal analysis and it is well known that the most basic signal analysis is Fourier analysis.

David Marr (1982) introduced the hypothesis that the main function of the ganglionary cells of the retina is to extract the qualitative discontinuities (zero-crossings) which are encoded in the signal and that the higher levels of visual processing are grounded in this early morphological organization of the image (primal sketch). In fact, it has been shown that the convolution of the signal by the receptive profiles of the ganglionary cells (which are essentially Laplacians of Gaussians), is a *wavelet analysis*, that is a *spatially localized* and *multiscale* Fourier analysis. Now, wavelet analysis is actually the best known device for extracting discontinuities. Let us explain very briefly the main idea in the one-dimensional case.

Consider the Hilbert space $L^2(\mathbb{R})$ of square integrable functions on $\mathbb{R}$. Fourier analysis provides an orthogonal decomposition of every $f \in L^2(\mathbb{R})$ relative to the orthonormal basis of trigonometric functions $e^{-i\omega x}$. That is, the Fourier transform (FT) $\hat{f}(\omega)$ of $f(x)$ is given by the formula :

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-i\omega x} dx .$$

It can be shown that $\hat{\hat{f}}(\omega) = f(x)$ and that the norms $\|f(x)\|$ and $\|\hat{f}(\omega)\|$ are equal (that is, the FT is an isometry).

The problem is that the information provided by $\hat{f}$ is *delocalized* (because the plane waves $e^{-i\omega x}$ are). In order to localize it, Gabor introduced the idea of the *Window Fourier transform* (WFT) :

$$Gf(\omega,u) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\omega x} g(x-u) f(x) dx = \frac{1}{\sqrt{2\pi}} f * \tilde{g}_{\omega,u} .$$

$Gf(\omega,u)$ is localized by a spatial "window" $g(x)$ translated along the x-axis. The WFT depends not only on the frequency ω but also on the position u. It generalizes the FT. The inverse transform is given by :

$$f(x) = \frac{1}{2\pi}\int_{\mathbb{R}^2} Gf(\omega,u)\, e^{i\omega x} g(u-x)\,d\omega dx \quad .$$

This is an isometry betweeen $L^2(\mathbb{R})$ and $L^2(\mathbb{R}^2)$, that is $\|f\| = \|Gf\|$. It is in general highly redundant and it is therefore possible to sample u and ω.

The problem with Gabor's WFT is that it operates at only *one* level of resolution. If the signal is a multiscaled one (e.g. fractal) this is a drastic limit.

The idea of the *wavelet transform* (WT) is to find decompositions of $L^2(\mathbb{R})$ using *a single* function $\psi(x)$ (the "mother" of the wavelets), its translated transforms $\psi(x-u)$ and its rescaled transforms $\psi_s(x) = \sqrt{s}\,\psi(sx)$ (or $\psi_s(x) = \frac{1}{s}\psi\left(\frac{x}{s}\right)$) . One gets therefore the following WT :

$$Wf(s,u) = \int_{\mathbb{R}} f(x)\psi_s(x-u)dx = f * \tilde{\psi}_s(u)$$

with $\tilde{\psi}(x) = \psi(-x)$. It is well defined if an *admissibility condition* C_ψ on the FT $\hat{\psi}$ is satisfied, which says that $\hat{\psi}(0) = 0$ and that $\hat{\psi}$ is sufficiently flat near 0 :

$$(C_\psi) \ : \qquad \int_0^\infty \frac{|\hat{\psi}(\omega)|^2}{\omega} d\omega < \infty$$

The main result of the theory is that convenient ψ *exist*. A typical example is Marr's wavelet. The *amplitude* $|Wf(s,u)|$ of the WT is an indicator of the *singularities* encoded in the signal. More precisely, the Lipschitzian order of f at x can be deduced from the asymptotic decreasing of the $|Wf(s,u)|$ in the neighborhood of x when the scale tends towards 0. As was emphasized by Stéphane Mallat (1989) :

> "the ability of the WT to characterize the type of local singularities is a major motivation for its application to detect the signal's sharper variations".[1]

As the WT is generally highly redundant,[2] it is possible to discretize it by sampling the u and ω variables. With such devices is it possible to *compress* an image in an *intrinsic* way, that is according to its specific structure.[3] The main fact we want to

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[1] For an introduction to the use of wavelet analysis in computational vision, see also Mallat-Zhong (1989). For a discussion of the link with morphological phenomenology, see e.g. Petitot 1989b, 1990, 1993b,c.

[2] When the redundancy is zero one speaks of orthogonal wavelets.

[3] Wavelet analysis can be refined — in particular for the applications to data compression problems — by means of wavelet *packet* algorithms and methods. Many wavelets are used in parallel so as to adapt in

stress here is that the compression of information — which is an information processing constraint — appears as identical with a morphological analysis — which is a geometrical objective fact.

Stephane Mallat has proved "Marr's conjecture": an image can be reconstructed from its qualitative discontinuities scanned at different scales. This shows that the morphological nucleus can be also recovered through signal analysis.

4. Multiscale algorithms for image segmentation (Mumford, Shah)

Multiscale and multichannel (colour, texture, etc.) edge detections can serve as input for image segmentation multiscale and space invariant algorithms ("space invariance" means that all points of the image are processed in the same way). The problem is to segment in an optimal way an image $I(x,y)$ defined on a domain W, that is to partition it in maximally homogeneous domains limited by boundaries K. For doing this we need a *functional* $E(W,K)$ which allows the comparison between two segmentations $E(W,K_1)$ and $E(W,K_2)$. E must contain two terms: a term which measures the variance of $I(x, y)$ on the connected components of $W—K$, and a term which controls the length, the smoothness, the parsimony and the location of the boundaries. The "energy" E proposed by Mumford and Shah (1989) is:

$$E(u,K) = \int_{W-K} |\nabla u|^2 dx + \int_W (u-I)^2 dx + \int_K d\sigma$$

Minimizing E is a compromize between:

(i) the homogeneity of the connected components of $W—K$: if u = cst then $\nabla u = 0$ and $\int_{W\text{-}K} |\nabla u|^2 dx=0$;

(ii) the approximation of I by u: if $u = I$ then $\int_W (u–I)^2 dx=0$

(iii) the parsimony and the regularity of the boundaries: they are measured by the global length L of K, $L = \int_K d\sigma$.

Such a variational algorithm optimizes the way in which you can *merge* neighboring pixels in homogeneous domains separated by qualitative discontinuities. It provides therefore a *variational* approach to the Verschmelzung/Sonderung duality.

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the best way the choice of the decomposition basis to the particular structure of the signal. The fit criterium is the minimizing of the information entropy. See e.g. Wickerhauser (1991).

5. The cognitive explanation II: the binding problem, the labeling hypothesis, temporal coding and weakly coupled oscillator networks

5.1. The binding problem and the labeling hypothesis

At the central processing level, the problem of parts and wholes, that is the problem of *constituency*, is also called the *binding problem*. The more promizing solution is now considered to be found in the *fine* temporal coding by means of coherent neural *oscillations*.

The main idea is that the binding of different features of an object (e.g. the segmentation of visual scenes) which are coded in a *distributed way* at the neural level, may be realized using a temporal coding. The *coherence* of features and constituents would be encoded in the *synchronization* (*phase locking*) of oscillatory neuronal responses to stimuli. And therefore *different phases* can code for different constituents. This hypothesis is also called the *labeling* hypothesis.

There is a large amount of experimental evidence concerning synchronized oscillations in the cortical (hyper)columns (in the frequency range of 40-70 Hz) which are sensible to the coherence of the stimulus (see e.g. Andreas Engel, Peter König, Charles Gray and Wolf Singer 1992).

The binding problem is evident. At the early stages of perception the features of the objects are extracted in a local, distributed and parallel manner. How these localized features and modularized constituents (parts) can be re-integrated in spite of their distributed encoding? You must avoid the "superposition catastrophe" that is their mere linear superposition.

For this you use a new parameter: temporal coincidence, synchronization. The binding resulting from functional coupling becomes then *dynamic*, purely transient. It is no more the consequence of a fixed anatomical wiring.

For corroborating this hypothesis

(i) you must have enough experimental evidence at disposal, and

(ii) you must show that the functional coupling of oscillators whose frequency is stimulus-dependent can effectively reflect the coherence of the stimuli patterns.

With such a local mechanism you can explain the global phenomena of fusion and separation:

Fusion ≡ Synchronization / Segmentation ≡ de-synchronization.

"According to this model, coherence in and between feature domains may be encoded by transient synchronization of oscillatory responses and thus permit a binding of *distributed* features of an object" (Engel *et al.* 1992). Via temporal coding, the integration of distributed representations is realized in a purely dynamical way without postulating higher level integrating "grand mother" neurons.

5.2. Networks of oscillators

The problem is now to model the fundamental fact of synchronization. For this we need the theory of networks of weakly coupled oscillators, the frequency of which depends on the intensity of the stimulus.

Even if it is a very complex system, a cortical (hyper)column can be approximated by a single oscillator. It is a sort of *mean field* approximation Let S be a system of formal neurons:

$$\dot{x_i} = -x_i + \sigma(\sum_j w_{ij} x_j + \theta_i)$$

where x_i is the state of activity of the i th neuron u_i, σ a sigmoid (gain function), w_{ij} the synaptic weights and θ_i the thesholds of the u_i. Averaging on the excitatory neurons and on the inhibitory ones, one gets a system of two equations for the mean activities X_E and X_I (Wilson-Cowan eq.). Under the retinal stimulus, the equilibrium state of this system bifurcates spontaneously, via a Hopf bifurcation, towards a cyclic attractor (attracting limit cycle). Moreover the frequency of this limit cycle depends on the intensity of the stimulus. One observes then a synchronization above the *homogeneous* parts of the stimulus.

The mathematical explanation of this phenomenon is difficult. Let us start with a network of N oscillators F_k (k=1,…,N) of frequency ω_k (period $T_k = 2\pi/\omega_k$). If θ_k are their phases and if φ_k are the differences of phases $\varphi_k = \theta_{k+1} - \theta_k$, the differential equations of the network are of the form:

$$\dot{\theta_k} = \omega_k - H(\varphi_1, \ldots, \varphi_{N-1}), \text{ e.g.}$$

$$\dot{\theta_k} = \omega_k - \frac{K}{N} \sum_{j=1}^{j=N} sin(\theta_k - \theta_j)$$

with the frequency ω_k depending on the intensity of the stimulus at position k.

A lot of works has been devoted to the analysis of such systems using qualitative dynamics (e.g. George Bard Ermentrout and Nancy Kopell) or statistical physics (e.g. Y. Kuramato). I mention only one simple exemple. Take a line of oscillators with linearly decreasing frequencies. Under the weak coupling hypothesis, one can show that *plateaus* are formed. This means that the oscillators effectively try to synchronize (phase locking), but as the total difference of phases is too large, they can only *partially* synchronize. Homogeneous synchronized zones are formed (plateaus), which are delimited by sharp discontinuities (jumps between plateaus). Now if there are discontinuities in the stimulus, they constitute of course preferential boundaries.

In a nutshell, the theory of weakly coupled oscillators:

(i) shows that such systems *enhance and complete* existing boundaries;

(ii) can generate new *virtual* boundaries (which are not in the inputs);

(iii) confirms the labeling hypothesis.

6. Return to the morphological nucleus

There is therefore a remarkable convergence of several different models of the morphological nucleus: physical, morphodynamical, geometrical-topological, sensorial (wavelet analysis), cortical (networks of oscillators). All these models *confirm and naturalize* the eidetic phenomenological pure description of forms.

	Totality (Wholes)	**Parts**
Phenomenological description	Verschmelzung	Sonderung
Topological-morphological description	Continuity	Discontinuity
Morphodynamical-physical explanation	Stability of internal attractors under spatial control	Bifurcation of internal attractors
Cognitive explanation I: wavelet analysis	Behavior of the amplitude of the wavelet transform of the signal	
Cognitive explanation II: oscillator networks	synchronized oscillations (phase-locking)	de-synchronized oscillations

III. SHEAF MEREOLOGY AND THE INTERNAL LOGIC OF TOPOI.

1. From morphological geometry to formal ontology and logic

I come now to the second part of my paper. It will be rather technical and concern the link of the morphological nucleus, as it is eidetically described by geometry, with logic, mereology and formal ontology. For establishing such a link the key concept will be that of *sheaf*. Why?

We have seen that at many converging levels, a form is essentially a set of discontinuities (a segmentation, an homogeneity breaking, that is a symmetry breaking) of a *covering relation* "quality → extension". We need therefore a deeper analysis of the concept of "Überdeckung". Now the geometrical concept of *sheaf* is exactly the mathematical one which analyzes the concept of covering relation.

Before tackling this point we must bear in mind that this concept is philosophically of utmost importance. I refer for this to the works of Kevin Mulligan, Barry Smith and Peter Simons.

(i) Covering relations yield prototypical examples of *dependent moments*, which are particular, monadic and static.

(ii) This dependence relation is unilateral and non conceptual. The fact that every qualitative moment depends *generically* (type) on some extension makes the dependence relation "quality → extension" *internal* at the generic level (it is an a priori law between essences in Husserl's sense). But the fact that it depends *specifically* (token) on its extension makes the relation *external* (it is a contingent functional dependence in Husserl's sense).

(iii) Individual independent things exist in spatio-temporal domains W. Their relations with their qualities are external.

(iv) But the relations between the qualities themselves, considered as points in manifolds of qualities $Q_1, ..., Q_n$ are internal.

In his paper *Internal Relations* (1992), Kevin Mulligan quotes Wittgenstein (*Remarks on Colours*):

> "A language-game: Report whether a certain body is lighter or darker than another. But now there's a related one: State the relationship between the lightnesses of certain shades of colour. The form of the propositions in both language games is the same "X is lighter than Y". But in the first it is an external relation and the proposition is temporal. In the second it is an internal relation and the proposition is timeless".

I will try to explain how this deep remark can be mathematically fully justified.

The main problem is to unify a *logical* axiomatics of dependence relations with the *geometric* eidetics of covering relations. In general, geometry is sacrificed to logic (it is an aspect of the reluctance towards the transcendental concept of synthetic a priori). Here we will do justice to geometry and give a faithful sheaf model of Husserl's synthetic a priori law of covering. Then we will use one of the main discoveries of the last twenty years logics, namely that every category of sheaves (in the mathematical sense of "category") yields a logic (which is called its "internal" logic). Such a logic explains why Wittgenstein is right, that is why it is the *same* linguistic expressions which are used for external relations of token dependence and internal relations of type dependence.

2. Fiber bundles and sections

There is a fundamental geometric structure which fits perfectly well with Husserl's eidetic pure description of the spreading [Ausbreitung] of a quality in an extension or, equivalently, of the covering [Überdeckung, Deckungszusammenhang] of an extension by a quality. It is the key geometrical concept of fiber bundle or *fibration.*

Let the spatial substrate (Ausdehnung) of the form be modeled by a differentiable manifold *W*. Let *Q* be the qualitative genus under consideration (e.g. the space of colors).

Q can be modeled by a manifold endowed with a categorization, that is with a decomposition in domains (categories) centered around central values (prototypes).

We have seen that a spreading-covering relation between W and Q can be naively defined as a map $q : W \rightarrow Q$ which, given any point $x \in W$, associates the value $q(x) \in Q$ of the quality at this point. This models Husserl's functional dependence. Verschmelzung is then expressed by the differentiability of q and Sonderung by the discontinuities of q. These discontinuities constitute a closed subset K of W which express geometrically the salient morphology profiled in W.

But this naive model is too naive. Indeed, we need to have *all* the space Q at hand at *every* point $x \in W$. This requisite is imposed by Husserl's pure description. But it is also a perceptive fact. It has been shown by many neurological experiments that the covering of extensions by qualities such as colors or by local geometrical elements such as directions are neurally implemented by (hyper)columns, that is by retinotopic structures where, "over" each retinian position, there exists a "column" implementing the same set of possibilities.

This leads to the fundamental and pervasive concept of *fibration* introduced by Whitney, Hopf and Stiefel and which concerns, in modern geometry and mathematical physics, all the situations where fields of non spatio-temporal entities functionally depend on space-time positions.

Mathematically, a fibration is a differentiable manifold E endowed with a *canonical projection* (a differentiable map) $\pi : E \rightarrow M$ over another manifold M. M is called the *base* of the fibration, and E its *total space*. The inverse images $E_x = \pi^{-1}(x)$ of the points $x \in W$ by π are called the *fibers* of the fibration. They are the subspaces of E which are projected to points.

In general a fibration is required to be *locally trivial*, that is to satisfy the two following axioms:

(F_1) All the fibers E_x are diffeomorphic with a typical fiber F.

(F_2) $\forall x \in M$, $\exists U$ a neighborhood of x such that the inverse image $E_U = \pi^{-1}(U)$ of U is diffeomorphic with the direct product $U \times F$ endowed with the canonical projection $U \times F \rightarrow U$, $(x,q) \rightarrow x$. (See figures 1, 2).

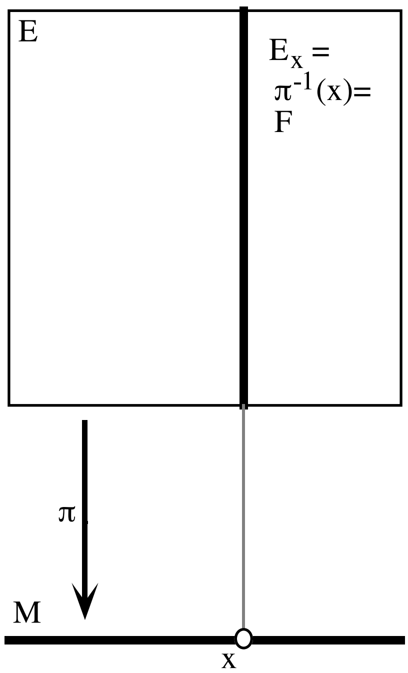

Figure 1

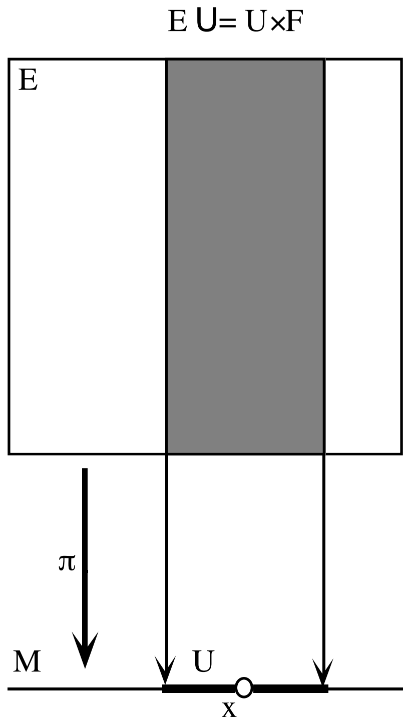

Figure 2

In our case, we have *M* = *W* and *F* = *Q*.

How can we interpret the concept of functional dependence in this new context? It corresponds to the key concept of a *section* of a fibration. Let $\pi : E \rightarrow M$ be a fibration and let $U \subset M$ be an open subset of M. A section s of π over U is a *lift* of U to E which is compatible with π. More precisely, it is a map $s : U \rightarrow E$, $x \in U \rightarrow s(x) \in E_x$, i.e. such that $\pi \circ s = \mathrm{Id}_U$. In general s is supposed to be continuous, differentiable, analytic. It can presents discontinuities along a singular locus. (See figures 3, 4).

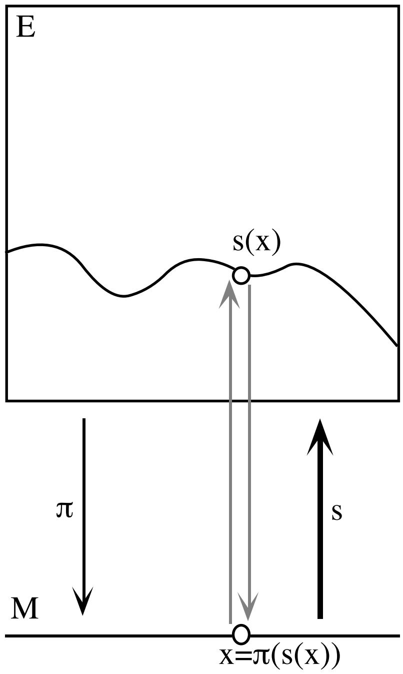

Figure 3

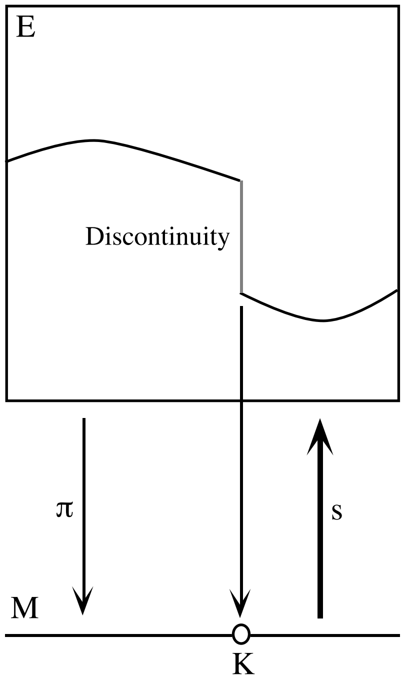

Figure 4

It is conventional to write $\Gamma(U)$ for the set of sections of π over U. If there exists a local trivialization of π over U (i.e. $E_U \to U \approx U \times F \to U$), it transforms every section $s : U \to E$ in a map $x \to s(x)=(x,f(x))$, that is in a map $f : U \to F$. Therefore, the concept of section generalizes the classical concept of map, that is of functional dependence.

We can now establish the link with Husserl's description.

(1) The functional dependences determined at the level of minimal specific differences [die niedersten spezifischen Differenzen] correspond exactly to *particular sections* $\sigma : W \to E$ of *particular* fibrations $\pi : E \to W$ with fiber Q. These sections model external relations of token dependence.

(2) The qualitative salient discontinuities are discontinuities of sections $\sigma \in \Gamma(U)$.

(3) The eidetic law "concretely determined by its material contents" corresponds to a particular fibration $\pi : E \to W$ of fiber Q *but without any particular given section*. Such a fibration models an *abstract* relation between the genus W and Q (first level of abstraction). It implicitely contains an infinite universe of potential functional dependences, namely, all the sets of sections $\Gamma(U)$ for $U \subset W$. It models internal relations of type dependence.

(4) The synthetic a priori law of dependence "quality $\to$ extension" corresponds to *the general mathematical structure* of fibration. It concerns the most abstract genus — the essences — of space and quality (second level of abstraction).

(5) Last but not least, the "analytic axiomatization" of this synthetic law in the framework of formal ontology corresponds to the *axiomatics* of fibrations.

This interpretation of Husserl's description in terms of fibrations uses only globally trivial fibrations and is therefore equivalent to a more classical one, using only functional dependences. But, as we will see, the concept of section will allow us to link the problem of covering relations (Überdeckung) with a very deep synthesis between geometry and logic.

3. Glueing and cohomology

The axiomatics of the concept of fibration essentially rests on the concept of glueing — of fusion, of merging, of collating — of sections.[4]

One constructs global sections by glueing local ones in the following way. Let $\pi : E \to M$ be a fibration of fiber F and let $\mathcal{U} = (U_i)_{i \in I}$ be an open covering of U which locally trivializes π. This means that there are diffeomorphisms $\varphi_i : U_i \times F \to \pi^{-1}(U_i) = E_{U_i}$ over the U_i which induce fiber diffeomorphisms $\varphi_{i,x} : F \to E_x = \pi^{-1}(x)$.

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[4] For an introduction to local/global, fibration, sheaf, toposes theories, see Petitot 1979, 1982, and, for more details, Mac Lane, Moerdijk 1992.

The condition of glueing is that $\forall i,j\in I$ such that $U_i\cap U_j \neq \emptyset$, then $\forall x\in U_i\cap U_j$, the automorphism of F $\theta_{i,j}(x) = (\varphi_{i,x})^{-1}\circ\varphi_{j,x} : F\rightarrow F$ belongs to a certain Lie group G of diffeomorphisms of F. This group is called the *structural group* of the fibration. For instance, in the case of a linear fiber bundle the fiber F of which is a vector space, the structural group G will be the linear group $GL(F)$. It is trivial to verify that if $U_i\cap U_j\cap U_k \neq \emptyset$, then $\theta_{i,j}\circ\theta_{j,k}\circ\theta_{k,i} = 1$

In fact we meet here a *simplicial structure* which lies at the basis of what is called the *cohomology* of fibrations. Let $\mathcal{U} = (U_i)_{i\in I}$ be an open covering of the base M. Let F and G be as above. The *skeleton* K of $\mathcal{U}$ is the simplicial structure over I defined in the following manner.

- the 0-simplexes are the indices $i\in I$;
- the 1-simplexes are the pairs $(i,j)\in I\times I$ such that $U_i\cap U_j \neq \emptyset$;
- the 2-simplexes are the triples $(i,j,k)\in I\times I\times I$ such that $U_i\cap U_j\cap U_k \neq \emptyset$; etc.

For any open set U of M, let $\Gamma(U) = \{\theta : U\rightarrow G,\ \theta$ differentiable map$\}$. If s is a p-simplex of K, a *p-cochain* is an element $\sigma\in\Gamma_s = \Gamma(\cap_{i\in s}U_i)$. The p-cochains form a group C^p. It is then easy to define a *coboundary* operator and therefore a cohomology theory. For instance, let $\varphi = (\varphi_i)$ be a 0-cochain. Its boundary is the 1-cochain $\partial\varphi = (\theta_{ij}=\varphi_i^{-1}.\varphi_j)$. If $\theta = (\theta_{ij})$ is a 1-cochain, its coboundary is the 2-cochain $\partial\theta = (\psi_{ijk}=\theta_{ij}.\theta_{jk}.\theta_{ki})$, etc.[5]

It is trivial to verify that $\partial^2 = 1$. One can therefore consider the group of *p-cocycles* Z^p, that is of p-cochains σ without boundary ($\partial\sigma = 1$) and of *p-coboundaries* B^p, that is of p-cochains of the form $\sigma = \partial\tau$ with $\tau\in C^{p-1}$. As $\partial^2 = 1$, $B^p \subset Z^p$ and one can therefore consider the quotient groups $H^p=Z^p/B^p$. They are called the *cohomology groups* of the fibration. One can then prove the following theorem.

<u>Theorem</u>. A fibration is characterized by a 1-cocycle $\theta = (\theta_{ij})$ ($\partial\theta = 1$ is the glueing condition). It is *globally* trivial iff θ is a 1-coboundary ($\theta_{ij}=\varphi_i^{-1}.\varphi_j$).

4. Sections and sheaves

At an abstract level, a fibration is characterized by the sets of its sections $\Gamma(U)$ over the open sets $U\subset M$. If $s\in\Gamma(U)$ is a section over U and if $V\subset U$, we can consider the *restriction* $s|_V$ of s to V. The restriction is a map $\Gamma(U)\rightarrow\Gamma(V)$. It is clear that if $V=U$, then $s|_V=s$ and that if $W\subset V\subset U$ and $s\in\Gamma(U)$, then $(s|_V)|_W = s|_W$ (transitivity of the restriction). We get therefore what is called a *contravariant functor* $\Gamma : \mathbf{O}^*(M)\rightarrow$**Sets** from the

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[5] In these formula, the products and inverses are those defined by G.

category $\mathfrak{O}(M)$ of the open sets of M in the category **Sets** of sets. (The objects of $\mathfrak{O}(M)$ are the open sets of M, and its morphisms are the inclusions of open sets.) [6]

Conversely, let Γ be such a functor — what is called a *presheaf* on M. To have a chance of being the functor of the sections of a fibration, Γ must clearly satisfy the two following axioms.

(S_1) Two sections which are locally equal must be globally equal. Let $\mathcal{U} = (U_i)_{i\in I}$ be an open covering of M. Let $s,s'\in\Gamma(M)$. If $s|_{U_i} = s'|_{U_i}$ $\forall i\in I$, then $s=s'$.

(S_2) Compatible local sections can be collated in a global one. Let $s_i\in\Gamma(U_i)$ be a family over $\mathcal{U} = (U_i)_{i\in I}$. If the s_i are compatible, that is if $s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j}$ when $U_i\cap U_j \neq \emptyset$, then they can be glued together : $\exists s\in\Gamma(M)$ such that $s|_{U_i} = s_i$ $\forall i\in I$.

(S_1) and (S_2) can be expressed in a purely categorical manner. For instance (S_2) says that the arrow

$$e \,:\, s \rightarrow \{s\,|_{U_i}\}_{i\in I}$$

is the *equalizer*

$$\Gamma(U) \xrightarrow{e} \prod_i \Gamma(U_i) \overset{p}{\underset{q}{\rightrightarrows}} \prod_{i,j} \Gamma(U_i \cap U_j)$$

of the two projections p, q corresponding to the inclusions $U_i\cap U_j\subset U_i$ and $U_i\cap U_j\subset U_j$.

In fact these axioms characterize a more general structure — and even more pervasive in contemporary mathematics — than the structure of fibration, namely the structure of *sheaf*. It can be shown that if the axioms (S_1) and (S_2) are satisfied, then one can represent the functor Γ by a general fibered structure $\pi : E\rightarrow M$ (called un “étale” space and which is not necessarily a locally trivial fibration) in such a way that $\Gamma(U)$ becomes the set of sections of π over U. In a nutshell, the fiber E_x — called in that case the *stalk* of the sheaf Γ at x — is the inductive limit (the colimit) :

$$E_x = \lim_{V\subset U\in \mathcal{U}_x}\{(\Gamma(U), \Gamma(V\subset U)\}$$

(where $\mathcal{U}_x$ is the filter of the open neighborhoods of x). The stalk E_x is the set of *germs* s_x of sections at x. E is the sum of the E_x. If $s\in\Gamma(U)$, it can be interpreted as the map $x\in U\rightarrow s(x)\in E_x$. The topology of E is then defined as the finest one making all these sections continuous.

The concept of sheaf expresses essentially *glueing conditions*, that is the way by which local data can be collated in global ones. It is the right mathematical tool for formalizing the covering relations between spatial locations and non spatial determinations. We will see now *that this geometric eidetics of local/global covering relations is, in an essential manner, linked to logic*.

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[6] We suppose that the reader is acquainted with category theory. For an introduction see e.g. Perruzzi [] and Petitot [].

5. The topos Sh(M)

In the following sections if A is a sheaf on M, $A(U)$ will denote the set $\Gamma_A(U)$ of sections of A over U.

It is easy to show that the sheaves on a base space M constitute a category **Sh**(M). Now the main point is

(i) that this category shares fundamental properties which are characteristic of what is called a *topos* structure and,

(ii) that *the topos structure is exactly what is needed for doing logic.*

5.1. Exponentials

Sh(M) is a *cartesian closed* category. This means that it has products and fiber products or pullbacks, a terminal object — classically denoted by 1 — and *exponentials* B^A. An exponential object is an object which "internalizes" in the objects of **Sh**(M) the morphisms $f : A \to B$. Such "internalization" of functorial structures are called *representable* functors. Technically, the functor $(\bullet)^A$ is the *right adjoint* of the functor $A \times (\bullet)$. This means that we have for every object C of **Sh**(M) a functorial isomorphism $\mathrm{Hom}(C,B^A) \cong \mathrm{Hom}(A\times C,B)$. E.g. for C=1, we get $\mathrm{Hom}(1,B^A) \cong \mathrm{Hom}(A,B)$. But an arrow $f : 1 \to B^A$ is like an "element" of B^A. In fact, if A is a sheaf, an arrow $s : 1 \to A$ is a *global section* of A, that is an element $s \in A(M)$.

If we take $C=B^A$ and Id_{B^A}, the right adjunction defines what is called a *counit* $\varepsilon : A\times B^A \to B$ such that for every $f : C \to B^A$ the associated $h : A\times C \to B$ is given by $h = \varepsilon\circ(1\times f)$.The counit generalizes the map $(x,f) \to f(x)$ in set theory and is therefore called the *evaluation map*.

The sheaf B^A is defined using the evident restrictions $A|_U$ and $B|_U$ to open sets: $B^A(U) = \mathrm{Hom}(A|_U, B|_U)$. It is called the "internal Hom" or the sheaf of germs of morphisms from A to B.

5.2. Subobject classifier

Sh(M) possesses also what is called a *subobject classifier* Ω, that is an object which "internalizes" the sets of subobjects, making the subobject functor representable. A subobject $m : S \hookrightarrow A$ is a monomorphism (an injective map in the case of the category of sets). This means that if f and g are two morphisms from an object R to S, then $m\circ f=m\circ g$ implies $f=g$. It is equivalent to say that the fibered product $S\times_A S$ defined by m is isomorphic with S. A subobject classifier is a monomorphism True : $1 \to \Omega$ such that every subobject $m : S \hookrightarrow A$ can be retrieved from True by a pull-back :

$$\begin{array}{ccc} S & \rightarrow & 1 \\ \downarrow & & \downarrow \text{True} \\ A & \underset{\varphi_S}{\rightarrow} & \Omega \end{array}$$

We get therefore a functorial isomorphism Sub(A) ≅ Hom(A,Ω). φ_S is called the *characteristic map* of the subobject S.

In the category **Sets** of sets, Ω = {0,1} is the classical set of boolean truth-values. Here — and it is perhaps the main difference between a topos like **Sh**(M) and the classical topos **Sets** —, Ω(U) *depends essentially on the topological structure. It expresses the localization of truth in a sheaf topos.*

By definition $\Omega(U) := \{W \subset U\}$. It is trivial to verify that Ω is a sheaf. The True map True : 1 → Ω is defined by True(U) : 1 → $U \in \Omega(U)$ that is by the *maximal* element of Ω(U): to be true over U is to be true "everywhere" over U. The global section $T \in \Omega(M)$ selected by True is nothing else than the whole base space M itself and True(U) is its localization to U. If S is a subsheaf of the sheaf A, its characteristic map $\varphi_S : A \to \Omega$ is given by the maps $\varphi_S(U) : A(U) \to \Omega(U)$ which map $s \in A(U)$ to the largest $W \subset U$ s.t. $s|_W \in S$. It is easy to verify that the monic map $S \hookrightarrow A$ is effectively the pull-back of True by φ_S. As Mac Lane says: φ_S gives "the shortest path to truth" $W \subset U$ for $S \hookrightarrow A$.

5.3. *Elements, properties and parts*

In a topos, the morphisms $a : B \to A$ are called *generalized elements* of A, or elements *defined on B* (this denomination comes from algebraic geometric and, more precisely, from Grothendieck's theory of schemes). Among the elements, the most important are those *defined on open sets U*, that is precisely the sections $s \in A(U)$. If $U \in \mathcal{O}(M)$, we can consider the Yoneda sheaf $\mathbf{y}(U) = \{V \subset U\}$ and verify that $A(U) \cong$ Hom($\mathbf{y}(U)$,A). The elements defined on the terminal object 1 are "global". We will see that an arrow $\theta : A \to \Omega$ is a "predicate" for A, that is a "property" of its generalized elements. Among all predicates, there is the predicate $\text{True}_A : A \to 1 \to \Omega$. It is easy to verify that an element $a : B \to A$ factorizes through a subobject $S \hookrightarrow A$ iff char$S(a)$ = $\varphi_S \circ a = \text{True}_B . \varphi_S$ is therefore the predicate of A which is true exactly for those elements of A which are in S. The *unicity* of φ_S expresses the *extensionality principle*.

Using the exponentials and the subobject classifier we can define the *parts* of an object A as another object $P(A) = \Omega^A$. We get the functorial isomorphisms:

$$\text{Sub}(A) \cong \text{Hom}(A,\Omega) \cong \text{Hom}(A\times 1,\Omega) \cong \text{Hom}(1,\Omega^A) = \text{Hom}(1,P(A)).$$

We have therefore $\Omega = P(1)$.

This shows that there are 3 equivalent descriptions of a subobject $m : S \hookrightarrow A$.

(i) its "extension" S: we will see that it can be symbolized as in **Sets** by $\{a | \varphi_S(a)\}$;

(ii) its characteristic map $\varphi_S : A \to \Omega$ which is a "predicate" of A;

(iii) the global section $s : 1 \to P(A)$ which is its "*name*".

The evaluation map $\varepsilon_A : A \times \Omega^A = A \times P(A) \to \Omega$ is a "membership" predicate: if $a : B \to A$ is an element of A, and if $s : 1 \to P(A)$ is (the name of) a subobject of A, then $\varepsilon_A \circ (a \times s) = \text{True}_{B\times 1}$ iff $\varphi_S \circ A = \text{True}_B$, that is iff a is an element of S.

5.4. Towards logic

The existence of an intuitionistic "internal logic" in a topos depends essentially on the fact that, being the set of the open sets W of the topological space U, $\Omega(U)$ is (fonctorially) an *Heyting algebra*. Ω is therefore a sheaf of Heyting algebras (an Heyting algebra object in $\mathbf{Sh}(M)$). The consequence is that the "external" set of subobjects Sub(A) and the "internal" one $P(A)$ are also Heyting algebras, the canonical isomorphism Sub(A) ≅ Hom(1, $P(A)$) being an isomorphism of Heyting algebras.

6. Topoi and logic

Now, the central fact is that a topos is exactly the categorical structure which is needed for doing logic. *But this logic is spatially localized.*

For details see e.g. Saunders Mac Lane, Ieke Moerdijk 1992.

6.1. Types and localization

We can associate with each topos $\mathbf{Sh}(M)$ a *formal language* $\mathcal{L}_M$ called its *Mitchell-Bénabou* language, and a *forcing semantics* called its *Kripke-Joyal* semantics. The crucial point is that a sheaf X can be considered as a *type* for variables x which are interpreted as *sections* $s \in X(U)$ of X. *We get therefore at the same time a typification and a localization of the variables*. This achievement fits perfectly well with Husserl's description and explain Wittgenstein's remark. It provides them with a correct mathematical status.

(i) Sections are tokens denoted by variables belonging to types (species, essences). They are "concretely" particularized by the specification of their localization U and by their specific values. But as an element of type X, s particularizes an abstract unilateral relation of dependence, the relation "quality→extension" which is constitutive of X.

(ii) The relations between particular sections $s \in X(U)$, $t \in Y(V)$ are *external*. The relations between X and Y are *internal*. Nevertheless the linguistic expressions which express them are formulas in the formal language $\mathcal{L}_M$ associated to $\mathbf{Sh}(M)$, and are *the same*.

6.2. Syntax

How are the terms and the formulas of $\mathcal{L}_M$ syntactically constructed? Here is a summary of their inductive construction.

A term σ of type X constructed using variables y, z of respective types Y, Z has a *source* $Y \times Z$ and is interpreted by a morphism $\sigma : Y \times Z \rightarrow X$ which expresses its structure.

(i) To each $X \in \mathbf{Sh}(M)$ considered as a type are associated variables x, x', … They are interpreted by the identity map $1_X : X \rightarrow X$.

(ii) Terms $\sigma : U \rightarrow X$, $\tau : V \rightarrow Y$ of respective types X and Y yield a term $<\sigma,\tau>$ interpreted by $<\sigma,\tau> : W = U \times V \rightarrow X \times Y$ of type $X \times Y$.

(iii) Terms $\sigma : U \to X$, $\tau : V \to X$ of the same type X yield the term $\sigma = \tau$ of type Ω interpreted by

$$(\sigma = \tau) \ : \ W = U \times V \to X \times X \overset{\delta_X}{\to} \Omega$$

where δ_X is the characteristic function of the diagonal subobject $\Delta : X \to X \times X$.

(iv) A term $\sigma : U \to X$ of type X and a morphism $f : X \to Y$ yield by composition a term $f \circ \sigma$ of type Y.

(v) Terms $\theta : V \to Y^X$ and $\sigma : U \to X$ of respective types Y^X and X yield a term $\theta(\sigma)$ of type Y interpreted by

$$\theta(\sigma) \ : \ W = V \times U \to Y^X \times X \overset{e}{\to} Y$$

where e is the evaluation map.

(vi) In particular, terms $\sigma : U \to X$ and $\tau : V \to \Omega^X$ yield a term $\sigma \in \tau$ of type Ω interpreted by

$$\sigma \in \tau \ : \ W = V \times U \to X \times \Omega^X \overset{e}{\to} \Omega \ .$$

(vii) A variable x of type X and a term $\sigma : X \times U \to Z$ of type Z and of source $X \times U$ yield a term of type Z^X interpreted by $\lambda x \sigma : U \to Z^X$.

(viii) Ω is the type of the *formulas* of $\mathcal{L}_M$. As it is an Heyting algebra, we get the logical operations of propositional calculus : $\varphi \wedge \psi$, $\varphi \vee \psi$, $\varphi \Rightarrow \psi$, $\neg \varphi$. It is easy to verify that if $\varphi(x,y) : X \times Y \to \Omega$ is a formula, we can write the subobject of $X \times Y$ classified by its interpretation in the "set theoretic" manner : $\{(x,y) \in X \times Y | \varphi(x,y)\}$.

(ix) One of the most remarkable facts of topoi theory is that it is possible to define *quantification* in a purely categorical manner. Let $f : A \to B$ be a morphism of $\mathbf{Sh}(M)$ and consider the "inverse image" functor $f^* : \mathrm{Sub}(B) \to \mathrm{Sub}(A)$ defined by composition with f. Its internal version is the morphism $P(f) : P(B) \to P(A)$. The fact is that $P(f)$ has *two adjoint functors* : a left adjoint one $\exists_f : P(A) \to P(B)$ and a right adjoint one $\forall_f : P(A) \to P(B)$. They generalize the two adjunctions in **Sets**. If $f : A \to B$, $S \subset A$ and $T \subset B$,

$$\exists_f(S) \ = \ \{b \in B \,|\, \exists a \in S \ (f(a) = b)\} \ = \ f(S),$$

$$\forall_f(S) \ = \ \{b \in B \,|\, \forall a \in A \ (f(a) = b \ \Rightarrow \ a \in S)\}$$

$$\left\{ b \in B \,|\, f^{-1}(b) \subset S \right\},$$

and $\begin{cases} f^*(T) \subset S \ \Leftrightarrow \ T \subset \forall_f(S) \\ S \subset f^*(T) \ \Leftrightarrow \ \exists_f(S) \subset T \end{cases}$.

We have:

$$S \in \exists_f(S)(U) \text{ iff } \{V \,|\, \exists t \in S(V) \, f(t) = s_{|V}\} \text{ covers } U,$$

$$S \in \forall_f(S)(U) \text{ iff } \forall V \subseteq U \, f^{-1}(s_{|V}) \subseteq S(V).$$

Now, let $\varphi(x,y) : X \times Y \to \Omega$ be a formula of two variables in $\mathbf{Sh}(M)$. Let $p : X \to 1$ the canonical projection and $P(p) : P(1) \to P(X)$. We get the adjunctions:

$$\Omega^X = P(X) \underset{\overset{\longrightarrow}{\exists_p}}{\overset{\overset{\forall_p}{\longrightarrow}}{\overset{P(p)}{\longleftarrow}}} P(1) = \Omega .$$

We consider $\lambda x\varphi(x,y) : Y \to \Omega^X = P(X)$ and we get the formula of source Y:

$$\forall x\varphi(x,y) = \forall_p \circ \lambda x\varphi(x,y) : Y \to \Omega .$$

These categorical constructs show that the formal language $\mathcal{L}_M$ is, at the "linguistic" level, exactly of the same nature as the classical formal language of sets theory. The main difference is that we have introduced a subtle dialectics *between the type of the variables and their localisation.*

6.3. Semantics

The Kripke-Joyal semantics of topoi is a *forcing* semantics generalizing Cohen's one. A variable x of type X denotes a *section* $s \in X(U)$, that is a morphism $s : U \to X$ where U is now the sheaf defined by U. The semantic rules are rules $U \Vdash \varphi(s)$ (U forces $\varphi(s)$). Let $s : U \to X$ and $\mathrm{Im}(s) \in \mathrm{Sub}(X)$ be the image of s. One defines

$$U \Vdash \varphi(s) := \mathrm{Im}(s) \subseteq \{x | \varphi(x)\},$$

that is $U \Vdash \varphi(s)$ iff $U \overset{s}{\to} X \overset{\varphi}{\to} \Omega$ factorizes through $\{x / \varphi(x)\}$:

$$U \Rightarrow \{x \mid \varphi(x)\} \to 1 \overset{\mathrm{True}}{\to} \Omega .$$

The semantic rules are:

(i) $U \Vdash \varphi(s) \wedge \psi(s)$ iff $U \Vdash \varphi(s)$ and $U \Vdash \psi(s)$.

(ii) $U \Vdash \varphi(s) \vee \psi(s)$ iff there exists an open covering $(U_i)_{i \in I}$ of U s.t. for every i, $U_i \Vdash \varphi(s|_{U_i})$ or $U_i \Vdash \psi(s|_{U_i})$ (intuitionistic rule for disjonction)

(iii) $U \Vdash \varphi(s) \Rightarrow \psi(s)$ iff, for all $V \subseteq U$, $V \Vdash \varphi(s|_V)$ implies $V \Vdash \psi(s|_V)$.

(iv) $U \Vdash \neg\varphi(s)$ iff there does not exist $V \subseteq U$, $V \neq \emptyset$ s.t. $V \Vdash \varphi(s|_V)$ (the negation is intuitionistic because Ω is a Heyting algebra and not a Boolean one).

(v) $U \Vdash \exists y\varphi(s,y)$ (y being of type Y) iff there exist an open covering $(U_i)_{i \in I}$ of U and sections $\beta_i \in Y(U_i)$ s.t. for every $i \in I$ $U_i \Vdash \varphi(s|_{U_i}, \beta_i)$.

(vi) $U \Vdash \forall y\varphi(s,y)$ iff for every $V \subseteq U$ and $\beta \in Y(V)$ we have $V \Vdash \varphi(s|_V, \beta)$.

7. Sheaf mereology

The concept of section is a key mathematical concept which permits to build up models for a large class of *dependent* parts. It provides therefore a good model (in the sense of Model theory) for axiomatic mereology,[7] what we shall call *a sheaf model*. This is why, to conclude this reflexion, I want to stress the relation between the axiomatics of fibrations and the piece of formal ontology proposed by Barry Smith (1993).

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[7] For mereology in the framework of formal ontology, see Poli 1992.

7.1. Smith's mereological and topological system.

In "Ontology and the logistic analysis of reality", B. Smith proposed an axiomatic for two primitives :∧

(i) the mereological primitive $x\mathbf{C}y$: "x is a *constituent* of y";

(ii) the topological primitive $x\mathbf{P}y$: "x is an *interior part* of y".

From $x\mathbf{C}y$ some other concepts can be immediately derived :

DC1. "x overlaps y"; $x\mathbf{O}y := \exists z(z\mathbf{C}x \wedge z\mathbf{C}y)$;

DC2. "x is discrete from y"; $x\mathbf{D}y := \neg x\mathbf{O}y$.

The first two axioms characterizing the **C** relation are :

AC1. $x\mathbf{C}y \Leftrightarrow \forall y(z\mathbf{O}x \Rightarrow z\mathbf{O}y)$;

AC2. $x\mathbf{C}y \wedge y\mathbf{C}x \Rightarrow x=y$.

This mereological system is extensional and **C** is an order relation.

A fondamental axiom is that, for every predicate φ of the x, one can define the *sum* — the fusion, the merging — of all the x which satisfy φ :

DC4. $[x{:}\varphi(x)] := \iota y(\forall w(w\mathbf{O}y \Leftrightarrow \exists v(\varphi(v) \wedge w\mathbf{O}v)))$.

Of course, we need axioms to guarantee that the matrix of DC4 is a definite description to which Russell's operator ι can be applied. AC1 implies unicity. For existence, we need :

AC3. $\exists x\varphi(x) \Rightarrow \exists y(y=[x{:}\varphi(x)])$.

One can then prove easily the following theorems.

TC3. $y=[x{:}\varphi(x)] \Rightarrow \forall x(\varphi(x) \Rightarrow x\mathbf{C}y)$;

TC4. $\exists x\forall y(y\mathbf{C}x)$ (the universe exists);

TC5. $y\mathbf{C}[x{:}\varphi(x)] \Leftrightarrow \forall w(w\mathbf{C}y \Rightarrow \exists v(\varphi(v) \wedge w\mathbf{O}v))$.

One can also define the following fundamental concepts.

$1 := [x{:}x=x]$ (the universe);

$x \cup y := [z{:}z\mathbf{C}y \vee z\mathbf{C}y]$ (union);

$x \cap y := [z{:}z\mathbf{C}y \wedge z\mathbf{C}y]$ (intersection);

$x' := [z{:}z\mathbf{D}x]$ (complement).

In what concerns the topological primitive **P**, the six following axioms are clearly needed.

AP1. $x\mathbf{P}y \Rightarrow x\mathbf{C}y$;

AP2a. $x\mathbf{P}y \wedge y\mathbf{C}z \Rightarrow x\mathbf{P}z$ (left monotonicity);

AP2b. $x\mathbf{C}y \wedge y\mathbf{P}z \Rightarrow x\mathbf{P}z$ (right monotonicity);

AP3. $x\mathbf{P}y \wedge x\mathbf{P}z \Rightarrow x\mathbf{P}(y \cap z)$ (condition on finite intersection);

AP4. $\forall x(\varphi(x) \Rightarrow x\mathbf{P}y) \Rightarrow [x{:}\varphi(x)]\mathbf{P}y$;

AP5. $\exists y(x\mathbf{P}y)$;

AP6. $x\mathbf{P}y \Rightarrow x\mathbf{P}[t{:}t\mathbf{P}y]$.

From these axioms it is possible to derive all the traditional topological concepts. For instance the *interior* of an object x is defined by :

DP6. $\mathbf{int}(x) := [y{:}y\mathbf{P}x]$.

For defining the *closure* of x one can define first the *boundary* relation $x\mathbf{B}y$ in the following manner. x *crosses* y if x overlaps y and its *complement* $1-y := [z{:}z\mathbf{D}y]$. x *straddles* y if when $x\mathbf{P}z$ then z crosses y. Finally $x\mathbf{B}y$ if $z\mathbf{C}y$ implies that z straddles y. One can then defines the closure of x by :

DP4. $\mathbf{cl}(x) := x\cup[y{:}y\mathbf{B}x]$.

7.2. Topology and mereology.

Let us now apply this general mereological axiomatics — which belongs to formal ontology — to the sheaf model modeling Husserl's pure eidetic description of the *Überdeckung* of extension by dependent qualities. The basic elements of our universe are sections of a sheaf defined by a contravariant functor $\Gamma : \mathcal{O}^*(M) \rightarrow \mathcal{S}$. Given such an object $s\in\Gamma(U)$ we must carefully distinguish between :

(i) its domain of definition $\mathrm{Dom}(s)=U\subset M$, which is a detachable part of a geometrical extensive whole (the base manifold M), and

(ii) its values $s(\mathrm{x})$, which belong to an intensive space of qualities (the fiber F).

It must be emphasized again that the key concept of section supports a *local/global* dialectic : restriction from global to local and glueing from local to global. The domains of sections correspond to the purely extensional (topological) part of the axiomatics. Their values correspond on the other hand to the truly mereological part.

As the base space M is a manifold, all the concepts of open set, interior part, boundary, closure, etc. are ipso facto well defined.[8] But the concepts of fibration and sheaf deepen what it is for a section s to be a *constituent* of another section t. There are in fact (at least) *two* meanings of constituency, a weak one and a strong one.

Weak sense.

$s\in\Gamma(U)\ \mathbf{C}\ t\in\Gamma(V) := U\subset V$ (that is the domains are included one in the other).

Strong sense.

$s\in\Gamma(U)\ \mathbf{C}\ t\in\Gamma(V) := (U\subset V)\wedge(t|_U=s)$ (that is the sections agree for the *Überdeckung*).

Of course, it is the strong sense which is the most interesting. With it, *overlaps* become glueing *conditions*. More precisely, the glueing condition $s|_{U\cap V} = t|_{U\cap V}$ is a condition of *maximal* overlap (that is of overlapping on $\mathrm{Dom}(s)\cap\mathrm{Dom}(t)$).

The strong sense of constituency is imposed by the AC2 axiom :

AC2. $x\mathbf{C}y\wedge y\mathbf{C}x \Rightarrow x=y$.

Indeed, if we retain only the weak sense, we must introduce an equivalence relation $x\equiv y := \mathrm{Dom}(x)=\mathrm{Dom}(y)$ and take the modified axiom :

AC2* $x\mathbf{C}y\wedge y\mathbf{C}x \Rightarrow x\equiv y$.

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[8] In some cases one can generalize the concept of section and define it for non open subsets of M. But in general "good" sections must, away of their singular set, share some properties of continuity, differentiability, analiticity, etc. These are all local properties which are well defined only on open sets.

This shows that AC2 is by no means evident for dependent part.

The strong sense of constituency is also imposed by the axioms AP1 and AP2a,b linking the primitives **C** and **P**. Indeed, if $s\in\Gamma(U)$, the unique plausible meaning for $s\mathbf{P}t$, $t\in\Gamma(V)$ is that $U\subset\mathbf{int}(V)$ and $t|_U=s$.[9] But then AP1 and AP2a,b imply immediately $s\mathbf{C}t$ in the strong sense.

Of course, we can also introduce a third meaning of constituency, a *mixed* one. The objects are now *sets S* of sections and we define :

$$S\mathbf{C}T := \forall s\in S\ \exists t\in T,\ s\mathbf{C}t \text{ in the strong sense.}$$

It is relevant to use the mixed sense if for instance we want to get a good definition of the complement $x' := [z{:}z\mathbf{D}x]$ of a section s. Indeed, a section t can be discrete from s for two completely different reasons :

(i) $(U=\mathrm{Dom}(s))\cap(V=\mathrm{Dom}(t)) = \varnothing$;

(ii) $(U=\mathrm{Dom}(s))\cap(V=\mathrm{Dom}(t)) \neq \varnothing$ but $t(x) \neq s(x)\ \forall x\in U\cap V$.

The complement s' of s is therefore a *set* of sections.

We can even introduce a fourth meaning if we take as objects *multivalued* sections. In that case a $s\in\Gamma(U)$ is no more a map $s : U\rightarrow E$ lifting the projection π. It is a map which associates to every point $x\in U$ a *subset* $s(x)$ of the fiber E_x over x. But, even if they can be interesting by themselves, such extensions of the concept of section are somewhat artificial. For a section, the natural meaning of being a constituent is to be a restriction of some larger section.

Now, the main point is that the mereological concept of *sum* (union and fusion), *splits* into two different concepts.

(1) The *union* of any two sections $s\in\Gamma(U)$ and $t\in\Gamma(V)$ can be defined as the section $s\cup t\in\Gamma(U\cup V)$ such that $s\cup t(x) = \{s(x),t(x)\}$ (we put $s(x)$, $t(x) = \varnothing$ if $x\notin U,V$). If $s(x) \neq t(x)$ for $x\in U\cap V$ then $s\cup t$ is a multivalued section.

(2) The *fusion* of two sections $s\in\Gamma(U)$ and $t\in\Gamma(V)$ is more restrictive. It requires the glueing condition $s|_{U\cap V} = t|_{U\cap V}$. The fundamental consequence is that, if $\varphi(s)$ is a predicate of sections, the sum $[s{:}\varphi(s)]$ is no longer a *single* element. It is the set of *maximal* sections satisfying φ.

In some cases, we can also suppose that there exists some algebraic “superposition” structure $u*v$ in the fiber F (think of the superposition of colors). We can then define the union of any two sections $s\in\Gamma(U)$ and $t\in\Gamma(V)$ as the section $r\in\Gamma(U\cup V)$ s.t. for every $x\in U\cup V$, $r(x)=s(x)*t(x)$. But this is no more a true union. It is rather a “sum” in an algebraic sense.

The concept of sum has to be redefined according to the fact that in sheaf theory the concept of union (of gluing) depends on the the concept of *prolongation* of a section. Let φ be a predicate for sections. Let us call φ-section a section satisfying φ. We look at

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[9] Of course **int** means here the topological interior. In general, V will be open and therefore $\mathbf{int}(V)=V$.

the possibility of *extending* a φ-section $s \in \Gamma(U)$ to a larger open set $U \subset V$. We look therefore at sections $t \in \Gamma(V)$ *s.t.* $U \subset V \wedge s\mathbf{C}t \wedge \varphi(t)$. t is a maximal φ-section if :

$$\forall r(\varphi(r) \wedge t\mathbf{C}r \Rightarrow r=t).$$

A maximal φ-section t satisfies the matrix of DC4 :

$$\forall w(w\mathbf{O}t \Leftrightarrow \exists v(\varphi(v) \wedge w\mathbf{O}v)),$$

but this is no more a definite description. Let $[x{:}\varphi x] := \Gamma_\varphi(M)$ be the *set* of maximal φ-sections. The TC3 theorems becomes :

$$\forall x(\varphi(x) \Rightarrow \exists z \in [x{:}\varphi(x)](x\mathbf{C}z)).$$

The universe is no more a single element. It is the set $\Gamma_m(M)$ of maximal sections. We have of course $\Gamma(M) \subset \Gamma_m(M)$: global sections are maximal ones. In general there are maximal sections which are not global but cannot be nevertheless extended.

In what concerns the topological part of Smith's system, the sheaf model permits also to clarify some difficulties. The AP1, AP2a,b and AP3 axioms are trivially satisfied. The AP4 axiom $\forall x(\varphi(x) \Rightarrow x\mathbf{P}y) \Rightarrow [x{:}\varphi(x)]\mathbf{P}y$ is also evident. Indeed, if *all* the φ-sections x are interior parts of *one* single section y, then the set $[x{:}\varphi(x)]$ of maximal φ-sections is reduced to one element x_m and $x_m\mathbf{P}y$. But, on the other hand, the "very strong" AP5 axiom $\exists y(x\mathbf{P}y)$ will not be satisfied in general, not because M cannot be an interior part of itself (it is an axiom for any topology that the global space is always clopen) but because there exist in general many global sections $t \in \Gamma(M)$.

The definition DP6 of the interior of an an object, $\mathbf{int}(x) := [y{:}y\mathbf{P}x]$, does not raise any problem. Let $t \in \Gamma(V)$. For the s s.t. $s\mathbf{P}t$ the sum $[s{:}s\mathbf{P}t]$ is a singleton $\{\mathbf{int}(t)\}$ with $\mathbf{int}(t) := t|_{\mathbf{int}(V)}$.[10] But the situation is not as staightforward for the definition DP4 of the closure of an object, $\mathbf{cl}(x) := x \cup [y{:}y\mathbf{B}x]$.

In fact, we meet here a very delicate point. As we have already stressed, all the classical topological concepts (interior, closure, boundary, etc.) are at hand in the sheaf model because the base space M is a manifold. But, as we have seen, the topological basis of a sheaf of sections constitute only the half of the structure. The other half is constituded by the values of the sections. And there is a worrisome problem concerning the extension of topological concepts to this last level. Indeed, if we put constraints of continuity, differentiability, analyticity, etc. on sections, then it is a well known fact that the problem of extending sections to the boundary of their domain is an extremely difficult one, and that, in general, it is even whithout solution.

Let us evoque briefly only one exemple concerning the theory of holomorphic dynamical systems, and in particular the (filled connected) Julia sets and the Mandelbrot set which have become so popular as typical "beautiful" fractals.[11] They give examples of what can be an infinitely complex compact, connected, closed set in $\mathbb{C}=\mathbb{R}^2$, the

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[10] In general the domain V of t will be open, and therefore $\mathbf{int}(t)=t$.

[11] For a breaf introduction to these technical topics, see e.g. Petitot 1992a.

complement of which is connected and simply connected in the Riemann sphere $\mathbb{C}\cup\{\infty\}$, and the interior of which is constituted by infinitely many open discs of different scales. Let K be such a closed set. A well known theorem of Riemann says that there exits a conformal map ψ from $\mathbb{C}-K$ to $\mathbb{C}-\Delta$ (where Δ is the closed unit disc). On the other hand, a deep theorem due to Carathéodory says that the inverse ψ^{-1} of ψ can be prolongated *continuously* — but *not* holomorphically in general — to the boundary $\partial\Delta=S^1$ of Δ. But in general ψ cannot be prolongated, even continuously, to the boundary ∂K of K. In short, it is in general difficult, even impossible, to extend maps defined on open sets to the boundary of their domain.

The definition DP4, $\mathbf{cl}(x) := x\cup[y{:}y\mathbf{B}x]$, is therefore problematic for sections in the sheaf model. In fact, it is meaningful only if the sheaf model is endowed with more restrictive structures (differentiable, holomorphic, etc.) than the simple topological one.

We can therefore conclude that the mereology of sections — in the sheaf model which axiomatizes Husserl's pure eidetic description — shows that some mereological axioms are "evident" only for purely *extensional* mereology, and are by no means "evident" for more sophisticated sort of (non extensional) mereological axiomatics.

7.3. Non well-pointed topoï.

Another non standard feature of sheaf mereology concerns the concept of *points*. In the classical topos **Sets** the points are the arrows $1\rightarrow A$, and **Sets** is a *well-pointed* topos in the sense that it has "enough" points : if $f,g : A\rightarrow B$ are two different maps then there exists a point $x\in A$ s.t. $f(x) \neq g(x)$. It can be proved that *a well-pointed topos is boolean* (Ω is a Boolean algebra) and two-valued (1 and $0 = \varnothing$ are the only subobjects of 1 i.e. there exist only two global truth-values).Therefore, a topos **Sh(M)** will *not* be well-pointed in general: there will not be "enough" points, i.e. *global sections*, for differentiating different arrows.

CONCLUSION: THE SYNTHETIC/ANALYTIC DISTINCTION

The formalization of Husserl's phenomenological description in terms of topoi theory allows to notably clarify the celebrated transcendental distinction between analytic and synthetic a priori. The formal language $\mathcal{L}_M$ corresponds to the logical analytic component of the formalization. But this is not the end of the story. The other component concerns *localization of truth*. The fact that truth-values are indexed on open sets and the forcing status of Kripke-Joyal semantics show that "space" is irreducible to logical analyticity and constitutes a *sui generis* geometrical dimension of the a priori stance. This geometrical dimension of truth corresponds exactly to what Husserl calls, after Kant, *synthetic a priori*. In that sense the concept of "synthetic a priori" is a perfectly sane one. That it has been dramatically misunderstood by logical positivism must not hide the fact that it is basic for ontology.

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