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An introduction to the Mumford–Shah segmentation model

Jean Petitot *

Mathematical Center of the École des Hautes Etudes en Sciences Sociales and CREA, École Polytechnique, 1, Rue Descartes, Paris 75005, France

Abstract

Some papers of this special issue concern recent results on mathematical models of segmentation. As they are rather technical we propose here a pedagogical introduction for the non-mathematical reader. We briefly present the variational model of image segmentation proposed by David Mumford and we summarize some fundamental results of De Giorgi's school.

Keywords: Segmentation; Mumford–Shah model; Variational models; Synchronization

1. The variational Mumford–Shah model

One of the main problem of natural and computational vision is to understand how signals can be transformed into geometrically well behaved observables. Let $I(x)$ (where x is in fact a bi-variable (x, y)) be a rough image defined on a domain W. It is an unstructured signal or sense data without any "good" geometrical structure. The question is: how can it be transformed into a well morphologically organized perceptual image? What is the "geometrical engine" providing its morphological structure? One of the key feature is the segmentation process partitioning W into domains W_i

1. on which the signal I is homogeneous and
2. which are delimited by a system of crisp and regular boundaries (qualitative discontinuities) K.

More than one thousand segmentation algorithms have been worked out by mathematicians and engineers, which merge local data into homogeneous regular regions bounded by regular crisp edges. The main problem is that the 2D regions and the 1D edges which compete are geometrical entities of *different* dimensions and interact in a very subtle way. Underlying these proliferating models there exists a deep unity. As was emphasized by Jean-Michel Morel (Morel–Solimini [11]):

> most segmentation algorithms try to minimize [...] one and the same segmentation energy.

The idea of introducing an energy is justified by the necessity of comparing one segmentation with another and of measuring how well it approximates the rough signal I. The one referred to by J-M. Morel is the so-called Mumford–Shah model (David Mumford is a Fields Medal in Algebraic Geometry who became a specialist of vision).

In a seminal paper "Bayesian rationale for the variational formulation", David Mumford [12] emphasizes that:

> one of the primary goals of low-level vision is to segment the domain W of an image I into parts W_i on which distinct surface patches, belonging to distinct objects in the scene, are visible.

The mathematical problem is to use low-level cues

> for *splitting* and *merging* different parts of the domain W

in an optimal way.

In Bayesian models, two parts exist: the prior model and the data model. Here, the prior model takes for an a priori the *phenomenological* evidence of what is qualitatively a segmentation, namely an approximation of the signal I by piecewise smooth functions u on $W - K$ which are discontinuous along a set of edges K. We must then introduce a way of selecting, from among all the

* Tel.: +33-1-55-55-86-23.
E-mail address: petitot@poly.polytechnique.fr (J. Petitot).
URL: http://www.crea.polytechnique.fr/JeanPetitot/home.html.

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allowed approximations (u, K) of I, the best possible one. For this, Mumford and Shah used an energy functional $E(u, K)$ which contains three terms:

1. a term which measures the variation and controls the smoothness of u on the open connected components W_i of $W - K$,
2. a term which controls the quality of the approximation of I by u,
3. a term which controls the length, the smoothness, the parsimony and the location of the boundaries K, and inhibits the spurious phenomenon of over-segmentation.

The MS "energy" [13] is:

$$E(u,K) = \int_{W-K} |\nabla u|^2 \, \mathrm{d}x + \lambda \int_W (u - I)^2 + \mu \int_K \mathrm{d}\sigma.$$

Due to the coefficients λ and μ, this MS-model is a multi-scale one: if μ is small, we get a "fine grained" segmentation, if μ is large, we get a "coarse grained" segmentation. The sensibility to contrast is $(4\lambda^2\mu)^{\frac{1}{4}}$, the scale $\lambda^{\frac{-1}{2}}$, the thresholds for ramp effects (the segmentation of a regular increase of I) $\left(\frac{\lambda^2\mu}{4}\right)^{\frac{1}{4}}$, and the resistance to noise $\lambda\mu$.

As some regularity properties of boundaries can in fact be deduced from the minimizing of E (see below), the third term of the MS-model is given in a more general setting (not a priori regular) by $H^1(K) = \int_K \mathrm{d}H^1$, where H^1 is the length of K in the Hausdorff sense defined by

$$H^1(K) = \sup_{\varepsilon \to 0^+} H^1_\varepsilon(K) \quad \text{with}$$

$$H^1_\varepsilon(K) = \inf \left\{ \sum_{i=1}^{i=\infty} \operatorname{diam} B_i : K \subseteq \bigcup_{i=1}^{i=\infty}, \operatorname{diam} B_i < \varepsilon \right\}$$

(we cover K in the less redondant way by small disks B_i, we approximate K by the diameters of the B_i, and we take the limit for vanishing diameters).

The MS-model can be interpreted probabilistically, using the equivalence

$$E(u,K) = -\log(p(u,K)),$$

where p is a probability defined on the space of possible segmentations (u, K).

As we see, minimizing E is a compromize between:

1. the homogeneity of u on the connected components W_i of $W - K$: if $u =$ cst then $\nabla u = 0$ and $\int_{W-K} |\nabla u|^2 \mathrm{d}x = 0$; minimizing this term forces therefore u to be as constant as possible on the W_i;
2. the approximation of I by u: if $u = I$ then $\int_W (u - I)^2 \mathrm{d}x = 0$; minimizing this term forces therefore u to be as close as possible to I;
3. the parsimony and the regularity of the boundaries: they are measured by the global length L of K, $L = H^1(K)$; minimizing this term inhibits therefore over-segmentation.

Such a variational algorithm optimizes the way in which neighboring pixels can be merged into homogeneous regions separated by qualitative discontinuities. It transforms the segmentation problem into a particular case of what is called in physics a "free boundary problem". It is extremely difficult to solve and actually not yet completely solved.

2. The Mumford–Shah conjecture

Many beautiful works have been dedicated to the MS-model by the Italian school (Ennio De Giorgi [6], Luigi Ambrosio [1,2], Gianni Dal Maso, Sergio Solimini [11], Antonio Leaci [6], Massimo Gobbino [8], Franco Tomarelli, Alessandro Sarti, Giovanna Citti, etc.), and in France by Jean-Michel Morel [11], Alexis Bonnet and Guy David [4]. [1]

If the singular set K is fixed, then u is the solution of the classical Neumann problem

$$\begin{cases} \Delta u = \mu(u - I) & \text{inside the components } W_i \text{ of } W - K, \\ \dfrac{\partial u}{\partial v} = 0 & \text{along the boundaries } \partial W \cup K. \end{cases}$$

On the other hand, in the simple case where the approximants u are locally constant, we have $\nabla u = 0$ and the MS-energy E reduces therefore to:

$$E(u,K) = \lambda \int_W (u - I)^2 + \mu H^1(K).$$

In that case, u is completely determined by K since u is equal to the the mean value of I on the components W_i of $W - K$. If $\bar{I}_i$ is the mean value of I on W_i, we have therefore:

$$E(u,K) = \lambda \sum_i \int_{W_i} (\bar{I}_i - I)^2 \, \mathrm{d}x + \mu H^1(K).$$

Mumford and Shah gave the complete solution to this simplified problem: minima of $E(u,K)$ exist and are reached for boundaries K which are piecewise C^1, whose curvature is bounded by $8\,\mathrm{osc}(I)^2$ (where the oscillation of I is defined by $\mathrm{osc}(I) = \mathrm{Max}\, I - \min I$) and whose singular points are reduced to triples points with 120° sectors inside W and points of orthogonal intersection on the boundary ∂W of W.

This is easy to see intuitively. Suppose we have shown that a piece of K is regular and let us parametrize it by $x = x(s)$ where s is the arc length (Fig. 1). As the length of the arc $A = \overline{x(-s)x(s)}$ is $2s$, we have of course $|x(s) - x(-s)| = 2(s - \varepsilon) \leqslant 2s$ (with equality for the

[1] For a synthesis, see [11].

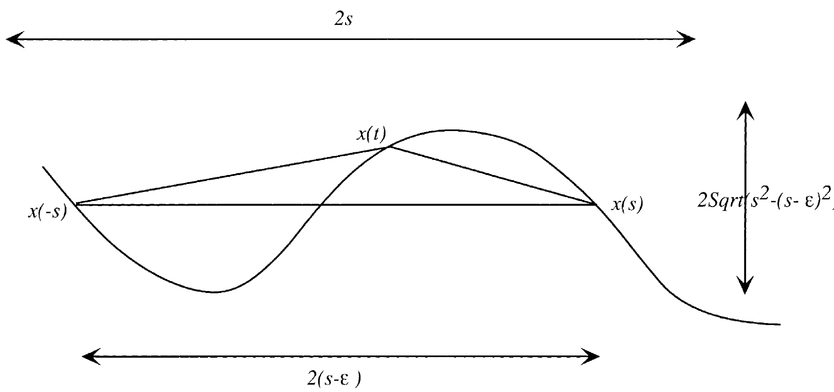

Fig. 1. The proof that the curvature of the boundaries K are bounded by 8 $\mathrm{osc}(I)^2$.

straight segment S). If we substitute the segment S for the arc A we get an energy variation

$$\Delta E = -\lambda\varepsilon + \Delta \int (u - I)^2\,\mathrm{d}x.$$

But if $\mathrm{osc}(I) = \mathrm{Max}I - \min I$ is the oscillation of I inside the rectangle centered on $x(0)$ with length $2s$ and width $2\sqrt{s^2 - (s-\varepsilon)^2}$, we have:

$$\left|\Delta \int (u - I)^2\,\mathrm{d}x\right| \leqslant \left(2s\sqrt{2s\varepsilon}\right)\mathrm{osc}\ (I)^2.$$

But as the segmentation K is supposed to be optimal, we must have $\Delta E \geqslant 0$, whence the inequality $\varepsilon \leqslant 8s^3\,\frac{\mathrm{osc}(I)^4}{\lambda^2}$ which implies the inequality on the curvature.

Now, let a be a triple point of K with angles different from 120°. One of the angles is <120°. Let C be a small circle of radius ε going trough a and crossing the two edges of this angle at v and w (Fig. 2). We substitute the three segments xa, xv, xw for the two segments av and aw and we compute the variation ΔE. We get a linear term Δl which is negative and of first order ε. Indeed, due to the triangular inequality, $H^1(av) < H^1(xa) + H^1(xv)$ and $H^1(aw) < H^1(xa) + H^1(xw)$. As the surface term $\Delta \int (u-I)^2\mathrm{d}x$ is of second order ε^2 and dominated by the linear term for small ε, we get $\Delta E < 0$, which is impossible by hypothesis.

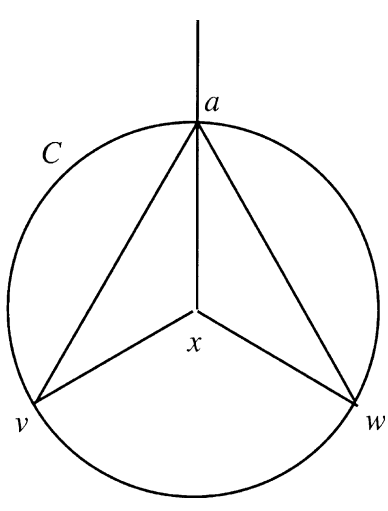

Fig. 2. The proof that the triple points of K are symmetric with 120° angles.

Mumford's conjecture says that in the general case where u is no longer locally constant, it is essentially the same thing but with a supplementary type of "end point" singularities (end points of branches of K also called "cracktips") whose normal form is given in polar co-ordinates by the formula:

$$u(r,\theta) = \left(\frac{2r}{\pi}\right)^{\frac{1}{2}} \sin\frac{\theta}{2} + C$$

for $-\pi < \theta < \pi$ (see Fig. 3).

This conjecture is partially proved and the proof is extremely difficult. The main problem is raised by the regularity of the boundaries and the topological structure of the connected components W_i of $W - K$.

One can prove first that K is closed and regular in the sense of Ahlfors: their exist constants c and C independent of the point $x \in K$ such that for every $x \in K$ and

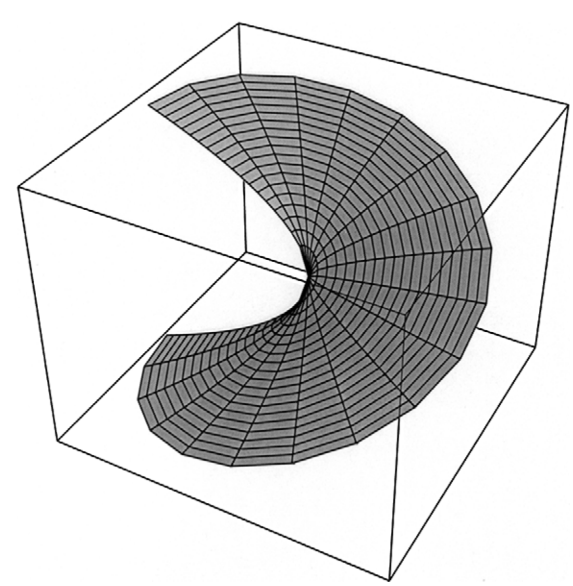

Fig. 3. The structure of a cracktip.

every disc $B(x,r)$ with center x and radius r, one has $cr \leqslant H^1(K \cap B(x,r)) \leqslant Cr$. K cannot therefore be fractal.

David and Semmes ([5]) have then shown a property of uniform rectifiability: for every $\varepsilon > 0$ there exists an $\alpha > 0$ such that K is ε-included in a regular curve γ (i.e. $H^1(\gamma - K) \leqslant \varepsilon H^1(\gamma)$) such that $H^1(\gamma) \geqslant \alpha r$.

Then Alexis Bonnet proved Mumford conjecture for the isolated *connected* components of K. He used a "blowing-up" method consisting in zooming on $x \in K$ for a minimizer (u,K) of E and looking at the "tangent" situation defined by the functional:

$$J = \int_{\mathbb{R}^2 - K} |\nabla u|^2 \, dx + H^1(K).$$

Indeed, in zooming on x we consider discs $B(x,r)$ with $r \to 0$. But the surface term $\int_B (u-I)^2 dx$ is of second order r^2 while the length term is of the first order r. As for the gradient term $\int_B |\nabla u|^2 dx$, if it would decrease faster than r then K would be regular at x and there will be no problem at all. We can therefore consider that we are in the interesting case where the surface term $\int_B (u-I)^2 dx$ can be neglected.

In fact we have to *localize* the functional J to discs $B_R = B(0,R)$ and to segmentations (u,K) such that the restriction $J_R(u,K) < \infty$ for every $R > 0$. We restrict then the comparison of (u,K) to competitors (v,G) such that

1. $(v,G) = (u,K)$ outside B_R,
2. if x, y are separated by K outside B_R then they are also separated by G.

Minimizing the localized functional in the space of such competitors defines the concept of *global minimizer*.

Bonnet has shown that if (u,K) is a minimizer of the MS-functional E then all its limits by blowing-up are global minimizers of J. His proof of Mumford conjecture for global minimizers (u,K) of J in the case K is connected shows that there exist only four possible situations:

(i) K is the null set and u is constant;
(ii) K is a straight line and u is constant on every side of K;
(iii) K is a symmetric triple point (120° angles) and u is constant in each of the three sectors;
(iv) K is a half straight-line and u is a cracktip.

More recently, Bonnet and David ([4]) proved that cracktips are global minimizers.

3. The links between variational models and diffusion equations

There exists a fundamental link between variational models of segmentation and diffusion partial differential equations (PDE). It is well known that the heat equation is the gradient flow associated to the energy:

$$E(u) = \frac{1}{2} \int_W |\nabla u|^2 \, dx$$

namely to the first term of the MS-model. Indeed, using Stokes theorem, it is easy to compute the functional derivative of E, $\nabla E = \frac{\delta E}{\delta u}$, defined by the formula

$$E(u+g) = E(u) + \int_W \frac{\delta E}{\delta u}(u) g(x) \, dx.$$

We get:

$$\nabla E = \frac{\delta E}{\delta u} = -\Delta u.$$

As the gradient flow associated to E is given by $\frac{\partial u}{\partial t} = -\frac{\delta E}{\delta u}$, we recover the heat equation $\frac{\partial u}{\partial t} = \Delta u$.

The MS-model corresponds also to a gradient flow but with discontinuities of the approximants u along K. We have $\frac{\partial u}{\partial t} = \Delta u$ on $W - K$ with the initial condition $u = I$ for $t = 0$, but we have also

$$\frac{\partial K}{\partial t} = \kappa - \left((\nabla u^+)^2 - (\nabla u^-)^2 \right),$$

where κ is the curvature of K (if K is sufficiently regular) and where ∇u^+ and ∇u^- correspond to the values of ∇u on the two sides of K. If (u,K) is a minimizer, $\frac{\partial K}{\partial t} = 0$ and therefore:

$$\kappa = (\nabla u^+)^2 - (\nabla u^-)^2.$$

4. The works of the Italian school

To appreciate the fundamental works of the Italian school concerning Mumford's conjecture, some preliminary elements of mathematical analysis are necessary. We have first to define precisely the type of functions we will accept.

Let X be a compact topological space endowed with a measure μ allowing to give a measure $\mu(U)$ to open subsets U of X and, through additive properties and techniques of approximation, to a large class of subsets (the measurable ones) V, and, through the association of the V with their characteristic functions χ_V, to a large class of functions on X. Technically, μ is by definition an element of the dual of the Banach space $\mathscr{C}(X)$ of scalar continuous functions on X. This means that if $f \in \mathscr{C}(X)$, $\mu(f)$ is a (real or complex) scalar, the mapping $f \mapsto \mu(f)$ being linear and continuous (there exists a constant $a > 0$ such that $|\mu(f)| \leqslant a\|f\|$ where $\|f\| = \sup_{x \in X} |f(x)|$ is the norm of f). For X locally compact, these definitions can be easily generalized by looking at the compact subsets of X.

Measurable functions are then defined as functions f which are μ-equivalent [2] to functions continuous on a partition of X in compact subsets K_n. Their measure $\mu(f)$ can therefore be defined. Integrable functions are measurable functions whose measure $\mu(f)$ is finite. Their space is referred to by $L^1(X)$. On the other hand, $L^\infty(X)$ is the space of bounded measurable functions. Of particular importance is the space $L^2(X)$ of measurable functions whose square is integrable. It was a great discovery that this space is naturally endowed with a structure of Hilbert space, that is with an infinite dimensional "Euclidean" structure.

We suppose that signals I in the MS-energy E can be identified with functions $I \in L^\infty(X) \cap L^2(X)$ (bounded functions of finite energy). We have to solve two problems:

1. the existence of solutions to the variational problem,
2. the structure of solutions if they exist.

In what concerns the existence theorem, it is due to De Giorgi ([6]). The basic technic is to approximate the functional E by functionals E_ε depending upon a parameter $\varepsilon \to 0^+$ in such a way that the discontinuity set K becomes the closure of the discontinuity set S_u of the limit u, for $\varepsilon \to 0^+$, of a sequence of solutions u_ε of E_ε.

To get "good" proofs, we need to work in "good" functional spaces. For technical reasons, the specialists have privileged the space $SBV(W)$ of piecewise differentiable functions u whose set of discontinuity S_u is an at most countable union of regular curves. A unitary normal vector ν_u can then be defined almost everywhere, as well as the values $u^+(x)$ and $u^-(x)$ of u on the two sides of the jump set S_u for $x \in S_u$.

More precisely, one says that a measurable function u is of bounded variation, $u \in BV(W)$, if u is integrable ($u \in L^1(X)$) and if its derivative (its gradient vector) $Du = (D_1u, D_2u)$ in the distributional sense (u need not to be differentiable in the classical sense) is of finite total variation. By definition $Du = (D_1u, D_2u)$ is the vectorial measure defined by [3]

$$\int_W u\frac{\partial \varphi}{\partial x_i} = -\int_W \varphi\, \mathrm{d}D_iu \quad (i = 1, 2),$$

where $\varphi \in C_0^1(W)$ is a continuous, differentiable test function with compact support. Du has finite total variation if:

$$|Du|(W) = \int_W \mathrm{d}|Du| < \infty.$$

If u is a measurable function, one says that u has an *approximative limit* $\bar{u}(x)$ at $x \in W$ if, for every neighborhood U of $\bar{u}(x)$, the image of almost all the ball $B_\varepsilon(x)$ is included in U when ε is sufficiently small. One has then $\bar{u}(x) = u^+(x) = u^-(x)$. The discontinuity set S_u of u is the set of $x \in W$ such that $\bar{u}(x)$ does not exist, that is where $u^-(x) < u^+(x)$.

A fundamental theorem, the Radon–Nikodym theorem, says that the vectorial measure Du is the sum of two parts, $Du = \nabla u + D^s u$, with ∇u the "regular" part (which is absolutely continuous relative to the Lebesgue measure $\mathrm{d}x : \nabla u = g\mathrm{d}x$ with g a continuous function) and $D^s u$ the "singular" part. Moreover, the singular part $D^s u$ is itself the sum of two parts, $D^s u = D^j u + D^c u$, with $D^j u$ the jump part concentrated on S_u and $D^c u = D^s u|_{W-S_u}$ the Cantor part. The function u belongs to $SBV(W)$ if the Cantor part $D^c u$ vanishes, that is if the singular part $D^s u$ is concentrated on S_u. We then have

$$D^s u = D^j u = (u^+ - u^-)\nu_u H^1|_{S_u}$$

and

$$|Du|(W) = \int_W \mathrm{d}|Du| = \int_W |\nabla u|\mathrm{d}x + \int_{S_u}(u^+ - u^-)\mathrm{d}H^1.$$

One of the main interest of the space $SBV(W)$ is Ambrosio's compacity theorem ([1]) stating that, if $u_\varepsilon \in SBV(W)$ is a sequence in $SBV(W)$ which is uniformly bounded in the sense that $\|u_\varepsilon\|_\infty + H^1(S_{u_\varepsilon})$ (the first term is the L^∞ norm of u_ε and $H^1(S_{u_\varepsilon})$ is the length of S_{u_ε}) and $\int_W |\nabla u_\varepsilon|\,\mathrm{d}x$ (the L^1 norm of $|\nabla u_\varepsilon|$) are bounded by constants independent of ε, then there exists a sub-sequence which converges in $L^1(X)$ to a function v which also belongs to $SBV(W)$.

De Giorgi fundamental existence theorem ([6]) says that there exists a minimum u of the Mumford–Shah functional $E(K, u)$ with K closed, $u \in C^1(W - K)$, $\sup|u| \leqslant \sup|I|$, K being an at most countable union of C^1 arcs.

For the proof, one looks at weak solutions depending only upon the approximant u (and no longer upon the set of discontinuity K) and prove the theorem for them using the compacity property of $SBV(W)$. One then shows that weak minimizers share good regularity properties.

The weak problem corresponds to the energy

$$E_0(u) = \int_W |\nabla u|^2\mathrm{d}x + \lambda\int_W (u - I)^2\mathrm{d}x + \mu H^1(S_u)$$

with $u \in SBV(W)$. One shows that if u is a minimizer of E_0 then the pair $(\bar{u}, K = \overline{S_u})$ is a minimizer of E for $\bar{u} \in C^1(W - \overline{S_u})$ and $H^1(\overline{S_u} - S_u) = 0$. To construct a weak solution, one looks for approximants u_ε of functionals E_ε which converge to E_0, the "good" concept of convergence being *Γ-convergence*. One says that a sequence of functionals E_ε *Γ*-converges to a functional E_0 if, for every convergent sequence of functions $u_\varepsilon \underset{\varepsilon\to 0^+}{\to} u$, $E_0(u)$ remains $\leqslant$ to the $\liminf_{\varepsilon\to 0^+}$ of $E_\varepsilon(u_\varepsilon)$ and if there

[2] Two functions are equivalent if their difference has measure 0.

[3] If μ is a measure on W, the measure $\mu(f)$ of a function f on W is classically written as an integral $\int_W f\,\mathrm{d}\mu$.

exists at least one convergent sequence of functions $u_\varepsilon \underset{\varepsilon\to0^+}{\to} u$ such that $E_0(u) \geqslant \limsup_{\varepsilon\to0^+} E_\varepsilon(u_\varepsilon)$.

De Giorgi studied in particular functionals of the type

$$E_\varepsilon(u) = \int_{W\times W} \arctan\left[\frac{(u(x+\varepsilon\xi)-u(x))^2}{\varepsilon}\right] e^{-|\xi|^2}\, d\xi\, dx,$$

where ξ represents complementary variables. Massimo Gobbino [8] has shown that such E_ε Γ-converge to the MS functional $\frac{\pi}{2}E$ with $\lambda = 1$ and $\mu = \sqrt{\pi}$. This beautiful result has been considerably improved by Alessandro Sarti and Giovanna Citti.

5. Oscillations in neural networks

Sarti's and Citti's work sheds a new light on the *neural plausibility* of the MS-model. Its framework is the theory of networks of oscillators. It is well known that the cortical columns composing the hypercolumns of the visual area $V1$ can behave as oscillators. Let S be a system of formal neurons whose activity is described by the Hopfield equations:

$$\dot{x}_i = x_i + \sigma\left(\sum_i w_{ij}x_j + \theta_i\right),$$

where x_i is the state of activity of the ith neuron u_i, σ a sigmoid (gain function), w_{ij} the synaptic weights and θ_i the respective thresholds of the u_i.

If we take the equations of this neural network and if we apply a mean field approximation by averaging on the excitatory and inhibitory connections, we get a system of two equations for the mean activities X_E and X_I (Wilson–Cowan equations). Under the retinal stimulus, the equilibrium state of this system can bifurcate spontaneously, via a Hopf bifurcation, towards a cyclic attractor (attracting limit cycle). Moreover, the frequency of this limit cycle depends upon the intensity of the stimulus. One observes then a synchronization above the homogeneous parts of the stimulus.

Since the pionneering works of Christoph von der Malsburg in the early eighties, many experimental evidences has accumulated concerning synchronized oscillations in the cortical hypercolumns (in the frequency range of 40–70 Hz) which are sensitive to the coherence of the stimulus (see e.g. the reference paper of Andreas Engel, Peter König, Charles Gray and Wolf Singer [7]). Even if they are quite controversial, [4] they are of great theoretical relevance since they yield a possible answer to the binding problem, the key idea being that frequency and phase locking of oscillatory neural responses can code the constituent structure and the mereological part/whole relations of the stimuli (see Atiya and Baldi [3]).

It is therefore an interesting and relevant issue to study networks of oscillators. At least three types of oscillators can be investigated:

1. harmonic oscillators and their variants (uniform limit cycles);
2. hysteresis cycles in fast/slow systems with a cubic slow manifold (Van der Pol cycles);
3. limit cycles with a discontinuous jump (pulse oscillators).

There exist also alternative choices for the couplings: couplings of the form $\sin(\theta_i - \theta_j)$ depending upon the differences of phases, coupling through pulses, etc. But in any case the mathematical analysis of such systems is very difficult. Let us give a basic example.

We start with a network of N oscillators F_i $(i = 1,\ldots,N)$ of frequency ω_i (period $T_i = 2\pi/\omega_i$). If θ_i are their phases and φ_{ij} their differences of phases $\varphi_{ij} = \theta_i - \theta_j$, the differential equations of the network are of the form:

$$\dot{\theta}_i = \omega_i - H(\varphi_{i,1},\ldots,\varphi_{i,N})$$

with the frequency ω_i depending upon the intensity of the stimulus at position i. It is a typical complex system which must be analyzed using the strong resources of statistical physics (Kuramoto, Daido, etc.) and qualitative dynamics (Kopell and Ermentrout [9] etc.). The most studied systems are of the simple form:

$$\dot{\theta}_i = \omega_i - \sum_{j=1}^{j=N} K_{ij}\sin\left(\theta_i - \theta_j\right),$$

where the K_{ij} are coupling constants. When there exists a single coupling constant K and when the network is totally connected, we get the 1987 Kuramoto model ([10]):

$$\dot{\theta}_i = \omega_i - \frac{K}{N}\sum_{j=1}^{j=N}\sin\left(\theta_i - \theta_j\right).$$

To study this limit case, Kuramoto took as *order parameter* the mean phase:

$$Z(t) = |Z(t)|e^{i\theta_0(t)} = \frac{1}{N}\sum_{j=1}^{j=N} e^{i\theta_j(t)}$$

and looked at the equivalent system:

$$\dot{\theta}_i = \omega_i - K|Z|\sin(\theta_i - \theta_0).$$

If the frequencies ω_i follow a random law $g(\omega)$ representing the statistical regularities of the environment (we use a rotating frame such that g becomes centered on its mean value 0), synchronization is a *phase transition*

[4] One of the main criticism raised against synchronization is that it is a too slow process and cannot explain very fast abilities such as shape recognition.

occurring for the critical value $K_c = 2/\pi g(0)$ of the coupling constant.

To prove this very striking result, Kuramoto looked first for solutions $Z =$ constant. After having discriminated the oscillators in two groups:

(i) the S-group of oscillators which can be synchronized because they satisfy the property

$$\dot{\theta}_i = 0 \text{ and therefore } \left|\frac{\omega_i}{KZ}\right| \leqslant 1,$$

(ii) the D-group of oscillators which cannot be synchronized because

$$\left|\frac{\omega_i}{KZ}\right| > 1,$$

he showed that only the S-group occurs in the synchronization process. Taking into account the fact that

$$Z = \int_0^{2\pi} n_0(\theta, t) e^{i\theta} \, d\theta,$$

where $n_0(\theta, t)$ is the equilibrium distribution of phases at time t and the fact that

$$n_0(\theta, t) d\theta = g(\omega) d\omega \quad \text{with } \omega = K|Z| \sin(\theta - \theta_0)$$

he get a self-consistency equation $Z = S(Z)$ which can be developed in a neighborhood of $Z = 0$. Whence the equation (which is in fact a normal form for this type of process):

$$\varepsilon Z - \beta |Z|^2 Z = 0$$

with $\varepsilon = \frac{K - K_c}{K_c}$ and $\beta = -\frac{\pi}{16} K_c^3 g''(0)$. The analysis of the stability of the solutions shows that the solution $Z = 0$, which is stable for $K \simeq 0$ (uncoupled oscillators), becomes unstable when K crosses the critical value K_c.

Under an adiabaticity hypothesis, Kuramoto proved that the (slow) evolution of the order parameter Z is ruled by the equation:

$$\xi \frac{dZ}{dt} |KZ|^{-1} = \varepsilon Z - \beta |Z|^2 Z.$$

And finally, he studied the fluctuations of the system in the neighborhood of the critical point, where they diverge and trigger the phase transition. These results show that synchronization is a typical phenomenon of emergent collective organization.

In what concerns the dynamical approach, I will mention only one simple but striking example due to Nancy Kopell and Bard Ermentrout [9]. Take a line of oscillators with linearly increasing frequencies. Under a weak coupling hypothesis, one can show that plateaus are formed. This means that the oscillators try to synchronize, but as the total difference of frequencies is too large, they can only do it partially. Homogeneous synchronized zones are formed (plateaus), which are delimited by sharp discontinuities (jumps between plateaus). Now if there are discontinuities in the stimulus, they constitute of course preferential precursors for the boundaries.

In a nutshell, the theory of coupled oscillators shows that such systems can enhance and complete existing boundaries, and generate virtual boundaries (which are not in the inputs).

6. The Mumford–Shah model as a synchronization model

The main result of Sarti and Citti is that, in the bidimensional case, such oscillator networks converge towards the Mumford–Shah variational model. This model is therefore endowed with a strong neuronal plausibility.

The idea is the following. We consider a 2D field of oscillators and we generalize the Kuramoto model. The phase $\theta(x, t)$ is now also a function of the spatial position x. Let ξ be the distance between oscillators. We look at a PDE of the form:

$$\frac{\partial \theta(x,t)}{\partial t} = \omega(x) + \frac{1}{|\xi|^2} \{K(x+\xi)[\varphi(\theta(x+\xi, t) - \theta(x,t))] - K(x)[\varphi(\theta(x,t) - \theta(x-\xi, t))]\},$$

where the function φ generalizes the function sin and where the sum Σ is taken on the $x+\xi$ and $x-\xi$ neighbors of x. To take into account the different ξ, we introduce a probability law on the connections. The simplest case is that of a Gaussian isotropic law. If we introduce the mesh ε of the lattice and if we encode in the coupling function $K(x)$ the anisotropy induced by the neural *functional architecture*, we get a model which converges towards the gradient flow associated to the Mumford–Shah variational model with the metric defined by $K(x)$.

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