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Conceptual analysis and computational synthesis in mathematical physics

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- Every science goes from empirical data to theoretical concepts using universal cognitive ressources : abstraction, categorization, taxinomies, inferences, correlations, causality, etc.

- It is the way of *conceptual analysis* : to “subsume” the multiplicity of empirical phenomena under the unity of concepts : transcendental analytic and regional ontology.

- The characteristic feature of mathematical physics is to solve the <u>inverse problem</u> of abstraction and conceptual subsumption.

- It is the *computational synthesis* going from categories and principles to the multiplicity of empirical phenomena.

- Constructing, using algorithms, a *virtual reality* which can *fit* the empirical reality (models, simulations).

- This is made possible by the <u>generativity</u> of mathematics.

The foundational problem

- To overcome the gap between
  - Conceptual analysis : categories and principles (conceptual comprehension),
  - Computational synthesis : laws and differential equations (operational explanation).

- For Newton's mechanics, Kant's *Metaphysische Anfangsgründe der Naturwissenschaft* are dedicated to this problem.
- See e.g. http://jeanpetitot.com/ArticlesPDF/Petitot_SHC_2014.pdf
- We want to give another example.

Heisenberg's “Umdeutung”

- Heisenberg supplies another example far beyond Kant's interpretation of classical kinematics and mechanics.
- "Über quantentheoretische Umdeutung kinematischer und mechanischer Beziehungen". *Zeitschrift für Physik*, 33 (1925) 879–893.
- *“Reinterpretation”* of the relations between kinematics and mechanics in QM.

- On the basis of experimental results on spectral rays and previous models of atoms (Sommerfeld, Bohr), Heisenberg “reinterprets” the transcendental foundations of mathematical physics.
- The new “regional object” is no longer motions but <u>spectra</u>.
- Spectral rays substitute positions on trajectories.

- To interpret rays as wavelengths or frequencies, i.e. as *magnitudes* with numerical values is already a great achievement (Maxwell electrodynamics)
- The key <u>empirical</u> regularity is the Balmer-Rydberg formula (1885-1900) for the wavelength of the emission spectrum of H.

$$\lambda = \left(\frac{R}{hc}\right)^{-1} \left(\frac{1}{m^2} - \frac{1}{n^2}\right)^{-1} \quad (m < n)$$

- Bohr model (1913) introduced stationary non radiative states. Hence a quantification of kinematic and mechanical variables:

$$r_n = \frac{4\pi\varepsilon_0\hbar^2}{m_e e^2}n^2 = r_0 n^2$$

$$r_0 = \text{ Bohr radius } = 0.0529 \text{ nm for H}$$

$$\begin{cases} v_n = \dfrac{\hbar}{m_e r_0 n} \\ p_n = \dfrac{\hbar}{r_0 n} \\ E_n = -\dfrac{e^2}{8\pi\varepsilon_0 r_0}\dfrac{1}{n^2} = -\dfrac{E_1}{n^2} \\ E_1 = \text{ } 13.6eV \text{ for the ground state of } H \end{cases}$$

- Transitions between energy levels

$$m \rightarrow n$$

are interpreted as photon emissions.

$$E_{photon} = E_n - E_m = E_1 \left(\frac{1}{m^2} - \frac{1}{n^2} \right) = h\nu$$

$$\nu = \frac{E_1}{h} \left(\frac{1}{m^2} - \frac{1}{n^2} \right)$$

- In the pre-Heisenberg models there is a quantification of orbits but the classical <u>Fourier description</u> of magnitudes along closed orbits is maintained.

Classical case: Fourier analysis ($\omega = 2\pi\nu$ is the orbital or angular frequency)

$$x(n,t) = \sum_{\alpha\in\mathbb{Z}} A_\alpha(n)\, e^{i\omega(n)\alpha t}$$

- Heisenberg starts with the *principle of reduction to observables* (i.e. to phenomena):

  > "Establish a basis for theoretical quantum mechanics founded exclusively upon relationships between quantities which in principle are observable."

- Heisenberg emphasizes strongly the change of regional object :
  - one does not know how to associate a <u>position</u> to an e^-
  - one knows how to associate a <u>frequency</u> to an e^-
- Spectral rays correspond to differences of (orbital) frequencies (Heis. notation) :

$$\omega\left(n, n-\alpha\right)$$

- Now, as in Kant's Phoronomy, the key problem is to interpret the composition of basic phenomena in terms of algebraic operations on quantities.

- As spectral rays substitute positions in classical mechanics (A. Connes, *NCG*, p.35),

  > “it is thus essential to find regularities that appear in these configurations.”

- Alain Connes strongly emphasized that Heisenberg reached the root of the inconstitencies of the new mechanics.
- The great discovery is that

  “physical quantities are governed by non commutative algebra” (*NCG*, p.34)
- We have physical quantities and we need an algebra of numbers to measure them.

- The basic phenomenal law of composition is the <u>Ritz-Rydberg combination principle</u> :

$$\begin{cases} \nu_{ij} = \nu_i - \nu_j \\ \nu_{ik} = \nu_{ij} + \nu_{jk} \end{cases}$$

- A. Connes : “the spectrum is naturally endowed with a partially defined law of composition.” (*NCG*, p.37)

There exists a *groupoid* Δ of frequencies (i, j) with the composition law $(i, j) \, . \, (j, k) = (i, k)$.
The observables X belong to the convolution algebra of Δ:

$$X_{ik} = \sum_j X_{ij} X_{jk}$$

- A. Connes : "Heisenberg's rules of algebraic calculation were imposed on him by the experimental results of spectroscopy." (*NCG*, p.38)

- Computations in Heisenberg's *Umdeutung*.
- In QM, quantities are of the form $a(n, t)$.

Quantum case: $a(n,t)$ is represented by the *set*

$$\left\{A_\alpha (n, n-\alpha)\, e^{i\omega(n,n-\alpha)t}\right\}$$

and no longer by a Fourier series :

$$a(n,t) = \sum_{\alpha\in\mathbb{Z}} A_\alpha (n)\, e^{i\omega(n)\alpha t}$$

- The key problem is the *functional calculus* : representation of $f(a, b)$.
- In classical mechanics,

$$\begin{aligned} M &= \text{phase space}, \\ \text{states} &= \text{points } x \in M, \\ \text{observable} &= \text{real function } a : M \to \mathbb{R} \end{aligned}$$

and the functional calculus is evident.

- In Bohr's model, the product of two observables has Fourier coefficients :

$$\text{Classical:}\ \ \omega\left(n,\beta\right)=\omega\left(n,\alpha\right)+\omega\left(n,\beta-\alpha\right)$$
$$C_{\beta}\left(n\right)=\sum\nolimits_{\alpha\in\mathbb{Z}}A_{\alpha}\left(n\right)B_{\beta-\alpha}\left(n\right)$$

(the sum becomes a product in the exponentials).

- In QM we have the <u>NC</u> product

Quantum: $\omega\left(n,n-\beta\right)=\omega\left(n,n-\alpha\right)+\omega\left(n-\alpha,n-\beta\right)$

$C_{\beta}\left(n,n-\beta\right)=\sum_{\alpha\in\mathbb{Z}}A\left(n,n-\alpha\right)B\left(n-\alpha,n-\beta\right)$

- William Fedak, Jeffrey Prentis : "The 1925 Born and Jordan paper on quantum mechanics", *Am. J. Phys.*, 77, 2, 2009.
- "In [this] construction Heisenberg uncovered the symbolic algebra of atomic processes."

$$\omega(n, n-\beta) = \omega(n, n-\alpha) + \omega(n-\alpha, n-\beta)$$

is "the backbone of the multiplication rule" and "allowed Heisenberg to algebraically manipulate the transition components."

Logic of judgements in QM

- From Heisenberg to Connes, quantum elements of Phoronomy, Dynamics, and Mechanics in the sense of Kant have been developped.
- But there is also Kant's *Phenomenology* concerning logic of judgements.
- It is a deep problem which is not solved by quantum logic.

- In QM there is a stronger <u>obstruction</u> than in CM to ascribe truth values to judgements ascribing numerical values to magnitudes.
- But the obstruction no longer comes from a relativity group but from the non-commutativity of the algebra of quantities.

- The problem is well known since Bell and Kochen-Specker no-go theorems.
- Value definiteness and non contextuality.
  K-S : For $dim(\mathfrak{H}) \geq 3$

  it is impossible to ascribe definite values to the observables a with the hypotheses :
  - to have a value $V(a) \Leftrightarrow$ to possess an intrinsic (non contextual) property $\Leftrightarrow$ the measure of a gives $V(a)$,
  - $FUNC : V\left(f\left(a,b\right)\right) = f\left(V\left(a\right), V\left(b\right)\right)$
    even just for *compatible* (i.e. commuting) observables.

- No-go theorems show that “measure” is not “valuation”.
- There exists a gap between:
  - “the measure of the observable a gives the result r when the system is in state ρ ”, and
  - “when the system is in state ρ , the quantity a <u>has</u> the value r .”
- There exists <u>no “element of physical reality”</u> associated to a measure.

- Simon Kochen-Ernst Specker (1967) :
- They worked out an exemple with $dim(\mathfrak{H}) = 3$ and 117 observables.
- Adan Cabello (1996) worked out a simpler example with $dim(\mathfrak{H}) = 4$ and 18 observables.
- To each observable is associated an orthonormal frame of eigenvectors and 4 projectors P_i (yes-no questions) with

$$V(P_i) = 0 \text{ or } 1$$

$$\sum_i V(P_i) = 1$$

- In each frame one P has $V(P) = 1$ and three P have $V(P) = 0$.
- Cabello found 9 orthonormal frames

$$9 \text{ x } 4 = 36$$

where each axis is counted <u>two</u> times

$$36 = 18 \text{ x } 2$$

- Non contextuality implies that the $V(P)$ are <u>intrinsic</u> and do not depend upon the frame.
- So we must have nine P with $V(P) = 1$. But this is impossible since they must be an even number.

u_1	(0, 0, 0, 1)	(0, 0, 0, 1)	(1, −1, 1, −1)	(1, −1, 1, −1)	(0, 0, 1, 0)	(1, −1, −1, 1)	(1, 1, −1, 1)	(1, 1, −1, 1)	(1, 1, 1, −1)
u_2	(0, 0, 1, 0)	(0, 1, 0, 0)	(1, −1, −1, 1)	(1, 1, 1, 1)	(0, 1, 0, 0)	(1, 1, 1, 1)	(1, 1, 1, −1)	(−1, 1, 1, 1)	(−1, 1, 1, 1)
u_3	(1, 1, 0, 0)	(1, 0, 1, 0)	(1, 1, 0, 0)	(1, 0, −1, 0)	(1, 0, 0, 1)	(1, 0, 0, −1)	(1, −1, 0, 0)	(1, 0, 1, 0)	(1, 0, 0, 1)
u_4	(1, −1, 0, 0)	(1, 0, −1, 0)	(0, 0, 1, 1)	(0, 1, 0, −1)	(1, 0, 0, −1)	(0, 1, −1, 0)	(0, 0, 1, 1)	(0, 1, 0, −1)	(0, 1, −1, 0)

- See Stefano Osnaghi's beautiful thesis : *“La sémantique des probabilités quantiques.”*

Transcendental logic and topos theory

- One of the main philosophical problems of scientific knowledge is formulated by Husserl's title *Erfahrung und Urteil* , “Experience and Judgement”
- On the “judgement” side we find *logic*.
- On the “experience” side we find *geometrical* structures of objects (vector spaces, relativity groups, Hilbert spaces, C*-algebras, etc.).

- To combine both sides, that is logic and geometry, is the transcendental problem since Kant.
- For the specific science of a specific regional object, logic must be constrained by an object structure.
- It is *“modal”* in Kant's sense w.r.t. a *“Phoronomy”* .

- We have seen in Part I with Kant that "alternative" judgements are apparently inconsistent because they must be *functorial* w.r.t. a groupoid expressing the "object structure".
- To day, one of the best framework for transcendental logic is topos theory, which
  - expresses the "object structure" by some category $\mathfrak{C}$,
  - formulates the theory by a certain type of functors $\mathfrak{C} \rightarrow$ **<u>Ens</u>**.

- Topos
  - pull backs,
  - terminal object,
  - exponentials i.e. internal Hom,
  - subobject classifier.
- Seaves, sites, sheaves.

- Saunders Mac Lane - Ieke Moerdijk (first sentence of *Sheaves in Geometry and Logic*, 1992) :

  "A startling aspect of topos theory is that it unifies two seemingly wholly distinct mathematical subjects : on the one hand topology and algebraic geometry and on the other hand, logic and set theory."

- The topos **<u>Ens</u>** is purely logic, without object structure.
- As claimed Ferdinand Gonseth,

  “la logique est la physique de l'objet quelconque.”

Bohr topos and K-S

- Interpretation of K-S in the late 90s by Chris Isham, Jeremy Butterfield, Andreas Döring, Chris Heunen, Bas Spitters, etc.
- The approach is mainly "foundational, structural and conceptual".
- How to ascribe truth values to judgements ?
- They use the "interplay between spatial and logical structures inherent to topos theory." (Heunen)

Bohr topos

Born rule

Example of *elementary* propositions ("yes" or "no" questions)

$$a \in \Delta$$

with $a \in \mathcal{B}_{s.a.}(\mathcal{H})$ and $\Delta \subset \mathbb{R}$ (Borel subset).

(B = bounded, s.a. = self-adjoint operator)

If $\psi \in \mathcal{H}$, $a \in \mathcal{B}_{s.a.}(\mathcal{H})$, and $[a \in \Delta]$ projector,

Born rule: the probability for $a \in \Delta$ to be true in state ψ is

$$\langle \psi, a \in \Delta \rangle = \|[a \in \Delta]\,\psi\|^2$$

A definite truth value $V(a \in \Delta)$ (not a probability) exists iff

$$\begin{cases} [a \in \Delta]\,\psi = \psi \text{ then } V = 1 \\ [a \in \Delta]\,\psi = 0 \text{ then } V = 0 \end{cases}$$

If we want more general truth values, since hidden variables don't work, we need a non classical truth object (subobject classifier) Ω.

Ω must *reflect* the object structure.

Logic must be "internalized" and become
a transcendental logic.

Classical case

M = phase space,
category (poset, frame i.e.distributive complete lattice)
$\mathcal{O}(M)$ of open subsets.
states = points $x \in M$,
observable = real function $a : M \to \mathbb{R}$,

$a \in \Delta$ true $\Leftrightarrow a(x) \in \Delta \Leftrightarrow x \in a^{-1}(\Delta)$.

A point x defines a valuation V_x on propositions $a \in \Delta$ with values in $\Omega = \{0, 1\}$:

$$\begin{cases} V_x(a \in \Delta) = 1 \text{ if } a(x) \in \Delta \\ V_x(a \in \Delta) = 0 \text{ if } a(x) \notin \Delta \end{cases}$$

Gelfand-Neimark-Segal

(M compactified)
The algebra $\mathfrak{A}_{s.a.}$ of observables $\subset$
the commutative C^*-algebra

$$\mathfrak{A} = C\left(M, \mathbb{C}\right), a : M \to \mathbb{R}.$$

A *state* is a map $\rho : \mathfrak{A} \to \mathbb{C}$ which is $\mathbb{C}$-linear, positive, normalized.
$\rho\left(a\right)$ is the value of the observable a in the state ρ.

Let $\Sigma =$ Spec$(\mathfrak{A})$, $\Sigma \simeq M$.
Gelfand duality: $\mathfrak{A} = C(\Sigma, \mathbb{C})$, i.e.

$$a \Leftrightarrow \widehat{a} : \Sigma \to \mathbb{C} \Leftrightarrow \widetilde{a} : C(\mathbb{R})_0 \to \mathfrak{A}$$

where $C(\mathbb{R})_0$ is the unitalization of the algebra of functions with compact support on $\mathbb{R}$ (compactification of $\mathbb{R}$).

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The map $\widetilde{a} : C(\mathbb{R})_0 \to \mathfrak{A}$ corresponds to the *functional calculus* FUNC on $\mathfrak{A}$, that is

$$\widetilde{a}(1) = a, \text{and } \widetilde{a}(f(x)) = f(a).$$

The state ρ defines a probability measure μ_ρ on Σ (Riesz theorem) and $\rho(a) = \int_\Sigma \widehat{a}(x)\, d\mu_\rho(x)$.

If the state ρ is not only $\mathbb{C}$-linear but also a morphism of *algebras*, then it is a point x of Σ, $\rho = \delta_x$.

Points $x \in \Sigma \Longleftrightarrow$ valuations $V_x : \mathfrak{A} \rightarrow \mathbb{C}$ defined by $V_x(a) = \widehat{a}(x)$.

Gleason theorem

If $\mathfrak{A}$ is no longer commutative, the states would a priori be only *quasi*-states $\rho : \mathfrak{A} \to \mathbb{C}$ which are $\mathbb{C}$-linear only on the commutative sub-C^*-algebra C of $\mathfrak{A}$.

But Gleason theorem (1957):
If $\mathcal{H}$ is a separable Hilbert space of dim ≥ 3 and if $\mathfrak{A} = \mathcal{B}(\mathcal{H})$ then every quasi-state is a state.

- In QM we need $\Omega \neq \{0, 1\}$.
- Quantum logic is not satisfactory.
- Isham's *et al.* strategy is to “localize” truth to *partial* and *contextual* valuations defined only on the commutative sub-C*-algebras *C*.
  - Philosophical relations with modal theory (Bas Van Fraassen, Richard Healey, Jeffrey Bub, etc.).
  - Mathematical relations with topos theory.

Let $\mathfrak{A}$ be the C^*-algebra describing the system.
Let $\mathcal{C}(\mathfrak{A})$ be the small category (poset) of *commutative* sub-C^*-algebra of $\mathfrak{A}$.
Let $\mathcal{T}(\mathfrak{A}) = Sh(\mathcal{C}(\mathfrak{A}))$ be the topos of sheaves over $\mathcal{C}(\mathfrak{A})$.

The natural topology of $\mathcal{C}(\mathfrak{A})$ is defined by the upper-sets U over C
(i.e. $U \subseteq C^{\uparrow}$ the maximal upper-set over C and if $D \in U$ and $D \subset E$ then $E \in U$).

If $\overline{F} \in \mathcal{T}(\mathfrak{A})$, it is equivalent to the *covariant* functor $F : \mathcal{C}(\mathfrak{A}) \to \mathbf{Ens}$ defined by $F(C) = \overline{F}\left(C^{\uparrow}\right)$.

Indeed, $C \subset D \Rightarrow D^{\uparrow} \subset C^{\uparrow}$ since

$$D \subset E \Rightarrow C \subset D \subset E \; i.e. \; C \subset E$$

The terminal object 1 of $\mathcal{T}(\mathfrak{A})$ is the constant functor.

Pullbacks are computed pointwise.

For the exponentials

$$F^G(C) = \mathrm{Nat}\left(G_{C^\uparrow}, F_{C^\uparrow}\right)$$
$$F^G(\mathbb{C}.1) = \mathrm{Nat}(G, F)$$

As the maximal upper-sets $C^\uparrow$ are sub-categories of $\mathcal{C}(\mathfrak{A})$, if $F \in \mathcal{T}(\mathfrak{A})$, we can consider the restrictions of F to the $C^\uparrow$. We note that $\mathbb{C}.1^\uparrow = \mathcal{C}(\mathfrak{A})$.

The subobject classifier is defined by

$$\Omega(C) \to \mathcal{U}_C = \{\text{upper sets } U \text{ over } C\}$$
$$\Omega(C \subset D) = \mathcal{U}_C \to \mathcal{U}_D, U \in \mathcal{U}_C \mapsto U \cap D^{\uparrow} \in \mathcal{U}_D$$

$$\Omega^F(C) \simeq \mathrm{Sub}(F_{C^{\uparrow}})$$
$$\Omega^F(\mathbb{C}.1) \simeq \mathrm{Sub}(F)$$
$$\Omega^F(C \subset D) = \text{ restriction of the } \mathrm{Sub}(F_{C^{\uparrow}}) \text{ to the } \mathrm{Sub}(F_{D^{\uparrow}})$$

The key point is that *inside* the topos $\mathcal{T}(\mathfrak{A})$, the *identity* functor $\underline{\mathfrak{A}}\colon \mathcal{C}(\mathfrak{A}) \to \mathbf{Ens}$, $C \mapsto C$, defines an *internal commutative C^*-algebra.*

$(\mathcal{T}(\mathfrak{A}), \underline{\mathfrak{A}})$ is called the *Bohr topos*.

As $\underline{\mathfrak{A}}$ is *commutative* we can apply Gelfand theory but *inside* $\mathcal{T}(\mathfrak{A})$.

We get an *internal* Gelfand spectrum $\underline{\Sigma}$ which synthetizes the Gelfand spectra Σ_C.

This NC "phase space" $\underline{\Sigma}$ can be described *externally* by a frame $\mathcal{O}(\underline{\Sigma})$ (distributive complete lattice) of "open sets", but this frame doesn't come from any classical topological space.

The interpretation of K-S in the Bohr topos

K-S means that the *internal* Gelfand spectrum $\underline{\Sigma}$ has *no* points (it is point free).

Indeed, a point is a morphism

$$\rho : 1 \to \underline{\Sigma}$$

(that is a morphism $\rho^{-1} : \mathcal{O}(\underline{\Sigma}) \to \Omega$).

Internally, observables a are elements of

$$\underline{\mathfrak{A}}_{s.a.} : C \to C_{s.a.}$$

If $a \in \underline{\mathfrak{A}}_{s.a.}$, its Gelfand transform is $\widehat{a} : \underline{\Sigma} \to \underline{\mathbb{R}}$ where $\underline{\mathbb{R}}$ is the constant sheaf $\mathbb{R}$.

$\widehat{a} \circ \rho : 1 \to \underline{\Sigma} \to \underline{\mathbb{R}}$ is a point of $\underline{\mathbb{R}}$.

We get a map

$$\underline{V}_\rho : \underline{\mathfrak{A}}_{s.a.} \to \mathrm{Pt}\left(\underline{\mathbb{R}}\right)$$

$$\underline{V}_\rho\left(C\right) = \underline{\mathfrak{A}}_{s.a.}\left(C\right) = C_{s.a.} \to \mathrm{Pt}\left(\underline{\mathbb{R}}\right)\left(C\right) = \mathbb{R}$$

One shows that $\underline{V}_\rho\left(C\right) = C_{s.a.} \to \mathbb{R}$ is a morphism of algebras and is therefore a *valuation*.
K-S implies it is impossible. Therefore a point ρ of $\underline{\Sigma}$ can't exist.

We can also say that $\underline{\Sigma}$ is a "fibration" $\underline{\pi}$ over $\mathcal{C}(\mathfrak{A})$ with fibers $\Sigma_C \to C$. A point $\rho : 1 \to \underline{\Sigma}$ is in fact a *gobal section* of $\underline{\pi}$ and K-S means that $\underline{\pi}$ has no global sections.

- Obstruction to ascribe truth values to judgements on numerical values of observables.
- It comes from the way logic is constrained by the framework-structures of a regional object.