Actuality of Transcendental Aesthetics for Modern Physics

in

1830-1930: Un siècle de géométrie, de C.F. Gauss et B. Riemann à H. Poincaré

et E. Cartan; épistémologie, histoire, et mathématiques

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1 Introduction

On 1935, 15-21 September, near the Institut Henri Poincaré, Hans Reichenbach opened at the Sorbonne the first *International Congress for the Unity of Science* – Congress he organized with Rougier, Carnap, Frank, and Neurath under the French title *Congrés International de Philosophie Scientifique*. He gave a celebrated keynote talk on the topic *L'empirisme logistique et la désagrégation de l'a priori* where he explained why logical empiricism constituted henceforth the only acceptable scientific philosophy. Concerning the concept of *a priori*, concept

> "qui englobe lui-même toutes les difficultés d'une théorie de la science empirique *et* mathématique",

he explained why and how

> "le développement de la science peut être considéré comme une décomposition constante des fondements du rationalisme et de l'*a priori*".

As is well known, his essential argument was that mathematical propositions being purely analytical (and then tautological) according to Russell's and Wittgenstein's thesis, they cannot provide any cognitive content for the empirical sciences:

> "tout ce que nous savons du monde est tiré de l'expérience, et les transformations des données empiriques sont purement analytiques".

Knowledge can grow only through induction. Consequently, any transcendental perspective had to be drastically banished. In spite of the disagreement of eminent philosophers and mathematicians such as Enriques, Cavaillès and Lautman, this dogmatic and militant point of view was accepted almost unanimously. Since these heroic times, modern epistemology has been dominated by these empiricist dogma, even if they have been progressively "liberalized" (cf. e.g. Quine's and Hempel's works). I want to take the view opposite to Reichenbach's verdict and explain why, contrary to accepted opinions, *the transcendental point of view about objectivity provides the effective physico-mathematical sciences with an adequate and natural epistemology.* In fact, I want to show that transcendental epistemology is nothing else than the rigorous philosophical thematization of a very special type of objective determination which is essential and pervasive in fundamental physics. Of course, such a thesis can be defended only if transcendentalism is no longer identified with its Kantian foundations. In the same manner as we can do axiomatics without confining ourselves to Euclide, or be empiricists without confining ourselves to Hume, we can be transcendentalists without confining ourselves to Kant.

2 Motivations

2.1 The lack of an objective epistemology

But why do we need again the *transcendental* concept of *a priori*? There is of course no necessity dictating any philosophical investigation about physics. Physics and Mathematics don't need epistemology. But *if* we try nevertheless to work out an epistemology, then our categorical imperative must be to work out a *plausible* one. I call an epistemology plausible if:

1. it concerns the *contents* – the internal, specific, technical and mathematical contents – of physical theories, and not exclusively the general structures (abstraction, induction, denotation, justification, verification, refutation, etc.) of physical knowledge;

2. it provides a philosophical concept of knowledge which is descriptively adequate for the *effective* structures of physical knowledge;

3. it leads to select as essential philosophical problems, problems which are also essential physical problems.

Now, it is easy to verify that the dominant trends in present-day epistemology are not plausible in this sense. I think that the reason for this strange state of affairs is that:

1. what we need for an epistemology to be plausible is a very particular and specific doctrine of *objectivity*, and
2. such a doctrine is not compatible with the tenets and the dogma of empiricism and positivism.

But why a plausible epistemology of physical objectivity had to be a transcendental one? How can we reinstate transcendentalism in its rights after more than seventy years of banishment? The reason is that there exist two main ways for epistemology:

1. on the one hand, the *objective* way which concerns for instance the question of the reality of mathematical entities (platonism, intuitionism, constructivism, etc.) or the question of the status of reality in Quantum Mechanics (strong objectivity VS weak objectivity, etc.);
2. on the other hand, the *cognitive* way which concerns the mental acts and cognitive processes which are characteristic of scientific activity (classification and categorization, abstraction, induction, denotation, relations between syntax and semantics, justification, etc.).

According to the positivist thesis, Kant's epistemology belonged to the cognitive way and the objective way was reducible to logics. According to the rationalist thesis, such a reduction is impossible. Now, the point is that, as we will see in a moment, the very nature of physical objectivity imposes drastic constraints to the objective way.

In fact, Kant was the first philosopher who has elaborated a plausible epistemology (for rational Mechanics). He discovered the *constitutive* nature of objectivity – "discovery of the constitutive element" which Hans Reichenbach himself acknowledged as an "eminent philosophical result" –: objective principles are *prescriptive* – not descriptive – principles which are *constitutive* of physical reality. But it is true that in Kant, this objective component was founded in a cognitive basis (representational theory, doctrine of faculties, etc.). As was stressed by Schlick: in Kant, the constitutive principles are characteristic of our representational consciousness. So, in Kant, a transcendental *subjectivism* became the foundational basis for the objective way.

Developments of physics (General Relativity and Quantum Mechanics) have made the objective component and the cognitive component enter in conflict. The result has been a major Krisis of epistemology. The positivist response to this Krisis has been to change the foundational basis of objective epistemology and to substitute transcendental subjectivism with a logical objectivism (Bolzano, Frege, Russell, etc.) radically overshadowing Kant's discovery of constitutive principles. But, in spite of its successive "liberalizations", logical positivism, and its project of "logical reconstruction" of reality, failed. As we will see, the reason for this failure is that, in physics, the relation of mathematics to empirical reality is *not* analogous with the relation of a "language" to an empirical "world". It is not a denotative relation between a symbolic syntax and an empirical semantics. For this reason, logicism cannot serve as a foundational basis for objective epistemology.

Two consequences have followed from such a failure:

1. the sceptic dissolution of objective epistemology (methodological anarchism, consensual, hence sociological, theory of truth, thesis of the unreachability of objective reality, etc.);
2. the fall back of the hard core of epistemology on purely cognitive epistemology (AI models, expert systems, computational Baconism, etc.).

In this context it is a categorical imperative to support objectivity. So, we want to rectify the positivist orientation and argue that there existed an alternative response to the main conflict opposing the objective and the cognitive components of transcendental epistemology. Logical positivism reduced, as we have seen, Kant's constitutive perspective (and in particular the synthetic a priori) to a cognitive innatism, a logical necessity or a consciousness of evidence. It is this reduction that led it to give up the synthetic a priori constitution of objectivity. But there existed another response to the main conflict. It was to methodologically *disjoin* the objective and the cognitive components *and to ask whether the objective component was correct for modern physics*. Such a methodological revival of transcendentalism was initiated by the neo-Kantian School (the Marburg School : Herman Cohen, Paul Natorp, and over all Ernst Cassirer). But, due to well known political diseases, this revival was not carried through. The conclusion of this complex story has been the lack of a truly philosophically based objective epistemology, at least in the anglo-saxon world and in France. In

Germany, the situation is different, due in particular to Kantian tradition and Weizsäcker's school.[1]

2.2 The need for an objective epistemology

We need an objective epistemology if we want to work out a plausible one. Consider for example the discussion about quantum reality. In Quantum Mechanics (QM) there exists a non marginal and a non eliminable interaction between the microphysical objects and the measure apparatus. This basic physical fact implies the basic philosophical fact that *physical objectivity cannot be an ontology*. Physics cannot describe any ontological reality. It can only objectively determine phenomena. As was stressed by many physicists (from Bohr and Heisenberg to B. d'Espagnat and G. Cohen-Tannoudji, see also the celebrated Max Jammer's *The Philosophy of Quantum Mechanics* [2]), physical theories cannot refer to a reality "an sich" but only to phenomena. As G. Cohen-Tannoudji writes:

> "Toute la problématique de la physique des particules est d'atteindre l'objectivité scientifique (matérialiste) malgré le caractère insécable de l'interaction entre l'objet et l'appareil de mesure".[3]

In QM, the experimental apparatus allow the manifestation (the "phenomenalization" process) of microphysical phenomena. This implies the disjunction between these phenomena and any underlying reality: microphysical elementary phenomena lack any underlying ontology.

> "C'est cette nouvelle conception des phénomènes qui est peut-être l'innovation la plus importante apportée par la théorie quantique. Les concepts quantiques ne se rapportent plus à l'objet en soi, mais ils se rapportent à des phénomènes" [4].

But how is it then possible to reach objectivity? What are the conditions of possibility of physical objectivity? In a same vein, B. d'Espagnat

[1]See e.g. Scheibe [1981]. See also works as those of H.J. Folse "Kantian Aspects of Complementarity" (Folse [1978]) or J. Honner "The Transcendental Philosophy of Niels Bohr" (Honner [1982]).
[2]Jammer [1974].
[3]Cohen-Tannoudji and Spiro [1986], pp. 80-81.
[4]Ibid., p. 141.

opposes two concepts of reality: the reality "an sich", which is an independent ontological reality, and the observable reality, which is empirical and phenomenal.

> "Ma thèse est que c'est la science elle-même qui – s'autocorrigeant sur ce point – fournit aujourd'hui au penseur de pressantes raisons d'accepter la dualité (philosophique) de l'être et du phénomène".[5]

We must go beyond the dogmatic conflict opposing, on the one hand, the "physicalist a priori" according to which physics possesses an ontological content, and, on the other hand, the "mentalist a priori" according to which physics is a mere cognitive construction. So, objective physical reality is by no means an independent ontological reality. But it is nevertheless a true objectivity. Physics is undoubtedly much more than a simple nominalistic systematization of phenomenal appearances. But what can then be the philosophical status of such an objectivity, which is neither an ontology nor a mere conceptual systematization? This problem is exactly the transcendental one. To solve it we need at least three things.

A non empirical definition of phenomena

As says G. Cohen-Tannoudji, microphysical phenomena are "horizons of reality". According to the non commutativity of the algebras of observables (Heisenberg's relations) there exist absolute limits imposed upon the epistemic accessibility of physical reality. These limits are absolute but not rigid. They constitute an horizon of discernibility, of predictability and of reality.

The integration of the horizon status of phenomena into the content of theoretical objectivizing concepts (system, state, causality, interaction, etc.)

This constraint is, in fact, a tremendous one. Actually, the content of theoretical concepts must be *mathematically* interpreted: in physics, there exists a sort of mathematical "hermeneutics" which constitutes the basis of the theories. Now, the constraint is that this mathematical interpretation must derive from an adequate mathematical formulation of the horizon status of phenomena.

[5] D'Espagnat [1985], p. VII.

The relativization of the concepts of truth and existence, and also of the modal categories of Possibility, Reality and Necessity, to such a process of objectivization

The ontological (metaphysical, dogmatic) use of these modalities leads unavoidably to *dialectical antinomies*. See for instance the typically dialectical debate between Bohr and Einstein, and, more generally, the debate about hidden variables (from von Neumann's theorem to Aspect's results): as it was stressed by Jerry Marsden:

> "Indeed, the entire point of the negative results concerning 'hidden variables' is that there is no 'objective underlying state' [in the sense of an ontological reality behind phenomena] of the system!".[6]

Then, we can say that QM has conquered a new concept of objectivity which is in fact of a constitutive and critical nature. "Weak" objectivity (as opposed to the "strong" objectivity of an independent reality) is no more than objectivity in the transcendental sense! As was claimed by Weizsäcker:

> "Bohr was essentially right, but he himself did not know the reasons why".[7]

These reasons are transcendental. Someone has spoken with humour of the "Königsberg's interpretation" of QM ...

We will also see (sections IV to VI) that the symmetry groups of relativity allow to make the mathematical descriptions independent of the conventional epistemic elements (coordinate frames, etc.) which overdetermine them. But these groups possess a rich geometry and this geometry becomes a *determining* component of the physical models. The fact that epistemic constraints on mathematical descriptions can become constitutive and determining for the explanation of physical contents implies that physical objectivity cannot be thought of as a relation between a cognitive epistemology and a substantialist ontology of things, events and states of affair. We need therefore a specific and adequate objective epistemology. The relevance of the transcendental approach is to provide the philosophical tools which allow us to apply the traditional philosophical concepts of reality, truth, necessity, essence/existence,

[6] Marsden [1974], p. 187.
[7] Weizsäcker [1979], p. 229.

a priori/a posteriori, substance, causality, etc. to this very special type of objective determination. We must convince ourselves that these concepts do not possess a scientific meaning antecedent to strategies of objective determination. It is only in their dogmatic metaphysical applications that they can operate in an absolute manner. But in science, they are always relative to a previously constructed concept of objectivity. Transcendental philosophy can be defined as the philosophical thematization of this specific type of relativity.

3 The actualization of transcendental philosophy

Our purpose here is not to comment Kant himself. But let us nevertheless emphasize the main points of an actualization of transcendental philosophy.

3.1 The uncoupling of the objective component of transcendental epistemology from its foundational cognitive basis

We think that transcendental epistemology is essentially correct for physics, at least in what concerns its general moments (Transcendental Aesthetics, Categories, Principles, etc.). This adequacy does not need to be rooted in any transcendental subjectivism, for exemple in a cognitive innatism. As we will see, the *synthetic a priori* does not need such a psychological basis.

3.2 The analogy between the axiomatic method and the transcendental method

We think that the transcendental method is analogous, for the explanation in objective sciences, to the axiomatic method of implicit definitions in semantic analysis. Let us make this parallel more explicit.

3.2.1 The axiomatic method

In semantic analysis, what we seek is a *definition* of concepts. There exist many levels of concepts, hierarchically organized, and definitions allow the reduction of higher levels to lower ones. This definitional reduction leads necessarily to *primitive* concepts (of lowest rank) which are undefinable. The main problem is then to go beyond this limit of definability. We meet here an alternative.

1. We can admit the existence of innate primitive concepts which are originally given. This is a classical metaphysical trend.

2. But we can also change our strategy in a radical manner and substitute to the undefinable content of primitive concepts *syntactic rules* which determine normatively their use. It is such "implicit definitions" which are provided by the axiomatic method.

3.2.2 The transcendental method

In objective sciences, what we seek is an "ontological" explanation of the empirical phenomena. There exist many levels of reality, hierarchically organized, and "ontological" explanations allow the reduction of higher level to lower ones. Lower levels operate as a foundation – a ground of reality and of explicative causality – for the higher ones. In that sense, causal reductionism is perfectly justified. The consequence of this is that at the higher levels, the underlying being ("l'être", "das Sein", the "esse") and the observed phenomena are unified in a substantial "ontology" of things, events and states of affairs. But this explicative reduction leads necessarily to a fundamental level which is no longer explainable using such a strategy. At the fundamental level, there exists a disjunction between the underlying being and the observed phenomena. The underlying being becomes a reality "an sich", a pure ground of reality *which lacks any causal explicative power*. The main problem is then to go beyond this limit of explanation. We meet here an alternative.

1. We can admit fundamental elementary phenomena which are irreducible data originally given and which support as such all the causal explicative power of physical theories. Such a phenomenalist empiricism leads to a conception wich is essentially "nominalistic".

2. But we can also change our strategy in a radical manner. We can elaborate an epistemology which starts from the disjunction "underlying being VS phenomenon" and concerns the conditions of possibility of

objectively determining phenomena with no underlying ontology. This is the transcendental strategy.

We can therefore say that transcendental objective epistemology is required for fundamental physics of elementary phenomena.

3.3 The evaluation and the rectification of the standard positivist critiques against Kantian doctrine

We give very briefly some examples.

(a) The confusion between the Kantian opposition between phenomenon and noumenon and the opposition between psychological internal representations and ontologically independent (external) entities. We agree with Henry Allison's and Gerold Prauss' "two aspect vew" for asserting that transcendentalism is methodological and concerns the epistemological conditions of objectivation, conditions which are neither internal (psychological) nor external (ontological).

(b) The thesis that Euclidean geometry is *cognitively* a priori. Kant admitted the logical possibility of non Euclidean geometries. For him, the primacy of Euclidean geometry derived from the inertia principle.[8] The very real point is not this one but that in physics the mathematical "hermeneutics" of theoretical concepts must be geometrical.

(c) The confusion between transcendental necessity and logical necessity. In Kant, necessity – the crowning category of Transcendental Analytic – is relative to the radical *contingency* of possible experience (third Postulate of empirical thinking: "Dessen Zusammenhang mit dem Wirklichen nach allgemeinen Bedingungen der Erfahrung bestimmt ist, ist (existiert) notwendig.").

(d) The thesis that the synthetic a priori characterizes some class of propositions. It is false (see Quine). Synthetic judgments are not some special sort of propositions. The synthetic a priori is a *function* in an objective process of constitution (see Cassirer).

(e) The claim that axiomatics renders geometry purely analytic and that, consequently, the synthetic a priori does not exist. As was stressed by Michael Friedman in his *Kant's Theory of Geometry* [9], the fundamental properties of

[8]See for instance J.E. Wiredu "Kant's Synthetic a priori in Geometry and the Rise of non Euclidean Geometries" (Wiredu [1970]), and R.J. Gomez "Beltrami's Kantian View of Non-Euclidean Geometry" (Gomez [1986]).

[9]Friedman [1985]. See also Brittan [1991]

the continuum are not expressible in a monadic first order logic such as that used by Kant.

> "So, for Kant, one cannot represent or capture the idea of infinity formally or conceptually"

and then there exists a clear distinction between "intuitive" and "discursive". Of course, with modern logic we can express purely logically "intuitive" properties like the property of density:

$$\forall x \forall y (x < y) \Rightarrow \exists z (x < z < y)$$

or the property of completion of $\mathbb{R}$ (convergence of Cauchy sequences):

$$\forall \varepsilon \exists N \forall m \forall n \, (m, n > N \Rightarrow |s_m - s_n| < \varepsilon) \Rightarrow \exists s \forall \varepsilon \exists N \forall n \, (n > N \Rightarrow |s_n - s| < \varepsilon)$$

But it would be dogmatic to conclude that the heterogeneity between intuition and discursivity has disappeared. Actually:

1. such quantifications on infinite sets raise all the difficult problems emphasized by Brouwer, Weyl or Wittgenstein;

2. non standard Analysis (from Veronese to Nelson, Reeb and Harthong) has shown that it is impossible to completely determine the continuum in a logical analytic manner.

This basic fact provides the modern sense of Kantian intuition. As was emphasized by the second Gödel, the theorems of limitation (Löwenheim-Skolem, incompleteness, indecidability, etc.) show that the continuum is an objective informationally infinite reality which transcends its symbolic (logical analytic) determination. For the continuum to possess a "good" structure in a ZFC-model of set theory, we must introduce new axioms, and in particular existence axioms for very large cardinal (measurable cardinals, etc.). As Gödel said, these axioms are like "physical hypothesis". They concern an "external" and "intuitively given" reality, and not a conceptual one.[10]

[10] See Petitot [1989], [1991a], [1991b]. For philosophical comments about non standard Analysis, see Salanskis [1989], [1991].

3.4 The actualization of the main constitutive moments of the transcendental doctrine

3.4.1 Transcendental Aesthetics

If phenomena are not ontologically based, how can we avoid psychologizing them as mere appearances? The answer is: *Transcendental Aesthetics*. The essential relevance of transcendental arguments is not of a "syntactic" nature. It does not concern the trivial fact that a linguistic framework is an a priori necessity for talking about knowledge and experience, but above all Transcendental Aesthetics. In Kant's *Kritik der reinen Vernunft* (KRV) Transcendental Aesthetics contains two "expositions" (*Erörterung* = clear representation of what belongs to a concept): the metaphysical one and the transcendental one.

The "metaphysical" exposition explains that phenomena appear in a specific medium of manifestation (space and time for sensible phenomena). This phenomenological medium provides *forms of intuition*. "Intuition" (Anschauung) has nothing to do here with a cognitive evidence (as in Descartes). It is not an intuitive knowledge. It expresses the fact that phenomena (*Erscheinungen*) are *given* in some receptive way as the trace of some unknown – and unknowable – transcendent reality, and that there do exist forms of this receptivity. We meet here the fundamental Kantian distinction between "gegeben" and "gedacht". Space is an actual infinite which is originally given. Its function is to "depsychologize" the psychological content of the concept of phenomenon. In his article "The Metaphysical Exposition: An Analysis of the Concept of Space" [11] P.M. Mc Goldrick has analysed the status of space as a *singular* concept possessing a reference which is unique in all its genus. As the universal substratum of sensible intuitions, the concept of space is neither a category subsuming empirical diversity under the unity of aperception nor an empirical (classificatory) concept. It cannot be abstracted from sensible experience. Using the Kripkean concept of "rigid designator", we can say that space is a rigid designator, "a term which designates the same object in all possible worlds". So, if space was equivalent to a Leibnizian order of co-existence between external objects, this equivalence would have to be necessary and, consequently, the concept of space would imply the concept of object. But this is false. Therefore, space is effectively an intuition. This argument agree with Gordon Brittan's definition of the

[11] McGoldrick [1985].

synthetic a priori as "true in all real possible worlds" (as opposed to "true in the actual real world" and "true in all logical possible worlds").[12] In that sense, the synthetic a priori concerns effectively the conditions for assertions about the actual world to be true or false.

The "transcendental" exposition concerns the *mathematical* determination of the forms of manifestation that is to say the conversion of pure intuitions into what Kant called *formal intuitions* (see the celebrated note of the §26 of the KRV). The main fact is that such a mathematical determination discloses an absolute limit of the conceptual description in physics, linked to the existence of *symmetries* (relativity) (Kant was the first philosopher who has philosophically understood and formulated the basic role of symmetries and relativity in physical theories). If one imposes cognitive foundations to objective epistemology, one is necessarily led to the conclusion that transcendental exposition must be cognitively determined and, therefore, that the mathematical determination of pure intuition must be univocal. And this is false. But if one frees objective epistemology from its cognitive basis, then one can easily accept that the mathematical determinability is actually *underdetermined* and depends historically upon the mathematical possibilities of determination. In that sense *conventionalism* (in Poincaré's sense) possesses a deep transcendental meaning and concerns the transcendental ideality of the forms of intuition.

3.4.2 Transcendental Analytic

The possibility of philosophically working out a concept of objectivity wich is neither an ontology nor a mere systematization of empirical data, relies upon the "Copernican revolution" – the critical "turning point" – that converts a descriptive conception of objectivity into a prescriptive one (it is essentially the same critical "turning point" that was accomplished by the second Wittgenstein for the symbolic objectivity of mathematics). Even if phenomena do not possess an underlying ontology, they can nevertheless be *legalized*. Objectivity is legality of experience. It is a juridical concept. But, for legalizing phenomena we need normative, prescriptive, "grammatical" (in the Wittgensteinian sense) concepts. We need forms of legality. These are provided by the categorial system that is to say by a *transcendental Analytic*. The main categories define positions (*Setzung*) of existence.

[12]Brittan [1978].

3.4.3 Transcendental Schematism and the construction of categories

But a position of existence is not still an objective determination. To objectively determine phenomena, we need a link between forms of manifestation (what is "gegeben") and forms of legality (what is "gedacht"). At the level of the KRV such a link is provided by the transcendental schematism which converts the categories into principles (*Grundsätze*). But in fact, the true link is given by what Kant called the *construction* ("*Konstruktion*") of categories. The construction is a mode of presentation ("*Darstellung*"). It means that one can interpret mathematically the schematized categorial contents using mathematics coming from the transcendental exposition of Transcendental Aesthetics. We think that it is in this very special sort of "mathematical hermeneutics*"* – not only for the forms of manifestation but also for the forms of legality themselves – that the synthetic a priori finds its true and deep transcendental meaning. We agree with Henry Allison's and Gerold Prauss' interpretation [13] according to which:

1. transcendental schematism must be not only temporal but also spatial (in fact spatio-temporal) and that it is much richer than Kant believed,

2. synthetic a priori judgments refer to transcendental schemata.

When they are so constructed, the categories and principles of the Analytic become *generators* of mathematical models for the empirical phenomena. So the "construction" unifies:

1. forms of manifestation, that is to say intuitive modes of phenomenological givenness;

2. forms of legality, that is to say categorial modes of existential position ;

3. basis for mathematical modelling.

This point is scientifically *and* philosophically fundamental. To understand the implication of mathematics in physical reality, we must avoid two dead ends.

[13] See for instance Allison [1981], [1983] and Prauss [1980], [1981a], [1981b].

1. A purely denotative point of view according to which mathematics provide a syntax and empirical phenomena provide a semantics. The relation between syntax and semantics in logical model theory cannot be used as an analogy for the relation between mathematics and the world.

2. An obscure Platonist "participation" of mathematics to reality.

We can avoid these two dead ends if we note:

1. that mathematics determine not the reality itself but *forms* of reality and that these forms are *ideal* components of reality;

2. that mathematics allow also a constructive interpretation of the categories of existence;

3. that through such a mathematical "hermeneutics", we can construct diversified and falsifiable models of empirical phenomena.

Our hypothesis is that it is exactly this type of strategy of determination which has become more and more systematic, sophisticated and pervasive in modern physics, even if it is not philosophically thematized by physicists themselves. It is clear that it is not compatible with a positivist (logical, predicative and denotative) conception.

4 The "Metaphysische Anfangsgründe der Naturwissenschaft"

In the *Anfangsgründe*, Kant applies his transcendental point of vue to newtonian Mechanics (that is to say to the regional object "motion" or "trajectory").[14] In modern language his interpretation can be summarized in the following manner. We must remind here the fundamental difference between the "mathematical" categories concerning *essence* and the "dynamical" categories concerning *existence*.[15]

[14]See for instance Howard Duncan "Inertia, the Communication of Motion, and Kant's Third Law of Mechanics" (Duncan [1984]).

[15]See for instance Jules Vuillemin's classical "Physique et Métaphysique kantiennes" (Vuillemin [1955]).

4.1 Phoronomy (Kinematics)

Phoronomy applies the "mathematical" categories of quantity and the principles called "Axioms of Intuition" (*Axiomen der Anschauung*) governing the "extensive" magnitudes. It concerns:

1. The problem of the *metric* of space (the link between the informal phenomenological space of the metaphysical exposition and the measure of distances by numbers). It clarifies the link between the transcendental exposition and the conversion of space as form of intuition into a formal intuition (determined as object). It explains also the inertia principle (geodesics are straight lines).

2. The Galilean relativity and the symmetry group which expresses it (of course the concept of group was not known to Kant).

4.2 Dynamics

Dynamics applies the "mathematical" categories of quality and the principles called "Anticipations of Perception" (*Antizipationen der Wahrnehmung*) governing the "intensive" magnitudes. It concerns:

1. The debate with Leibniz and the metaphysical tradition concerning the "substantial" *interiority* of matter. For Kant, such an interiority is "noumenal" (*an sich*) and therefore must be excluded from Mechanics (but this very difficult problem is reopened in the *Opus Postumum* where Kant speculates about the system of the fundamental primitive internal forces governing the derivative mechanical forces and, therefore, the external movements of matter).

2. In space-time, matter expresses itself through intensive magnitudes such as velocity and acceleration. These intensive magnitudes are *differential* data. And to be compatible with Phoronomy, they must possess an intrinsic meaning, that is to say they must be covariant. Kant was also the first philosopher who understood clearly – even if it was at an elementary level – the constitutive role of invariance and covariance principles in physical objectivity.

4.3 Mechanics

Mechanics applies the "dynamical" (physical) categories of relation (substance = *Inhärenz und Subsistenz*, causality = *Causalität und Dependenz*, community, reciprocity and interaction = *Gemeinschaft*) and the principles called "Analogies of Experience" (*Analogien der Erfahrung*). We meet here the true sense of the critical turning point: the separation of physical objectivity from any substantial ontology. The *construction* of the category of substance (schematized as transcendental principle of temporal permanence: "*die Beharrlichkeit des Realen in der Zeit*") leads to its complete reinterpretation: substance is the general principle underlying the *conservation laws* of physical magnitudes. We see very well here why physics cannot be a predicative description of some empirical states of affairs. We see also how a priori principles can become generators of a great diversity of mathematical models: covariance principles and conservation laws express themselves through fundamental *equations* whose *solutions* are models for phenomena. In Mechanics, causality – i.e. the *force* – is expressed by Newton's law and reciprocity by the law of equality of action and reaction.

4.4 Phenomenology

Phenomenology applies the categories of modality and the principles called "Postulates of empirical thinking" (*Postulate des empirischen Denkens überhaupt*). According to Galilean relativity, motion cannot be a real (*wirklich*) predicate of matter. It is only a *possible* (*möglich*) one. Absolute spatial and temporal positions or absolute velocities are not observable. It is for this reason that the "empirical realism" of space and time cannot be ontologically strengthened up to a "transcendental realism". The celebrated "transcendental ideality" of space and time is no more than the correct philosophical formulation of the fact that principles of relativity are, and must be, constitutive of physical objectivity. But forces (causality) are real and are governed by necessary laws. Necessity is not here a logical modality but a transcendental one. It is a conditional necessity, relative to the radical contingency of possible experience.

The *Anfangsgründe* show that the transcendental structure of rational Mechanics – as a science determining the regional object "trajectory" – *coincides* with its fundamental physical characteristics and problems. It is in that sense that Kantian objective epistemology is plausible. Our research program is to work out the equivalent of the *Anfangsgründe* for *modern* physical theories. Our leading idea is that there exists some historical permanence of the transcendental structure of physical objectivity, even if the mathematical and physical contents of the transcendental moments develop themselves historically. As we have seen, transcendental moments are in fact "hermeneutical" ones and this mathematical hermeneutics is of course an historical one. We will now give some examples.

5 Noether's theorem and symplectic Mechanics

In classical Mechanics, an extraordinary confirmation and deepening of the Kantian link between Transcendental Aesthetics and the reinterpretation of substance by conservation laws is provided by the link between relativity groups, symmetries and conservation laws. In that sense Nœther's theorem possesses an eminent transcendental meaning. In fact, we think that the direct legacy of Transcendental Aesthetics is to be found:

1. in the constitutive role of global geometrical symmetry groups (see the works of Lie, Poincaré, Cartan, Weyl) [16];
2. in Noether's theorem ;
3. more generally in the possibility of reducing conservation laws to purely geometrical a priori; this possibility of reduction provides one of the true modern physical meanings of the synthetic a priori.

5.1 Noether's theorem

5.1.1 Lagrangian version

Let S be a mechanical system, M its configuration space (M is a differentiable manifold with tangent bundle TM) and $L : TM \to \mathbb{R}$ its Lagrangian. The action integral is

$$\phi(\gamma) = \int_{t_0}^{t_1} L(q, \dot{q})dt$$

[16] For a transcendental approach of these fundamental geometric traditions, see for instance Boi [1989].

γ being a path from (q_0, t_0) to (q_1, t_1). The least action principle $\delta\phi = 0$ implies the well known Euler-Lagrange equations :

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0$$

which are Newton's equations $\dot{p} = \partial L/\partial q$ for the generalized moments $p = \partial L/\partial \dot{q}$ and the generalized forces $\partial L/\partial q$. Let φ_s be a one parameter subgroup of diffeomorphisms of M (i.e. a flow) and suppose that the φ_s are symmetries of L, that is to say that L is φ_s-invariant for all the φ_s in the sense that $L(v) = L(D_q\varphi_s(v))$ for every tangent vector $v \in T_qM$ (T_qM is the tangent vector space of M at q and $D_q\varphi_s$ is the tangent linear map of φ_s at q). Then Noether's theorem asserts that S admits *a law of conservation* (a first integral):

$$I : TM \to \mathbb{R},\ (q, \dot{q}) \mapsto I(q, \dot{q}) = \frac{\partial L}{\partial \dot{q}} \cdot \left.\frac{d\varphi_s\,(q)}{ds}\right|_{s=0} = p.X_\varphi(q)$$

where X_φ is the vector field generating the flow φ_s.

5.1.2 Hamiltonian version

Let $H : T^*M \to \mathbb{R}$ be the Hamiltonian of S, where T^*M is the phase space of S, that is the cotangent bundle of the configuration space M. T^*M is a *symplectic* manifold endowed with the canonical (closed and non degenerate) 2-form $\omega = \omega_0$ (in canonical local coordinates q for M and p for T_q^*M, ω_0 has the Darboux form $dp \wedge dq$). Let $F : T^*M \to \mathbb{R}$ be an observable. The Hamiltonian field X_F generated by F is defined by $i_{X_F}\omega = dF$, where $i_X\omega$ is the 1-form derived from the 2-form ω by contraction with the field X. Suppose that H is invariant under the flow $\exp(tX_F)$ generated by X_F. Then F is a first integral of the hamiltonian flow X_H, that is to say the Poisson bracket of F and H vanishes:

$$\{F, H\} = \omega(X_F, X_H) = i_{X_F} i_{X_H}\omega = 0.$$

The three classical examples of Noether's theorem are provided by the geometrical relativity group of classical Mechanics. They are summarized in the following table:

Non observability of some absolute entities	*Symmetry groups*	*Conservation laws*
Origin of time	Temporal translations	Energy
Origin of space	Spatial translations	Momentum
Privileged direction	Rotations	Angular momentum

The transcendental meaning of Noether's theorem is to prove the correlation between two things:

1. some mathematical (non physical, non observable) entities which are necessary for the mathematical description of physical phenomena: these entities are conventional epistemic elements submitted to relativity principles (transcendental ideality),

2. physical magnitudes which are observable and measurable and constitute the "substantial" content of the theory.

This correlation between the impossibility of observing certain absolute entities and the possibility of observing conserved correlated magnitudes is mediated here by the Lagrangian or the Hamiltonian. But it is clear that its philosophical meaning is typically transcendental. As Gilles Cohen-Tannoudji says,

> "la relativité signifie l'impossibilité d'effectuer des mesures *absolues*"

and Noether's theorem establishes

> "la correspondance entre *relativité* (non observabilité de certaines entités absolues), *symétries* (invariance par transformation de symétrie) et *lois de conservation* (conservation et donc observabilité de certaines quantités)".[17]

Actually this confers its true physical meaning to Transcendental Aesthetics. A phenomenon is something which is conditioned by forms of manifestation. Therefore it is a *relational* entity relative to its means of observation. To describe it, we must mathematically determine the forms of manifestation.

[17]Cohen-Tannoudji, Spiro [1986], pp. 106-107.

But this introduces some conventional (epistemic) elements. Objectivity essentially means that we can eliminate these over-determining elements. But such an elimination is expressed by new mathematical structures (in particular by symmetry groups). These structures are therefore constitutive of objectivity.

5.2 The Momentum map

In fact the transcendental nature of Noether's theorem can be mathematically deepened using the works of B. Kostant, J.M. Souriau, V. Arnold, A. Weinstein, R. Abraham, J. Marsden and many other symplectic geometers.[18] With the *momentum map* (or moment map) formalism, it becomes possible to deduce *directly* first integrals from relativity groups even without specifying any Lagrangian. The phase space T^*M of S is a symplectic manifold (P, ω). Let G be a Lie group acting symplectically upon P via an action $\Phi : G \times P \to P$ such that $\Phi_g(x)$ belongs to the symplectic group $\mathrm{Sympl}(P, \omega)$ for each $g \in G$. There are essentially two manners for generating vector fields on P: from symmetries and from observables.

1. Let $\mathfrak{g} = T_eG$ be the Lie algebra of G and let $\xi \in \mathfrak{g}$. Through the action Φ, the one parameter subgroup $\exp(t\xi)$ acts upon P as the flow $\varphi_\xi(t) = \Phi_{\exp(t\xi)}$. Let $\xi_P = \frac{d}{dt}\varphi_\xi(t)|_{t=0}$ be the infinitesimal generator of the flow $\varphi_\xi(t)$. $\xi_P \in \mathcal{X}(P)$, $\mathcal{X}(P)$ being the algebra of vector fields on P, and $\xi \mapsto \xi_P$ is a morphism of Lie algebras $\mathfrak{g} \to \mathcal{X}(P)$.

2. Let $F : P \to \mathbb{R}$ be an observable. F generates the hamiltonian field X_F such that $i_{X_F}\omega = dF$.

The idea is then to associate with every $\xi \in \mathfrak{g}$ – in an automatic and natural way – an observable F_ξ such that :

- $X_{F_\xi} = \xi_P$,

- the F_ξ are first integrals for every hamiltonian H which is G-invariant (i.e. for which $H(x) = H(\Phi_g(x)),\ \forall g \in G$).

[18] See for instance Abraham-Marsden [1978], Arnold [1976], Souriau [1975], Weinstein [1977]

Let us suppose that such an association exists. If we fix some $x \in P$, we can consider the system $J(x)$ of the values $F_\xi(x)$ (x constant, ξ variable) of the first integrals F_ξ. $J(x)$ is a *linear form* on $\mathfrak{g}$, i.e. an element of the dual $\mathfrak{g}^*$. So we get a map – *the momentum map*:

$$\begin{aligned} J: \quad & P \to \mathfrak{g}^* \\ & x \mapsto J(x): \quad \mathfrak{g} \to \mathbb{R} \\ & \qquad\qquad\qquad \xi \mapsto J(x)(\xi) = F_\xi(x) \end{aligned}$$

Noether's theorem. If $H : P \to \mathbb{R}$ is G-invariant, J is a set of first integrals for X_H.
Remark. If $(P, \omega) = (T^*M, \omega_0)$ and G acts *upon* M, then we have

$$\begin{aligned} J: \quad & T^*M && \to \mathfrak{g}^* \\ & \alpha \in T_q^*M && \mapsto J(\alpha) : \xi \to J(\alpha)(\xi) = F_\xi(\alpha) = \alpha(\xi_M(q)). \end{aligned}$$

We have therefore $F_\xi = P(\xi_M)$ where the map $P(X)$ is defined for $X \in \mathcal{X}(M)$ by

$$\begin{aligned} P(X): \quad & T_q^*M && \to \mathbb{R} \\ & \alpha \in T_q^*M && \mapsto \alpha(X(q)). \end{aligned}$$

5.3 The symplectic structure of the orbits of the co-adjoint representation

We have just seen that first integrals can be directly associated to relativity groups. But the synthetic a priori reduction of the conservations laws can be taken even further. In fact, as was shown by Kirillov, Souriau, Kostant, Weinstein and Arnold, it is possible to apply this construction to the Lie groups themselves and to deduce Hamiltonian systems *directly from the symmetries.*
Let G be a Lie group. G acts upon itself by left translations $L_g : h \to gh$. Let us consider the adjoint and co-adjoint representations of G. The adjoint representation is given by:

$$\begin{aligned} Ad: \quad & G \to \mathrm{Aut}\,\mathfrak{g} \\ & g \mapsto Ad_g = T_e(R \circ L_g) \end{aligned}$$

where $R \circ L_g$ is the inner automorphism of G associated with g. It is the infinitesimal expression of the *non commutativity* of G. If we take the linear tangent map T_eAd of Ad at e, we get the map :

$$\begin{aligned} ad: \quad & \mathfrak{g} \to \mathfrak{g} \\ & \xi \mapsto ad_\xi : \eta \mapsto ad_\xi(\eta) = [\xi, \eta] \end{aligned}$$

which returns the Lie bracket of G.

The co-adjoint representation is given in a dual manner by:

$$\begin{aligned} Ad^* : \quad & G \longrightarrow \mathrm{Aut}\,\mathfrak{g}^* \\ & g \mapsto Ad^*_g : \alpha \mapsto Ad^*_g(\alpha) : \eta \mapsto Ad^*_g(\alpha)(\eta) = \alpha(Ad_g \eta). \end{aligned}$$

If we take the linear tangent map $T_e Ad^*$, we get the map:

$$\begin{aligned} ad^* : \quad & \mathfrak{g} \rightarrow \mathfrak{g}^* \\ & \xi \mapsto ad^*_\xi : \alpha \mapsto ad^*_\xi(\alpha) : \eta \mapsto ad^*_\xi(\alpha)(\eta) = \alpha(ad_\xi(\eta)) = \alpha([\xi, \eta]). \end{aligned}$$

To go further, we need the concept of a *Poisson structure* (A. Weinstein). If (P, ω) is a symplectic manifold, the Poisson bracket:

$$\{f, g\} = \omega(X_f, X_g) = \sum_i \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right)$$

defines a Poisson structure (PS) on P. This means that $\{f, g\}$ is a Lie bracket on the algebra $\mathcal{F}(P)$ of differentiable functions on P, which is related to the algebra product in $\mathcal{F}(P)$ by a derivation law $\{fg, h\} = f\{g, h\} + \{f, h\}g$. These particular PS are called symplectic. It is possible to reformulate all Hamiltonian Mechanics only in terms of PS. If $H \in \mathcal{F}(P)$ is an Hamiltonian on P, the hamiltonian field X_H (considered as a derivation on $\mathcal{F}(P)$) is defined by $X_H(f) = \{f, H\}$. Likewise, if $\theta : \mathfrak{g} \rightarrow \mathcal{F}(P)$ is a morphism of Lie algebras, the momentum map is given by

$$\begin{aligned} J : \quad & P \rightarrow \mathfrak{g}^* \\ & x \mapsto J(x) : \xi \mapsto J(x)(\xi) = \theta(\xi)(x) \end{aligned}$$

Let us then consider the commutative diagram of fibrations:

$$\begin{array}{ccccc} & T^*P & \xrightarrow{\Pi} & TP & \\ {\scriptstyle dH} & \uparrow\downarrow & & \downarrow\uparrow & {\scriptstyle X_H} \\ & P & \xrightarrow{Id_P} & P & \end{array}$$

where Π is the morphism defined by $X_H = \Pi \circ dH$. The symplectic PS are *non degenerate* in the sense that Π is an *isomorphism* (of fibrations). This is not the case in general. Nevertheless, in general, the image of Π will be an *integrable foliation* S on P. It can be shown that the leaves $S(x)$ of S are symplectic manifolds and that the canonical inclusions $S(x) \hookrightarrow P$ respect the PS. Weinstein's theorem asserts that any PS is *locally* the direct product of a symplectic PS and a "transversal" PS which is tangent to the PS of a Lie algebra $\mathfrak{g}^*$. The PS on the Lie algebras $\mathfrak{g}$ are those which transform all the momentum maps $J : P \to \mathfrak{g}^*$ in Poisson maps. They are degenerate. Their symplectic leaves are the orbits of the co-adjoint representation of a Lie group G with Lie algebra $\mathfrak{g}$. These orbits are endowed with the Kirillov symplectic structure defined in the following way. Let $\alpha \in \mathfrak{g}^*$ and $G_\alpha = \left\{ Ad_g^* \alpha \right\}_{g \in G}$ be its orbit. Let $\theta \in T_\alpha G_\alpha$. We can suppose that $\theta \in \mathfrak{g}^*$ for, $\mathfrak{g}^*$ being a vector space, we have $T_\alpha \mathfrak{g}^* \cong \mathfrak{g}^*$. When θ is considered as a velocity vector at α, it is identified with the $\eta \in \mathfrak{g}$ such that $ad_\eta^* \alpha = \theta$. The Kirillov structure is then given by the Lie bracket $\sigma(\theta_1, \theta_2) = \alpha([\eta_1, \eta_2])$. Using the momentum map $J : T^*G \to \mathfrak{g}^*$ of the action of G on itself by left translations, it can be shown that this symplectic structure comes, by means of an operation of "reduction", from the canonical symplectic structure ω_0 on T^*G. So it is possible to generate Hamiltonian systems directly from symmetries. This fact allows the reduction to a priori principles of whole parts of Mechanics.

5.4 Example: the reduction to a priori principles of the Eulerian motion of a non ponderous solid around a fixed point (noncommutative kinematics)

A beautiful example of the previous construction was given by Arnold.[19] Let S be a non ponderous solid moving around a fixed point. The configuration space of S is the rotation group $G = SO(3)$. We have $\mathfrak{g} = so(3) \cong \mathbb{R}^3$ with $[x, y] = x \times y$ (wedge product). Through this identification, the adjoint representation becomes simply the rotations in $\mathbb{R}^3$. Let x be the position of S in a fixed coordinate frame e, X the position of S in a moving coordinate frame E, R the rotation of S (that is $x = RX$), ω and Ω the rotation velocities of S relative to e and E ($\omega, \Omega \in \mathfrak{g} = \mathbb{R}^3$ and $\omega = R\Omega$), and (m, M) the angular moment of S relative to e and E ($m, M \in \mathfrak{g}^* \cong \mathfrak{g} \cong \mathbb{R}^3$ and $m = RM$). If we express the conservation law $m = 0$ in the frame E, we get the *Euler equation* $\dot{M} = [M, \Omega]$ which is the Euler-Lagrange equation for the momentum $p = \partial L / \partial \dot{q} = M$ and the force $\partial L / \partial q = [M, \Omega]$. But, v being the velocity of a point x and μ its mass, we have $v = [\omega, x]$,

[19] Arnold [1976]. See also Abraham-Marsden [1978].

$m = [x, \mu v] = \mu[x, [\omega, x]]$ and then

$$M = \mu[X, [\Omega, X]] = -\mu(adX)^2(\Omega) = A\Omega.$$

The linear operator A is associated with the kinetic energy

$$T = 1/2(A\Omega|\Omega) = 1/2(M|\Omega)$$

($(\bullet|\bullet)$ = scalar product). It is the inertia tensor at X. By summation over X, we get the inertia tensor I. By diagonalisation, we get the inertia axes A_1, A_2, A_3 and the eigenvalues I_1, I_2, I_3. The Euler equations are then $I_i\Omega_i = (I_j - I_k)\Omega_j\Omega_k$, with (i, j, k) a cyclic permutation of $(1, 2, 3)$. It is easy to deduce from them the Poinsot description of the motion of S: the ellipsoid of inertia $\mathcal{E} = \{\Omega|(I\Omega|\Omega) = 1\}$ rolls without sliding on a fixed plan orthogonal to m.

As was shown by Arnold, this classical theory is completely reducible to a priori principles. The Lagrangian of S is given by the kinetic energy T. It is defined on TG and it is left invariant. According to Noether's theorem, there exist 4 first integrals: the energy T and the three components of M. If S does not possess particular symmetries (i.e. if the I_j are different from one another), then these first integrals are independent. For given T and M, the invariant manifolds are therefore *surfaces* Σ. But $SO(3)$ is a parallelizable manifold (its tangent fibered bundle $TG \cong \mathbb{R}^3 \times SO(3)$ is globally trivial), hence orientable. It is also a compact manifold. Then, every Σ is also compact. But if $T \neq 0$, the velocity vector field is everywhere $\neq 0$. Now, it is well known that the only compact and orientable surfaces where such vector fields exist are the tori. Then the Σ are invariant tori (it is a particular case of the Liouville-Arnold theorem on the integrable Hamiltonian systems) and the rotation of S is the composition of two periodic motions (Poinsot motion). If the periods are non commensurable, it is a *quasi periodic* motion which is purely inertial. Its trajectories are the geodesics of G for the Riemannian structure derived from the kinetic energy. Let $g(t)$ be the motion of S in the configuration space $G = SO(3)$. The velocity vector $\dot{g}(t) \in T_gG$. We can transport $\dot{g}$ in $\mathfrak{g} = T_eG$ by a right translation or by a left translation. We get $\omega = R^*_{g^{-1}}\dot{g}$ and $\Omega = L^*_{g^{-1}}\dot{g}$ (where R^*_h (resp. L^*_h) is the linear tangent map of the right (resp. left) translation R_h (resp. L_h)). We then have $\omega = R^*_{g^{-1}}L^*_g\Omega = Ad_g\Omega$: ω and Ω are related by the adjoint representation. The angular momentum (m, M) is the momentum associated to (ω, Ω). Therefore $m, M \in \mathfrak{g}^*$ are related by the co-adjoint representation.

The kinetic energy $T = 1/2(A\Omega|\Omega) = 1/2M(\Omega)$ defines a metric on $\mathfrak{g}$ which can be transported on G by left translations. This metric μ on G is, by definition, left invariant. Let $H : T^*M \to \mathbb{R}$ be the Hamiltonian on G defined by T. Euler equations $\dot{M} = [M,\ \Omega]$ are the geodesic equations for μ. They are the equations of the Hamiltonian systems defined by T on these symplectic manifolds that are the orbits of the co-adjoint representation of $G = SO(3)$ in $\mathfrak{g}^* \cong so(3)^* \cong \mathbb{R}^3$ (these orbits are the spheres $||M|| = \text{cst}$). It is in that sense that, as was stressed by R. Abraham and J. Marsden,

> "the Euler equations (...) are purely geometrical or kinematical".[20]

This shows that there exist Hamiltonian systems which are completely deducible from symmetry (relativity) groups. Such a reduction constitutes a reduction of the transcendental moment of Mechanics to the transcendental moment of Kinematics or, conversely, a transformation of relativity principles into *dynamical* ones.

6 General relativity, the shift of transcendental aesthetics and the construction of the category of force

It was asserted by nearly all philosophers of science that general relativity (GR) marked the definitive collapse of the transcendental approach in modern physics. Our point of view is the opposite of such a claim. In fact, we think that GR yields an extraordinary confirmation of the transcendental approach, provided we accept the idea that the constitutive moments of objectivity can modify their content. Our thesis is that with GR the synthetic a priori shifts *from the metrical level to the differentiable one* (that is to say the symmetry group of the theory changes radically). The geometrical a priori no longer lie at the metrical level but at the underlying differentiable level (level which was made autonomous by the pioneering works of Riemann and Clifford). It is only if we believe that the synthetic a priori is cognitively rooted that we are committed to reject a transcendental interpretation of GR. If, on the contrary, we seek for a correct epistemology of GR then we

[20] Abraham, Marsden [1978], p.327.

are irresistibly led to a transcendental one (see for example the works of Ernst Cassirer, Oscar Becker, and Hermann Weyl).

6.1 The shift of transcendental moments in GR

Let $\mathcal{E}$ be space-time endowed with a locally Minkowskian metric $g_{\mu\nu}$. If e_α is a basis of the tangent space $T_x\mathcal{E}$ and if ω^α is the dual basis of $T_x^*\mathcal{E}$, the Riemann curvature tensor $\mathcal{R}$ is given by $R^\alpha{}_{\beta\gamma\delta} = \langle\omega^\alpha, [\nabla_\gamma, \nabla_\delta]e_\beta\rangle$ (where ∇ is the covariant derivative and, for $\alpha \in T_x^*\mathcal{E}$ and $v \in T_x\mathcal{E}$, $\langle\alpha, v\rangle = \alpha(v)$). For $\alpha \in T_x^*\mathcal{E}$ and $u, v, w \in T_x\mathcal{E}$, $\mathcal{R}(\alpha, u, v, w) = \langle\alpha, R(v, w)u\rangle$ with $R(v, w) = [\nabla_v, \nabla_w] - \nabla_{[v,w]}$. By contraction, we get the Ricci curvature tensor $R_{\mu\nu} = R_\mu{}^\alpha{}_{\alpha\nu}$. By a second contraction, we get the scalar curvature $R = R_\mu{}^\mu$. Einstein curvature tensor is then given by $G = \text{Ricci} - \frac{1}{2}gR$. It satisfies the contracted Bianchi identities $\nabla G \equiv 0$.

As was shown by Hilbert (1921) and developped later by Arnowitt, Deser and Misner (1962), it is possible to derive the metric $g_{\mu\nu}$ from a variational principle using the scalar curvature as Lagrangian density. The action is then:

$$S = \frac{1}{16\pi}\int R\sqrt{|g|}d^4x$$

We consider the tridimensional geometry g^3 of space-like sections Σ of $\mathcal{E}$ (hypersurfaces of simultaneity). The configuration space is the space $\mathcal{M}$ of metrics – in fact the space of embeddings $\Sigma \to \mathcal{E}$ – , and a metric on $\mathcal{E}$ is a path in $\mathcal{M}$. Such a path is solution of an (infinite) hamiltonian system.[21]. More precisely we can compute g^4 and R^4 in a slice $\Sigma_t - \Sigma_{t+dt}$ if we know g^3, the distance (N, Ndt) between Σ_t and Σ_{t+dt}, R^3 and the extrinsic curvature K of the embedding $\Sigma \to \mathcal{E}$. We get that way a (very complex) Hamiltonian system on $T^*\mathcal{M}$:

$$\begin{cases} \frac{\partial g}{\partial t} = \frac{\delta}{\delta\pi}(N\mathcal{H} + N\mathcal{J}) \\ \frac{\partial\pi}{\partial t} = -\frac{\delta}{\delta g}(N\mathcal{H} + N\mathcal{J}) \end{cases}$$

where $\pi^{ij} = \delta S/\delta g_{ij}$ are the conjugate momenta of g_{ij} and where:

$$\begin{cases} \mathcal{H}(g, \pi) = \frac{1}{\sqrt{g}}(Tr\,\pi^2 - \frac{1}{2}(Tr\,\pi)^2) - \sqrt{g}\,R^3 \\ \mathcal{J}^i(g, \pi) = \nabla_j^3\pi^{ij} \end{cases}$$

(if $F(f) = \int\mathcal{F}(f(x))dx$ is a functional, the functional derivative $\delta F/\delta f$ is given by : $F(f_0 + g) = F(f_0) + \int\delta F/\delta f(x)|_{f=0}g(x)dx$).

[21] See Misner-Thorne-Wheeler [1973] and Abraham-Marsden [1978].

This shows that in GR, the metric is no longer an a priori component but on the contrary a physical phenomenon which has to be determined. It is for this very reason – as it was deeply anticipated by Clifford – that metric can absorb forces. In transcendental terms, we can say that in GR there is a conversion of the *kinematical* moment concerning metric – and therefore the category of Quantity and the Axioms of Intuition – into the *dynamical* moment concerning forces – and therefore the category of Causality and the Analogies of Experience. But this does not mean at all that GR falsifies transcendental epistemology. In fact, the equivalence principle and the general covariance principle possess an eminent transcendental meaning. Philosophically, the two most important points are:

1. That the Transcendental Aesthetics shifts from the metrical and global level to the local and differentiable one: the relativity group of the theory is no longer the Galilean or the Poincaré group but the group $\mathrm{Diff}(\mathcal{E})$ of diffeomorphisms of space-time. This implies that the synthetic a priori concerns henceforth this level.

2. Such a shift of Transcendental Aesthetics allows the construction of the category of force, and, even more, its reduction to a priori principles.

6.2 The a priori determination of Einstein equations proposed by Wheeler

Actually, in *Gravitation* John Archibald Wheeler (with Charles Misner and Kip Thorne) has proposed an *a priori determination* of Einstein's field equations. Of course, such a determination is not a complete reduction. But nevertheless it reduces drastically the contingency of the empirical content of Mechanics. Wheeler starts from the equivalence and locality principles according to which motion of matter expresses space-time curvature. Then, the source of the curvature must be the tensor T of energy-momentum. This source must satisfy a *physical* principle of conservation $\nabla T \equiv 0$. Wheeler requires that this conservation be "automatic", that is to say due to purely *mathematical* reasons. We must therefore put $G' = kT$ where G' is a tensor of the same type as T, constructed in a purely geometrical way and satisfying *for a priori reasons* the identity $\nabla G' \equiv 0$. If we require that $G' \equiv 0$ when $\mathcal{E}$ is flat (vanishing of the cosmological constant) and that G' be a linear function of the Riemann curvature tensor $\mathcal{R}$, then the Einstein tensor G is the only solution. The analysis of examples and of the newtonian limit shows that $k = 8\pi$. So we get Einstein's equations $G = 8\pi T$. Now, $\nabla G \equiv 0$ is the *Bianchi identity* which is a consequence of the *cohomological* theory of differential forms ($d^2 = 0$, $\partial^2 = 0$ and Stokes' theorem). This means that the synthetic a priori belongs henceforward to the local and differentiable level and, in particular, to the cohomological one. Wheeler is very explicit (quite lyric) about this fundamental possibility for geometrodynamics to derive physical equations from purely mathematical a priori.

> "This conservation is not an accident. According to Einstein and Cartan, it is 'automatic'; and automatic, moreover, as a consequence of exact equality between energy-momentum and an automatically conserved feature of the geometry".
> "Thus simply is all of general relativity tied to the principle that the boundary of a boundary is zero. No one has ever discovered a more compelling foundation for the principle of conservation of momentum and energy. No one has ever seen more deeply into that action of matter on space, and space on matter, which one calls gravitation. In summary, *the Einstein theory realizes the conservation of energy-momentum as the identity, 'the boundary of a boundary is zero'.*"[22]

Here again, conservation laws (which guarantee the physical accessibility to physical magnitudes) are correlated with purely geometrical a priori.

6.3 The Critique of Wheeler by Adolf Grünbaum

Philosophically, it is crucial to understand this shift of Transcendental Aesthetics in GR. Otherwise we are led to strange conclusions. A good example is given by Grünbaum's critique of Wheeler. For instance, in his *Philosophical Problems of Space and Time* – chapter 22: *General relativity, Geometrodynamics and Ontology* [23] – Grünbaum tries to show that there exists a contradiction between, on the one hand, GR as it is treated by Wheeler's geometrodynamics (GMD) according to Clifford's slogan "Physics is Geometry" and, on the other hand, the "Hypothesis" of Riemann ("*Über die Hypothesen, welche der Geometrie zugrunde liegen*"). According to him, GMD is "an

[22]Misner-Thorne-Wheeler [1973], p. 380.
[23]Grünbaum [1973].

all-out geometrical reductionism and absolutism" which *ontologizes* GR and transforms metric into an *intrinsic* property of space-time. Now, Grünbaum wants to maintain the thesis of the *conventionality* of metric. According to Riemann's metric hypothesis (RMH), we choose the metric in such a manner that:

1. the class U_p of photon trajectories consists of metrically null curves,
2. the class U_m of free fall motions consists of geodesics.

> "An indefinite Riemann metric (tensor) is to be SO CHOSEN that with respect to *one and the same such metric ds*
>
> (a) The photon trajectories at any given world point are to BECOME *metrically null*.
>
> (b) The Space-Time-trajectories belonging to the proper subclass U_m of U are to be TURNED INTO time-like GEODESICS via the intra-theoretic defining equation $\delta \int ds = 0$."[24]

These two classes U_p and U_m are empirically given. It is only the conventional choice of the metric which qualifies them geometrically.

> "It follows that if RMH is true, then the metrical structure of space-time – and thereby the very constitution of the only autonomous substance in the GMD monistic ontology – depends crucially for being what it is not only on the physically-determinate membership of U, but also on the *intrinsically*-UNFOUNDED, humanly-stipulated ascriptions of metrical *geodesicity* and *nullity* to the appropriate members of U!!".
>
> "Granted RMH, human stipulations enter ontologically – not just verbally! – into making the metric geometry of space-time be what it is, and thereby these stipulations paradoxically generate the character of the only autonomous 'physical' substance recognized by GMD. For on the assumption of RMH, human conventions are indispensable to the GMD metric geometry".[25]

It is clear that Grünbaum wants to go beyond the dogmatic conflict between an ontological realism and a nominalistic ("verbal", non geometrical)

[24] Grünbaum [1973], p. 757, Grünbaum's emphasis.
[25] Ibid., p. 758.

conventionalism. His conception is clearly objective in the transcendental sense. According to him, Wheeler's GMD must be criticized because it is "ontological" and runs into a contradiction:

> "there is the outright inconsistency that space is and also is not intrinsically metric".

But he misses the main point. In GR, the dialectic between "is" and "is not" is exactly the same as the "is" and "is not" in the sentence "a body is and also is not moving" in classical Mechanics. As Einstein said, acceleration (and therefore force) becomes as relative as velocity. In GR, according to the fact that metric has become a physical phenomenon, the modality of metric is no longer that of reality but that of possibility. The shift of Transcendental Aesthetics renders irrelevant the classical problem of the conventionality of metric. It is the *differentiable* structure (with its underlying topological stucture and its cohomological correlates) which is now conventional (non physical) and nevertheless physically determinant, that is to say synthetic a priori.

7 Quantum Mechanics and generalized Transcendental Aesthetics

Quantum Mechanics (QM) was also used by positivists as a weapon against transcendentalism. But we think that, in fact, there exists also a clear and deep transcendental structure of QM. We have seen at the beginning of this work that a plausible epistemology of QM must be based upon a non ontological concept of "weak" objectivity which is of a transcendental nature. Bohr's operationalism is in fact transcendental . This thesis can be developed in a rather technical way.

7.1 Quantum Transcendental Aesthetics

In an objective process of constitution, the function of Transcendental Aesthetics is characterized by four requisites:

1. to determine *forms* of manifestation which permit to delete the *subjective* contents of the concept of phenomenon;

2. to manifest *relativity* properties violating the principles of substantial ontology;

3. to provide a mathematical basis for the construction of the dynamical (physical) categories of substance, causality and interaction;

4. to lead to a non metaphysical interpretation – only physical and specifically relative to the process of constitution – of the modal categories of possibility (virtuality), reality (actuality) and necessity.

For classical Mechanics, special relativity and general relativity, Transcendental Aesthetics is a spatio-temporal one. But, according to the previous definition, it can perfectly exist non spatio-temporal generalized Transcendental Aesthetics. It is the case in QM. If we try to recognize in QM such an instance associated with the specific quantum concept of phenomenon, we are immediately led to the concept of *probability amplitude*. As was emphasized by G. Cohen-Tannoudji,

> "the recourse to statistics permits to take into account in quantum phenomena the inseparability between the object and the measure apparatus, *without explicitly introducing the measure apparatus*".[26]

Quantum states are sets of *possibilities* – a "potentia" – which are *actualized* by measure operations. This *relational* status of the concept of quantum state (often stressed by Bohr) and its interpretation as a new type of relativity, were very well formulated by Vladimir Fock [27]:

> "The probabilities expressed by the wave function are the probabilities of some result of the interaction of the micro-object and the instrument (of some reading on the instrument). The wave function itself can be interpreted as the reflection of the potential possibilities of such an interaction of the micro-object (prepared in a definite way) with various types of instruments. A quantum mechanical description of an object by means of a wave function corresponds to the relativity requirement with respect to the means of observation. This extends the concept of relativity with respect to the reference system familiar in classical physics".

[26] Cohen-Tannoudji and Spiro[1986], p. 152.
[27] See Jammer [1974], p. 202.

The axioms governing the use of probability amplitudes give an example of a radically non classical Transcendental Aesthetics. Of course, one of the main problems is to establish the link with the spatio-temporal Transcendental Aesthetics. It is a very difficult one. It is well known (Gelfand's spectral theory) that if $\mathcal{A}$ is a commutative C^*-algebra (that is to say a Banach algebra over $\mathbb{C}$ which is involutive and where the norm is $||x||^2 =$ spectral radius of (x^*x)), then $\mathcal{A} \cong C(X)$ the algebra of continuous complex functions on the space $X = \mathrm{Sp}(\mathcal{A})$ of characters of $\mathcal{A}$: $X = \{\chi : A \to \mathbb{C} \mid \chi \not= 0\}$ (X is also the space of maximal ideals of $\mathcal{A}$). So, in the classical case, there exists a *duality* between space and observables. In QM it is no longer the case and we must use *noncommutative geometry* (in Alain Connes' sense) to find again a duality between the (noncommutative) algebras of observables and some sort of "underlying space".[28]

7.2 Theories with hidden variables

The basic idea of theories which hidden variables is to interpret a quantum – an then *dispersive* – state ψ as a statistic set of *non dispersive* ideal states. If A is an observable and $\psi(A) = \langle A\psi \mid \psi \rangle$ (where $\langle \bullet | \bullet \rangle$ is the scalar (Hermitian) product in the Hilbert space $\mathcal{H}$ of the system), one tries to express $\psi(A)$ by an integral of the form:

$$\psi(A) = \int_\Omega f_\omega(A)\ d\mu(\omega)$$

where Ω is a space of hidden variables, μ a probability measure on Ω and the f_ω are ideal non dispersive states.[29] So, we try to associate to ψ a probability measure ν_ψ on Ω and to an observable A a function $\mathbf{A} : \Omega \to \mathbb{R}$ in such a way that, if $\mu_{\psi,A}$ is the distribution on $\mathbb{R}$ for the measure of A when the system is in the state ψ, then we have for every $E \subset \mathbb{R}$, $\mu_{\psi,A}(E) = \nu_\psi(\mathbf{A}^{-1}(E))$. But a celebrated theorem of von Neumann shows that this is essentially impossible:

Von Neumann's theorem. If a C^*-algebra of observables (self-adjoint operators) admit hidden variables then it is *commutative* (i.e. classical).

Mackey, Gleason, Bell, have refined von Neumann's theorem. They have shown that some too strong hypothesis could be removed. But the negative result remains: as is now well known, there exist no *local* theories admitting hidden variables. We can recall here Marsden's verdict:

[28] See Connes [1990].

[29] See e.g. Marsden [1974].

"The entire point of the negative results concerning 'hidden variables' is that there is no 'objective underlying state' [in the ontological sense of the term 'objective'] of the system!".

It is henceforth the science itself which imposes an objective epistemology which can philosophically thematize the fact that a "weak" objectivity is nevertheless a true (but non ontological) objectivity.

7.3 Gauge theories and the construction of the category of Interaction

We consider that, philosophically speaking, one of the main achievement of quantum field theory (QFT) is to have constructed (in the transcendental sense) the third dynamical category of *interaction* (GR constructed the second dynamical category of causality i.e. the force) using a typically transcendental principle, Weyl's gauge principle converting gauge invariances into dynamical principles. In QFT we start with Lagrangian densities $\mathcal{L}(\varphi, \partial_\mu \varphi)$ where the fields $\varphi(x, t)$ possess *internal symmetries*. The action functional is:

$$S(\Gamma) = \int_\Omega \mathcal{L}\ d^4x = \int_{t_1}^{t_2} \int_{\mathbb{R}^3} \mathcal{L}(\varphi, \partial_\mu \varphi)\ d^3x\ dt$$

computed along the paths $\Gamma = \varphi(x,t) : \varphi_i = \varphi(x, t_1) \to \varphi_f = \varphi(x,\ t_2)$. The axioms of QM governing probability amplitudes lead to Feynman's path integrals:

$$\langle \varphi_f \mid \varphi_i \rangle = \int_\Gamma \exp(iS(\Gamma)/\hbar)\ \mathcal{D}\Gamma$$

It is well known that this formula (which is analogous with the partition functions Z in statistical mechanics) encodes a considerable amount of information. It is possible to derive from it innumerable explicit, quantitative and predictive models, using appropriate mathematical tools as:

1. perturbative developments;

2. Wick's theorem: all moments of a Gaussian probability can be expressed as functions of its moments of order two;

3. stationary phase theorem: an oscillatory integral $e^{i\tau\varphi(x)}$ concentrates, for $\tau \to \infty$, on the critical points of the phase φ;

4. methods of the renormalisation group.

We meet here a perfect example of an objective determination leading from constitutive principles to very diversified and specific models: the constitutive principles (relativity, symmetry) provide Lagrangians, wich in turn provide Feynman's integrals, which provide themselves models.

Constitutive a priori, relativity, symmetries	→	Lagrangians	→	Action
	→	Path integrals	→	Specific models of phenomena

Gauge theories have shown that if we *localize the internal symmetries* and if we impose the invariance of the Lagrangians for these supplementary local symmetries, we can reconstruct in a purely mathematical way the interaction Lagrangians (reduction to a priori principles).[30] The standard example is that of the (minimal) coupling between an electron and the electro-magnetic field. We have:

the Dirac Lagrangian: $\mathcal{L}_D = \overline{\psi}(x)(i\gamma^\mu \partial_\mu - m)\psi(x)$;

the Maxwell Lagrangian: $\mathcal{L}_{EM} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - J^\mu A_\mu$;

the Interaction Lagrangian: $\mathcal{L}_{Int} = e\overline{\psi}(x)\gamma^\mu A_\mu \psi(x) = -J^\mu A_\mu$ (where $J^\mu(x) = e\overline{\psi}(x)\gamma^\mu\psi(x)$ is the electro-magnetic current generated by the field ψ).

• The Dirac Lagrangian $\mathcal{L}_D$ is invariant under the internal global symmetry $\psi \to e^{-ie\theta}\psi$.

• The Maxwell Lagrangian $\mathcal{L}_{EM}$ is invariant under gauge transformations $A \to A + d\Lambda$ (for Maxwell's equations are $F = dA$, $dF = 0$, $d^*F = 4\pi^*J$ and we have the cohomological a priori $d^2 = 0$).

• If we let $\theta = \theta(x)$ depends upon the spatial position x, then $\mathcal{L}_D$ is no longer invariant. But the supplementary term $e\overline{\psi}\gamma^\mu\partial_\mu(\theta(x))\psi$ can be exactly balanced by the gauge transformation $A \to A + d\theta$. This is equivalent to the substitution of ∂_μ by the covariant derivative $D_\mu = \partial_\mu + ieA_\mu(x)$. The *vector potential* A_μ is geometrically interpreted as a *connection* on a vector bundle over space-time. The EM field is then identified with the *curvature* of this connection.

[30] For a presentation of Gauge Theories, see for instance Itzykson-Zuber [1985], Le Bellac [1988], Manin [1988], Quigg [1983].

In the non abelian case, let G ($G \neq U(1)$, $G = SU(2)$, $SU(3)$, etc.), be the non abelian Lie group of internal symmetries. Let ξ_a be the generators of the Lie algebra $\mathfrak{g}$ of G. The Lagrangians are invariant under the internal global symmetry $U(g) = \exp(-i\theta_a\xi_a)$. If we let the $\theta_a = \theta_a(x)$ depend upon x, they are no longer invariant. We recover invariance if we introduce gauge fields $A_a^\mu(x)$. We get in that way a field $\mathcal{A}^\mu(x)$ with values in the Lie algebra $\mathfrak{g}$.

• The gauge transformations $A_\mu \to A_\mu + \partial_\mu\theta$ are generalized by:

$$A_a^\mu \to c_a{}^{bc}\theta_b A_c^\mu + \partial^\mu\theta_a$$

(where the $c_a{}^{bc}$ are the structure constants of $\mathfrak{g}$), i.e.

$$\mathcal{A}'_\mu = U(g)\ [\mathcal{A}_\mu + iU(g)^{-1}\partial_\mu(U(g))]\ U(g)^{-1}$$

• The covariant derivative $D_\mu = \partial_\mu Id + ie\mathcal{A}_\mu$ is transformed into $D'_\mu = U(g)D_\mu$ which balances the term $\partial_\mu(U(g))\varphi$ in $\partial_\mu\varphi \to U(g)\partial_\mu\varphi + \partial_\mu(U(g))\varphi$.

• The field $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ is generalized in $\mathcal{F}^{\mu\nu} = \partial^\mu\mathcal{A}^\nu - \partial^\nu\mathcal{A}^\mu + i[\mathcal{A}^\mu, \mathcal{A}^\nu]$.

• The default of commutativity introduces non linearities and a self-coupling of the gauge fields.

Using Feynman's algorithm it has been possible to work out a true ontogenesis of physical reality which converts the synthetic a priori in explicit models. The mathematical constraints are so strong (renormalizability, elimination of anomalies, Higgs mechanism of symmetry breaking for giving a mass to the gauge bosons, etc.), that it becomes possible to infer from very few (but crucial) empirical data to the choice of a symmetry group.

7.4 Superstring theory

In superstring theory this fact is even more evident.[31] One supposes that elementary particules are non punctual entities but strings. Let $\sigma \in [0, \pi]$ be a parametrization of the string. If τ is its proper time, the parametrization of its world leaf is $X_\mu(\sigma, \tau)$ endowed with the metric $g_{ab} = g_{\mu\nu}\ \partial_a X^\mu \partial_b X^\nu$ ($a, b = \sigma$ or τ). This leads to the introduction of new Lagrangians, for instance the Polyakov Lagrangian:

$$L = -\sqrt{g}\ g^{ab}\ \partial_a X_\mu \partial_b X^\mu$$

[31] For a presentation of Superstring Theories, see for instance Green-Schwarz-Witten [1987] or Kaku [1988].

with $g = |det(g_{ab})|$. In that way, we get *a new symmetry group*. Feynman's interaction graphs are substituted by Riemann surfaces (which are topological configurations of interaction). One has then to compute functional integrals of the following type :

$$Z = \sum_{topologies} \int_{metrics} \mathcal{D}g_{ab} \int_{leaves} \mathcal{D}X^{\mu} \ \exp(iS/\hbar)$$

To avoid redundancies corresponding to gauge invariances, we must know the spaces and the measures occurring in the integration. For doing that, we need Riemann surfaces theory. For example we need Teichmüller theory of moduli spaces for knowing exactly what are the automorphisms of a Riemann surface (what are its diffeomorphisms which are not isotopic to the identity, what are the complex structures compatible with a given differentiable structure, etc.).

We need also the solution of the Schottky problem. Let S be a Riemann surface of genus g. It is well known since Riemann that it is possible to find a basis $(a_i, b_i),\ i = 1, ...,\ g$ of the homology of S and a basis $(\omega_j),\ j = 1, ..., g$ of the space of differentiable 1-forms which are the simplest possible, that is to say which satisfy: $\int_{a_i} \omega_j = \delta_{ij}$ and $\int_{b_i} \omega_j = \Omega_{ij}$, the matrix $\Omega = (\Omega_{ij})$ of periods being symmetric and of imaginary part positive definite: $Im\Omega > 0$. But if $g > 3$, the space of such matrices Ω has a dimension $\frac{1}{2}g(g+1)$ which is greater than the dimension $3g - 3$ of the moduli space of S. Therefore, we must characterize the Ω which can be the period matrices of Riemann surfaces. This is the Schottky problem. It has been solved only in 1984.

All these sophisticated and deep mathematical results are necessary to express the independence of objectivity relatively to the new conventional elements introduced in superstring theory. Renormalization constraints and the elimination of anomalies impose for instance the dimension of space-time (10 or 26) (elimination of the *conformal* anomaly) and the gauge group ($O(32)$ or $E_8 \otimes E_8$) (elimination of the *chiral* anomaly). The chiral anomaly is defined in the following manner. On a spin manifold of dimension $D = 4$, Dirac equation is defined and chirality corresponds to the invariance under the transformation $\psi \to e^{i\gamma_5 \varepsilon}\psi$. This invariance implies the conservation law of the axial current $\partial_\mu J^{\mu,5} = \partial_\mu(\overline{\psi}\gamma^5\gamma^\mu\psi) = 0$. It can be shown that

$$\begin{cases} \partial_\mu J^{\mu,5} & = & -\frac{1}{16\pi^2}\ \varepsilon^{\mu\nu\sigma\rho}\ \text{Trace}(F_{\mu\nu}F_{\sigma\rho}) \\ J^{\mu,5} & = & -\frac{1}{4\pi^2}\varepsilon^{\mu\alpha\beta\gamma}\ \text{Trace}(A_\alpha\partial_\beta A_\gamma + \frac{2}{3}A_\alpha A_\beta A_\gamma) \\ & = & \overline{\psi}\,\xi_a\gamma^5\gamma^\mu\,\psi \end{cases}$$

where F is the field and A the vector potential. More intrinsically we have:

$$\partial_\mu J^{\mu,5} = -\frac{1}{16\pi^2}\ \text{Trace}(F \wedge F^*)$$

where F (the field) is the curvature 2-form $F = dA + A \wedge A$ and A the 1-form $A = A^a_\mu \xi_a dx^\mu$ (the ξ_a are the gauge fields). The 4-form equal to the square of this curvature 2-form is an exact differential: $\text{Trace}(F \wedge F^*) = d\omega$, ω being the Chern-Simons 3-form $\omega = \text{Trace}(AdA + \frac{2}{3}A^3)$.

When one introduces the corrections which are necessary to make the Feynman integrals converge, one breakes this symmetry and produces an anomaly which is deeply linked with the index of the Dirac operator (which is a topological number). In fact the anomaly is a cohomology class. Let S be the spin bundle of the manifold M and D the Dirac operator on the smooth sections of S. The symmetry group G is here the group of diffeomorphisms of S (considered as a fiber bundle). Let $\text{Det} : G \to \mathbb{C} - \{0\}$ be the determinant of the operator D. The anomaly α is the pull-back by Det of the generator dz/z of the first integral cohomology group $H^1(\mathbb{C} - \{0\}, \mathbb{Z})$ of $\mathbb{C} - \{0\}$. We have by definition $\alpha \in H^1(G, \mathbb{Z})$. It can be shown that the elimination of the chiral anomaly imposes the dimension $n = 496$ for the gauge group, and this imposes $O(32)$ or $E_8 \otimes E_8$.

We must return in dimension 4 if we want to recover the phenomenology. For instance – using a Kaluza-Klein device – we can compactify 16 dimensions (starting from $D = 26$) using the lattice of the roots of the Lie gauge group $E_8 \otimes E_8$ and then compactify again 6 dimensions: $M^{10} \to M^4 \times K^6$. Physical constraints of preservation of super-symmetry impose for example that there exists on K a spinor field ξ which is constant for the covariant derivation (i.e. $D_i \xi = 0$). This fact imposes drastic constraints upon the geometry of K^6: the Ricci curvature must be $= 0$, the holonomy group must be $= SU(3)$ (and non $O(6) \cong SU(4)$), the first Chern class $c_1(K)$ of K must be $= 0$, there must exist a Kähler metric on K, etc. In fact, according to a celebrated theorem of Calabi and Yau, a Kähler manifold K^{2n} with $c_1(K) = 0$ admits necessarily a Kähler metric with holonomy group $SU(n)$ (and not $O(2n)$). To relate this compactification $M^{10} \to M^4 \times K^6$ to the compactification $M^{26} \to M^{10}$ using the lattice of $E_8 \otimes E_8$, the most direct method consists in identifying a part of the gauge fields of $E_8 \otimes E_8$ to the spin connection of M^{10}. This relate the gauge fermions to the geometry of K^6.

More precisely, we start with the spin connection $\omega = \omega^{ab}_\mu dx^\mu M_{ab}$ of K^6 which is given by the covariant derivative of spinors $D_\mu \psi = (\partial_\mu + \omega^{ab}_\mu M_{ab})\psi$

(where M_{ab} is a representation of the generators of the Lorentz group) and we embed ω in the gauge connection of the Yang-Mills theory with gauge group $E_8 \otimes E_8$. This allows to satisfy the *Bianchi identities*:

$$\mathrm{Trace}(R \wedge R^*) - \frac{1}{30}\,\mathrm{Trace}(F \wedge F^*) = 0$$

which relate the Riemann curvature tensor R to the Yang-Mills tensor F. As for the spin connection the holonomy group is $SU(3)$, we must consider $SU(3)$ as a gauge subgroup of $E_8 \otimes E_8$. The simplest manner for doing this is to consider $SU(3) \otimes E_6 \subset E_8$. If we consider the decomposition of the adjoint representation of E_8 (which is of dimension 248) relatively to the subgroup $SU(3) \otimes E_6$ we find :

$$248 = (3, 27) \oplus (\overline{3}, \overline{27}) \oplus (8, 1) \oplus (1, 78)$$

that is to say 27 sets of fields which transform according to the representation $3 \oplus \overline{3}$ of $SU(3)$ and a set of octets which transform according to the adjoint representation 8 of $SU(3)$. But such an octet must be counted 3 times relatively to $3 \oplus \overline{3}$. There exists therefore in the representation $3 \oplus \overline{3}$ of $SU(3)$ a total redundancy equal to $27 + 3 \times 1 = 30$. The Bianchi identities can therefore be satisfied. We see perfectly well with this technical example how general a priori principles lead to very precise models and predictions which can be experimentally falsified. The epistemological point is that such falsifiable models and predictions are remote mathematical consequences of a transcendental type of objective determination.

8 Conclusion

1. The more mathematics and physics unify themselves in modern physico-mathematical theories, the more an objective epistemology becomes necessary. Only such a transcendental epistemology is able to thematize correctly the status of the mathematical determination of physical reality.
2. There exists a *transcendental history* of the synthetic a priori and of the construction of physical categories.
3. The transcendental approach allows to supersede Wittgenstein's and Carnap's anti-Platonist thesis according to which pure mathematics are physically applicable only if they lack any objective content and reduce to mere prescriptive and normative devices. In fact, pure mathematics are prescriptive-normative in physics because:

1. the categories of physical objectivity are prescriptive-normative, and
2. their categorial content is mathematically "constructed" through a Transcendental Aesthetics.

Only a transcendental approach make compatible, on the one hand, a grammatical conventionalism of Wittgensteinian or Carnapian type and, on the other hand, a Platonist realism of Gödelian type. Mathematics are not a grammar of the world but a mathematical hermeneutics of the intuitive forms and of the categorial grammar of the world.
4. The transcendental approach allows also to reconcile the objective validity and the historical value of scientific theories. It allows to go beyond the epistemological antinomy opposing dogmatic positivism (there exists an absolute value of objective truth) and sceptic post-positivism (there exists an historico-anthropological relativity of truth). As we have seen, truth, reality, necessity are moments of the procedures of constitution and determination. They are relative to them, and therefore relative to an *historical* (non cognitive) synthetic a priori.

References

[1] Abraham, R., Marsden, J. 1978. *Foundations of Mechanics*, Benjamin Cummings, New-York, Reading.

[2] Allison, H.E., 1981."Transcendental Schematism and the Problem of the Synthetic a priori", *Dialectica*, 35, 1-2, 57-83.

[3] Allison, H.E., 1983. *Kant's Transcendental Idealism. An Interpretation and Defense*, New-Haven, Yale University Press.

[4] Ameriks, K., 1982. "Current German Epistemology: The Significance of Gerold Prauss", *Inquiry*, 25, 1, 125-138.

[5] Arnold, V., 1976, *Méthodes mathématiques de la mécanique classique*, Moscou, Mir.

[6] Bohr, N., 1935. *Physical Review*, 48, 696.

[7] Boi, L., 1989. "Idéalisation et objectivation, ou des rapports entre géométrie et physique", *Fundamenta Scientiae* , 10, 1, 85-114.

[8] Bouveresse, J., 1987. *La Force de la Règle*, Paris, Editions de Minuit.

[9] Brittan, G., 1978. *Kant's Theory of Science*, Princeton University Press.

[10] Brittan, G., 1991. *Algebra and Intuition*, Department of Philosophy, Montana Sate University, (to appear).

[11] Cassirer, E., 1910. *Substanzbegriff und Funktionbegriff*, Berlin. *Substance et Fonction* (trans. P. Caussat), Paris, Editions de Minuit, 1977.

[12] Châtelet, G., 1985. "Le retour de la monade", *Fundamenta Scientiae*, 6, 327- 345.

[13] Cohen-Tannoudji, G., Spiro, M., 1986. *La Matière-Espace-Temps*, Paris, Fayard.

[14] Connes, A., 1990. *Géométrie non commutative*, Paris, InterEditions.

[15] d'Espagnat, B. 1985. *Une incertaine réalité*, Paris, Gauthier-Villars.

[16] Duncan, H., 1984. "Inertia, the Communication of Motion, and Kant's Third Law of Mechanics", *Philosophy of Science*, 51, 93-119.

[17] Ehlers, J., 1973. "The Nature and Structure of Spacetime", *The Physicist's Conception of Nature*, (J. Mehra ed.), 71-91, Dordrecht, Reidel.

[18] Folse, H.J., 1978. "Kantian Aspects of Complementarity", *Kant-Studien*, 69, 58-66.

[19] Friedman, M., 1985. "Kant's Theory of Geometry", *The Philosophical Review*, XCIV, 4, 455-506.

[20] Gomez, R.J., 1986. "Beltrami's Kantian View of Non-Euclidean Geometry", *Kant-Studien*, 77, 1, 102-107.

[21] Green, M. B., Schwarz, J. H., Witten, E., 1987. *Superstring Theory*, Cambridge University Press.

[22] Grünbaum, A., 1973. *Philosophical Problems of Space and Time*, Dordrecht-Boston, Reidel.

[23] Honner, J. 1982. "The Transcendental Philosophy of Niels Bohr", *Studies in History and Philosophy of Science*, 13, 1, 1-29.

[24] Itzykson, C., Zuber, J.B., 1985. *Quantum Field Theory*, Singapour, Mc Graw-Hill.

[25] Jammer, M., 1974. *The Philosophy of Quantum Mechanics*, New-York, John Wiley and Sons.

[26] Kaku, M., 1988. *Introduction to Superstrings*, New-York, Springer.

[27] Kant, E, 1980-1986. *Oeuvres philosophiques* (F. Alquié ed.), Paris, Bibliothèque de la Pléïade, Gallimard.

[28] Kant, E., 1781-1787. *Critique de la Raison pure*, (trans. A.J.L. Delamarre et F. Marty), Paris, Pléïade, Gallimard, 1980.

[29] Kant, E., 1786. *Premiers Principes métaphysiques de la Science de la Nature*, (trans. J. Gibelin), Paris, Vrin, 1971.

[30] Kant, E., 1790. *Critique de la Faculté de Juger*, (trans. A. Philonenko), Paris, Vrin, 1979.

[31] Kant, E., 1796-1803. *Opus Postumum*, (trans. F. Marty), Paris, Presses Universitaires de France, 1986.

[32] Kant, I., 1781-1787. *Kritik der reinen Vernunft*, Kants gesammelte Schriften, Band III, Preussische Akademie der Wissenschaften, Berlin, Georg Reimer, 1911.

[33] Kant, I., 1786. *Metaphysische Anfangsgründe der Naturwissenschaft*, Kants gesammelte Schriften, Band IV, Preussische Akademie der Wissenschaften, Berlin, Georg Reimer, 1911.

[34] Kant, I., 1790. *Kritik der Urtheilskraft*, Kants gesammelte Schriften, Band V, Preussische Akademie der Wissenschaften, Berlin, Georg Reimer, 1913.

[35] Lautman, A., 1937-1939. *Essai sur l'unité des mathématiques et divers écrits*, Paris, Bourgois, 1977.

[36] Le Bellac, M., 1988. *Des phénomènes critiques aux champs de jauge*, Paris, InterEditions - C.N.R.S.

[37] Manin, Y. I., 1988. *Gauge Field Theory and Complex Geometry*, Berlin, New- York, Springer.

[38] Marsden, J., 1974. *Applications of Global Analysis in mathematical Physics*, Berkeley, Publish or Perish.

[39] Mc Goldrick, P.M., 1985. "The Metaphysical Exposition: An Analysis of the Concept of Space", *Kant-Studien*, 76, 3, 257-275.

[40] Misner, C.W., Thorne, K.S., Wheeler, J.A., 1973. *Gravitation*, San Francisco, Freeman.

[41] MNS, 1989. *La Mathématique non standard*, (H. Barreau, J. Harthong, eds.), Paris, Editions du CNRS.

[42] Paty, M., 1988. *La Matière dérobée*, Paris, Editions des Archives Contemporaines.

[43] Petitot, J., 1979-1982. "Infinitesimale", "Locale/Globale", "Unità delle matematiche", *Enciclopedia Einaudi*, VII, 443-521; VIII, 429-490; XV, 341-352; XV, 1034-1085, Turin, Einaudi.

[44] Petitot, J., 1987a. "Refaire le Timée. Introduction à la philosophie mathématique d'Albert Lautman", *Revue d'Histoire des Sciences*, XL, 1, 79-115.

[45] Petitot, J., 1987b. "Mathématique et Ontologie", *La scienza tra filosofia e storia in Italia nel Novecento*, (F. Minazzi, L. Zanzi, eds.), 191-211, Rome, Edizione della Presidenza del Consiglio dei Ministri.

[46] Petitot, J., 1988. "Logique transcendantale et ontologies régionales", *Colloque de Cerisy: Rationalité et Objectivités.*

[47] Petitot, J., 1989. "Rappels sur l'analyse non standard", *MNS [1989]*, 187-209.

[48] Petitot, J., 1990. "Premiers principes métaphysiques d'une Science de la Forme", *Colloque de Cerisy autour de la Critique de la Faculté de Juger.*

[49] Petitot, J., 1990a. "Logique transcendantale, Synthétique a priori et Herméneutique mathématique des objectivités", *Fundamenta Scientiae*, 10, 1, 57-84.

[50] Petitot, J., 1990b. "Logique transcendantale et Herméneutique mathématique: le problème de l'unité formelle et de la dynamique historique des objectivités scientifiques", *Il pensiero di Giulio Preti nella cultura filosofica del novecento*, (F. Minazzi ed.),155-172, Milano, Franco Angeli.

[51] Petitot, J., 1990c. "Note sur la querelle du déterminisme", *La Querelle du Déterminisme*, (K. Pomian ed.), 202-227, Paris, *Le Débat*, Gallimard.

[52] Petitot, J., 1991a. "Idéalités mathématiques et Réalité objective. Approche transcendantale", *Hommage à Jean-Toussaint Desanti*, (G. Granel ed.), 213-282, Mauvezin, Editions TER.

[53] Petitot, J., 1991b. "Continu et Objectivité. La bimodalité objective du continu et le platonisme transcendantal", *Le Labyrinthe du Continu*, (J.M. Salanskis, H. Sinaceur eds.), Colloque de Cerisy, Paris, Springer, 239-263.

[54] PQG, 1988. *Physique quantique et géométrie* (Colloque André Lichnérowicz, D. Bernard, Y. Choquet-Bruhat eds.), Paris, Hermann.

[55] Prauss, G., 1980. *Einführung in die Erkenntnistheorie,* Darmstadt, Wissenschaftliche Buchgesellschaft.

[56] Prauss, G., 1981a. "Time, Space and Schematisation", *The Philosophical Forum*, XIII, 1, 1-11.

[57] Prauss, G., 1981b. "Kants Theorie der ästhetischen Einstellung", *Dialectica*, 35, 1-2, 265-281.

[58] Quigg, C., 1983. *Gauge Theories of the Strong, Weak, and Electromagnetic Interactions*, Reading, Benjamin-Cummings.

[59] Salanskis, J.M., 1989. "Le potentiel et le virtuel", MNS [1989], 275-303.

[60] Salanskis, J.M., 1991. *L'Herméneutique formelle*, Paris, Éditions du CNRS.

[61] Scheibe, E., 1981. "Invariance and Covariance", (J. Agassi, R. S. Cohen, eds.), *Scientific Philosophy Today*, 311-331, Dordrecht, Reidel.

[62] Souriau, J.M., 1975. *Géométrie symplectique et physique mathématique*, Colloques Internationaux du C.N.R.S., 237, Paris.

[63] Stegmüller, W., 1979. *The Structuralist Views of Theories*, Berlin, New-York, Springer.

[64] Vuillemin, J., 1955. *Physique et Métaphysique kantiennes*, Paris, Presses Universitaires de France.

[65] Wang, H., 1987. *Reflections on Kurt Gödel*, Cambridge, M.I.T Press.

[66] Weinstein, A., 1977. *Lectures on Symplectic Manifolds*, C.B.M.S., Conf. Series, Am. Math. Soc., 29, Providence.

[67] Weizsäcker, C.F. von, 1979. *Die Einheit der Natur*, Munich, Hauser.

[68] Weyl, H., 1922. *Space-Time-Matter*, New-York, Dover.

[69] Wiredu, J.E., 1970. "Kant's Synthetic A Priori in Geometry and the Rise of Non-Euclidean Geometries", *Kant-Studien*, 61, 1, 5-27.