From the Curry-Howard correspondance to a toposic interpretation of perceptive judgments

Jean Petitot

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The Curry-Howard correspondance

In computer science the high level logical typing of low level compiled programs is essentially given by the well known *Curry-Howard correspondance*. The low level computations are described by λ-terms of a *λ-calculus*. In the most simple type-free λ-calculus we construct inductively λ-terms by iterating two basic operations :

(i) the *application* MN of a λ-term M to another λ-term N,

(ii) the *abstraction* $\lambda x.M$ transforming the free occurrences of the variable x in M in slots for others λ-terms.

The basic rule of the λ-calculus (which corresponds to the execution of the program described by the λ-term) is the *β-reduction* consisting on applying a λ-term λ*x*.*M* to another λ-term *N* by substituting *N* to the free occurrences of *x* in *M* :

$$(\lambda x.M)N \to_{\beta} M[x:=N]$$

The *normalization* of a λ-term is a sequence of β-reductions which stops at a β-irreducible λ–term. The normalizable λ-terms describe therefore *effective* computations.

The expressivity of the λ-calculus is very rich since all recursive functions are λ-definable.

The fundamental link with logic comes from the *typing* of the λ-terms M by types μ (notation M:μ). Intuitively, if M:μ is a λ-term of type μ and if x:σ is a variable of type σ, then the abstraction $\lambda x.M:\sigma\rightarrow\mu$ is of the type $\sigma\rightarrow\mu$ of the functions from sources σ to goals μ. In the same way, if $M:\sigma\rightarrow\tau$ is a λ-term of functional type $\sigma\rightarrow\tau$ and if N:σ is of type σ, then MN:τ is of type τ. One can show that normalization preserves typing and that all the typable λ-terms are normalizable.

The types correspond in fact to the formulae of the *implicational fragment of intuitionistic propositional logic*. They can be given a *categorical* interpretation by the objects of a category where this logic is interpretable. The right structure for that is the structure of *topos*.

The Curry-Howard correspondance is given by the following translation :

λ-calculus, programming	**logic, proof theory**
low level	high level
code	source
compilation	decompilation
execution of the program	specification of the program
proof encoding	theorem, program typing
instruction	logical rule

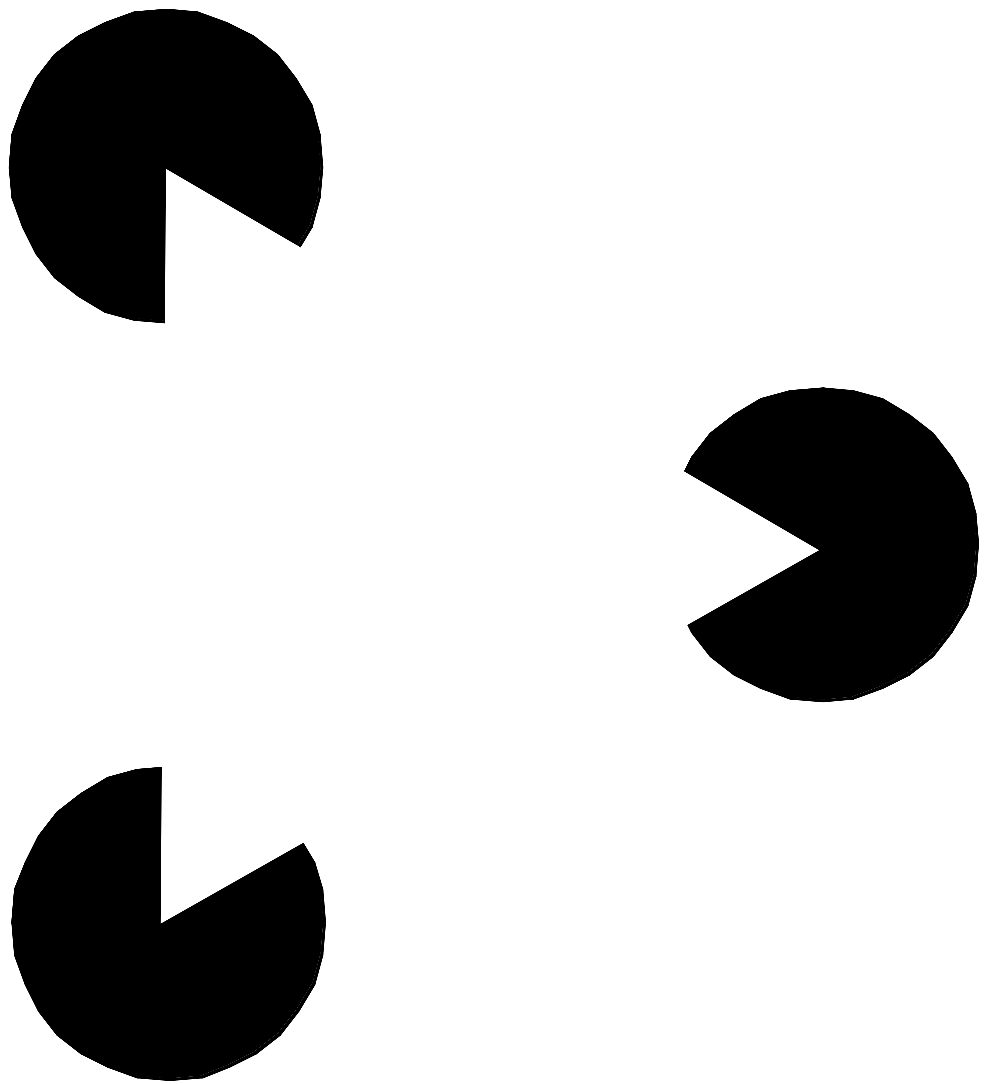
We jump now to visual perception. The low level computations are now neuronal and the high level structures are perceptive judgments.

The functional architecture of V1

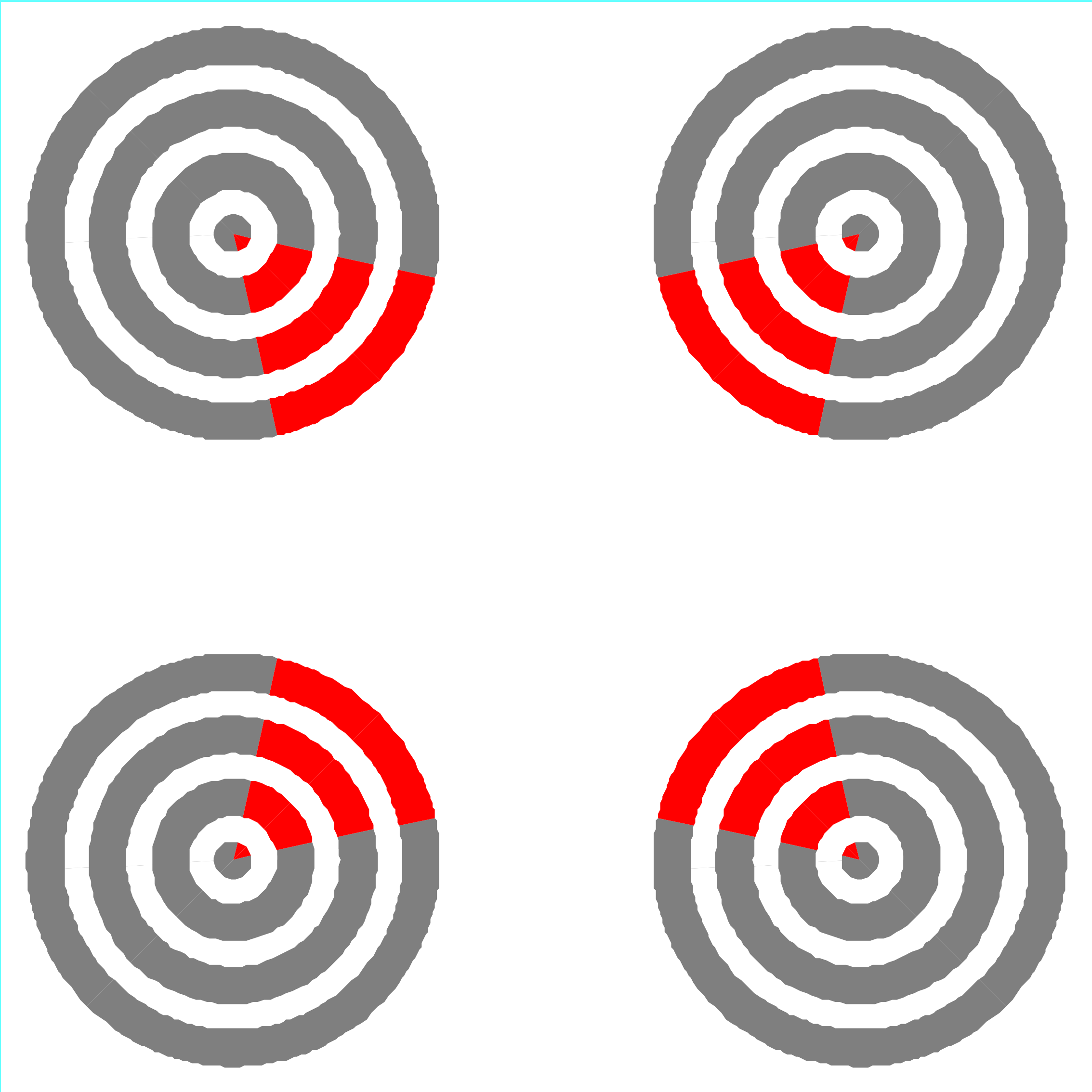
With my colleague Jacques Ninio, we have explored the shape of *curved* Kanizsa modal subjective contours and proposed a *variational model* whose *simplest* (very insufficient) version is based on the *functional architecture* of V1. The idea goes back to Shimon Ullman (1976: « Filling-in the gaps: the shape of subjective contours and a model for their generation », *Biological Cybernetics*) who says that.

> « A network with the local property of trying to keep the contours " as straight as possible " can produce curves possessing the global property of minimizing total curvature. »

A classical Kanizsa illusory contour



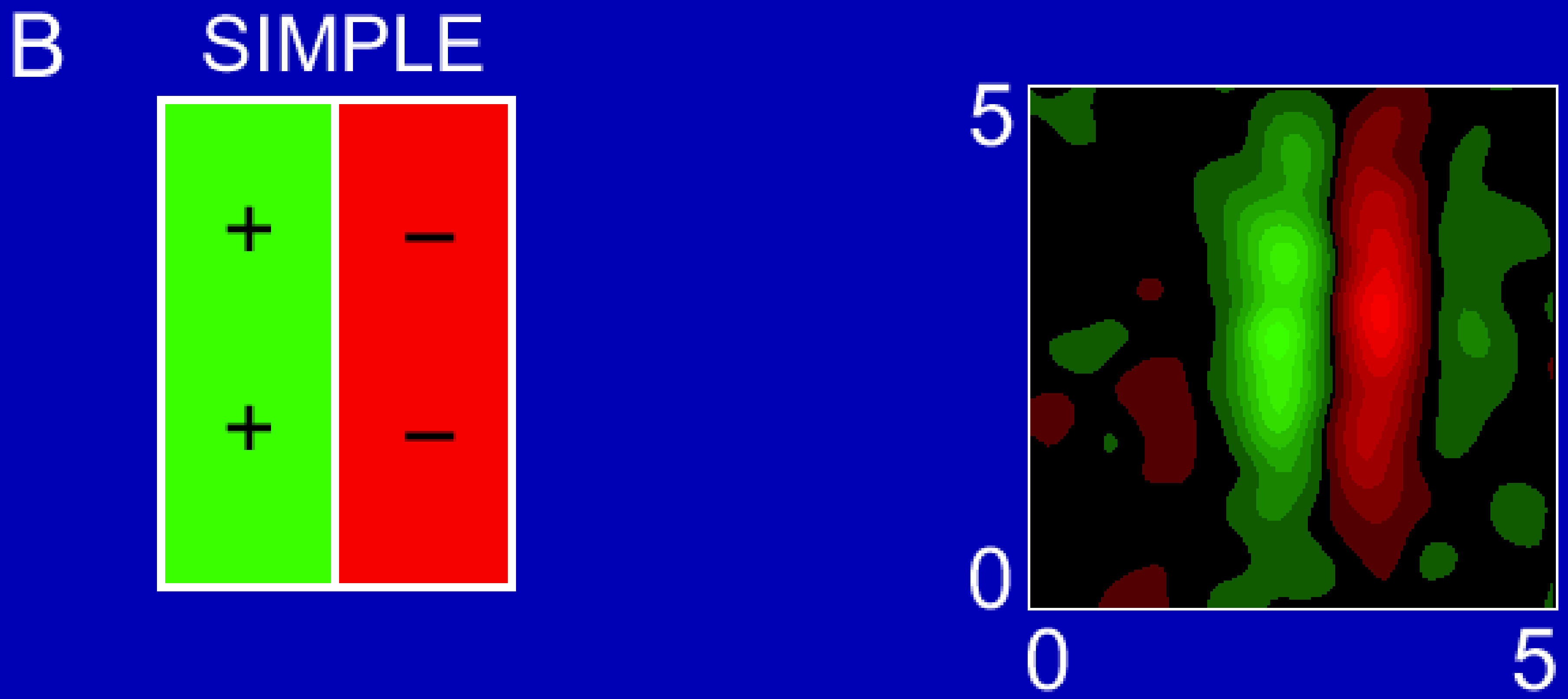
A curved Kanizsa square with watercolor effect



Simple cells in V1

The simple cells of V1 are neurons which detect pairs (*a*,*p*) of a retinian position *a* (their receptive field) and a preferential orientation *p*. This is due to the very specific form of their receptive profile which is a third derivative of Gaussian (as was shown by David Marr in the late 1970s the ganglionary cells of the retina have a receptive profile which is a Laplacian of Gaussian). They act as filters on the optical input by convolution.

The next slide shows the *level curves* (positive and negative, i.e. ON and OFF) of the receptive profile of an orientation neuron as they were recorded by Gregory DeAngelis.


B
SIMPLE
+
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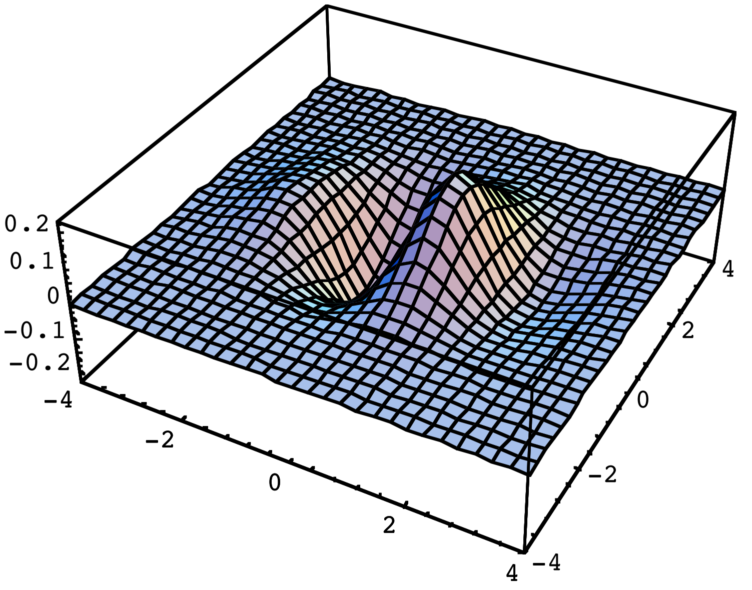
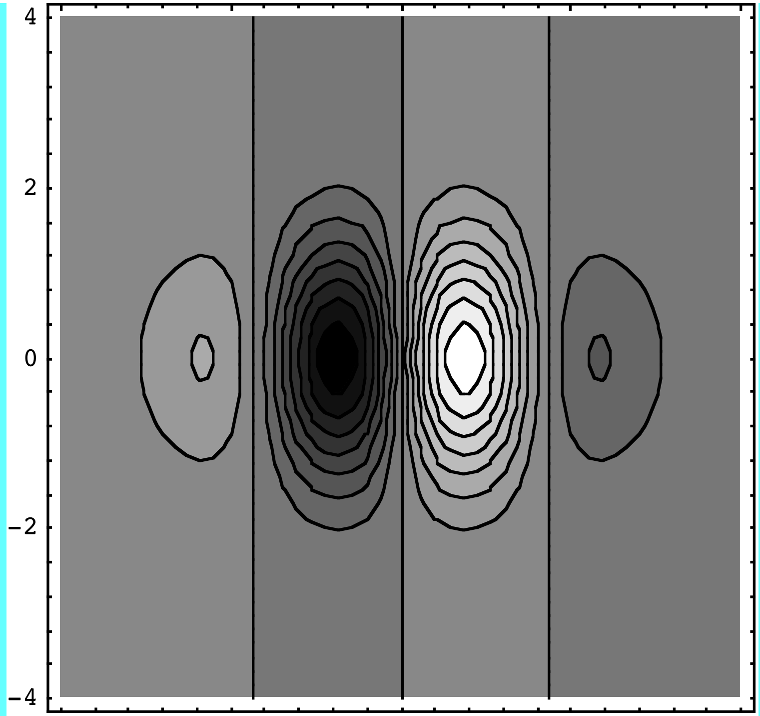
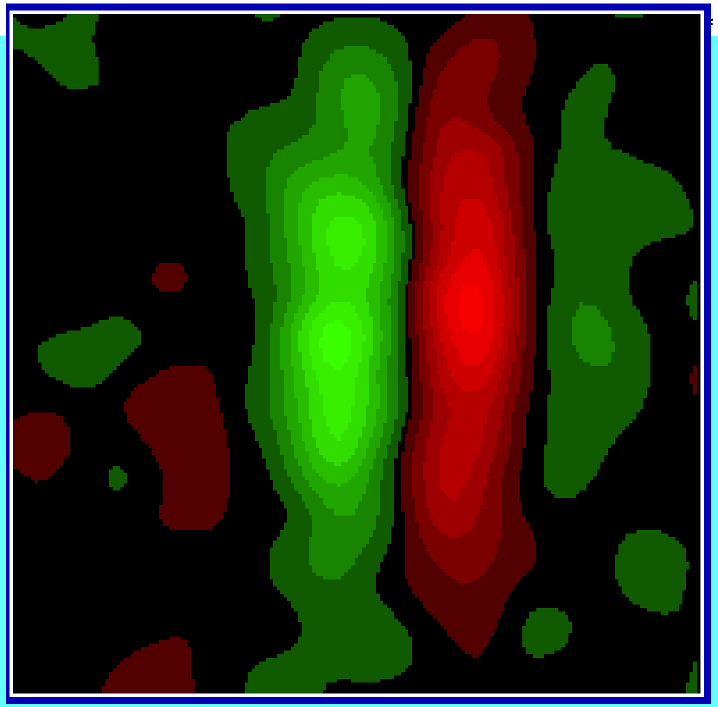




Der3Gaussx3

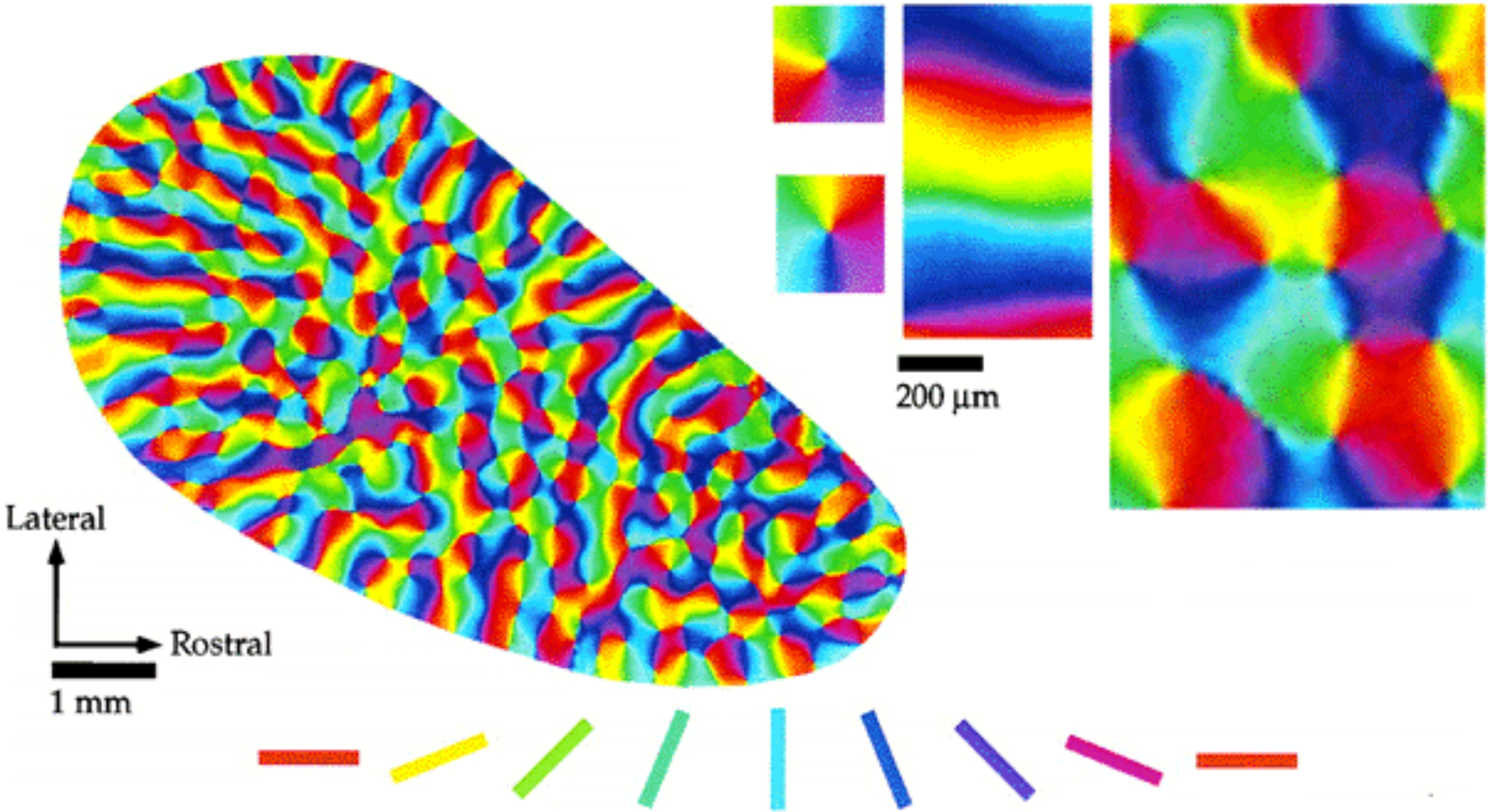


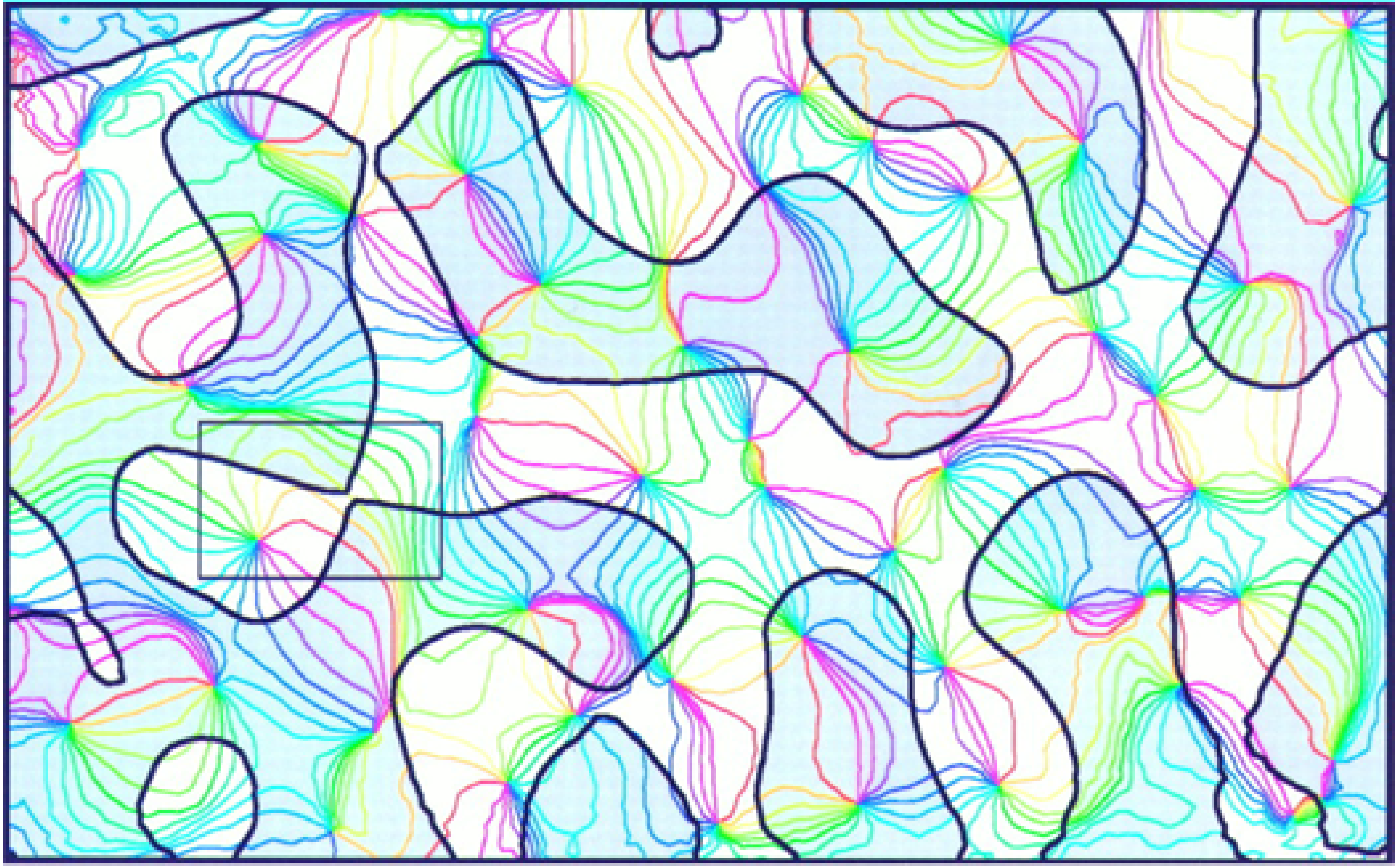
Pinwheels

Recent experiments have shown that the hypercolumns are geometrically organized in *pinwheels*. The cortical layer (4C) is reticulated by a network of singular points which are the centers of the pinwheels. Around these singular points all the orientations are represented by the rays of a "wheel" and the wheels are glued together in a global structure.

In the next slide the orientations are coded by colors and iso-orientation lines are therefore coded by monocolor lines.

William Bosking, Ying Zhang, Brett Schofield, David Fitzpatrick (Dpt of Neurobiology, Duke) 1997, « Orientation Selectivity and the Arrangement of Horizontal Connections in Tree Shrew Striate Cortex », *J. of Neuroscience*.


200 μm
Lateral
Rostral
1 mm



Fibrations and parallel transport

The fonctional architecture associating retinotopically to each position *a* of the retina *R* an exemplar P_a of the space of the orientations at *a* implements a very well known geometrical structure, namely the *fibration* $\pi : R\times P\rightarrow R$ with base *R* and fiber *P*.

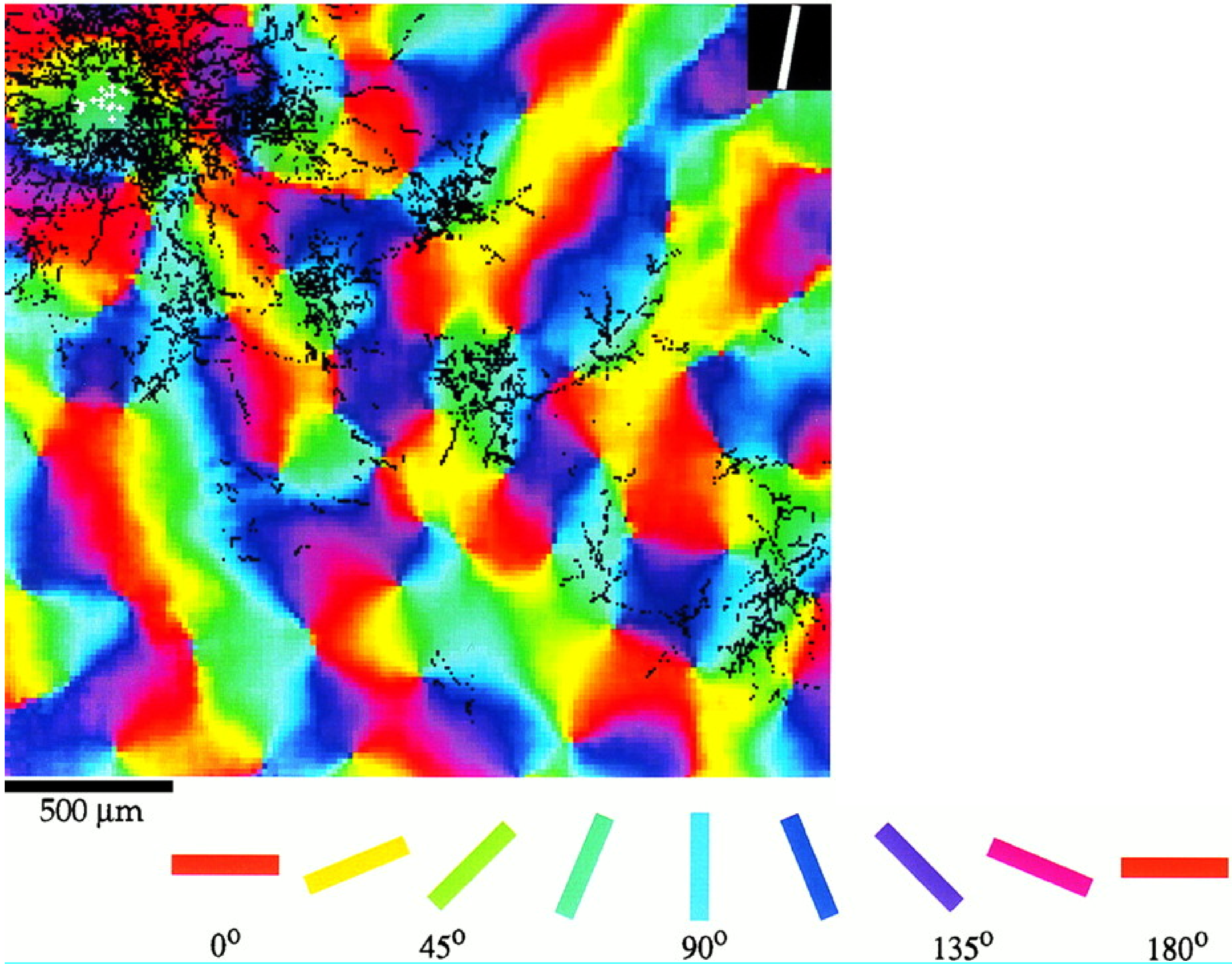
But such a "vertical" structure is not sufficient. To implement a *global coherence*, the visual system must be able to *compare two* retinotopically neighboring fibers P_a et P_b over two neighboring points *a* and *b*. This is a problem of *parallel transport*. It has been solved at the empirical level by the discovery of *"horizontal" cortico-cortical connections*.

Cortico-cortical connections connect neurons of the *same* orientation in neighboring hypercolumns. This means that the system is able to know, for *b* near *a*, if the orientation *p* at *a* is the same as the orientation *q* at *b*.

The retino-geniculo-cortical "vertical" connections give an internal meaning to the relations between (*a*,*p*) and (*a*,*q*) (*different* orientations *p* and *q* at the *same* point *a*) .

The "horizontal" cortico-cortical connections give an internal meaning to the relations between (*a*,*p*) and (*b*,*p*) (*same* orientation *p* at *different* points *a* and *b*).

The next slide shows how biocytin injected locally in a zone of specific orientation (green-blue) diffuses via horizontal cortico-cortical connections. The key fact is that even if the short range diffusion is isotropic, the *long range diffusion* is on the contrary highly *anisotropic* and restricted to zones of the *same* orientation (the same color) as the initial one.



500 µm
0°
45°
90°
135°
180°

Moreover cortico-cortical connections connect neurons coding pairs (*a*,*p*) and (*b*,*p*) *such that p is the orientation of the axis ab*.

"The system of long-range horizontal connections can be summarized as preferentially linking neurons with co-oriented, co-axially aligned receptive fields". (William Bosking)

The integrability condition

This functional architecture explains the *integration* of contours. If you take a sequence of local stimuli (a_i,p_i), the corresponding orientation neurons are activated and each of them *coactivates* via horizontal connections neighboring aligned neurons. Therefore if it is possible to *interpolate* between the positions a_i by a curve γ in such a way that the p_i are *tangent* to γ at the a_i (this is an *integrability condition*), then the cortex will activate γ (saliency and pop-out).

And if the distance between the a_i is important, then we must optimize the interpolation by using a *variational* trick. It is the case for Kanizsa illusory modal contours.

Of course, a correct model would have to take into account other areas such as V2 which have a top down feedback influence on V1. In fact, it can be shown that the *same* boundary conditions for a Kanizsa contour generate *different* illusory contours in the case of a triangle and in the case of a square.

Perceptive judgments.
Husserl

We now turn to the question of perceptive judgments.

Husserl was an extraordinary precursor. In *Erfahrung und Urteil* he tries to clarify phenomenologically what he calls *the "perceptive genealogy" of predication*. According to him, classical logic *conceals* the way predicative judgments such as "*S* is *p*" are rooted in *ante*-predicative and *pre*-judicative (non conceptual) perceptive experience.

> "The formal character of logical Analytics consists in the fact that it does not consider the material quality of what is given prior to judgments, and that it looks to substrates only in what concerns the categorial form they take in the judgement."

Husserl points out a fundamental conversion from the *synthetic* (geometric) perceptive unity unifying an extension *W* with its qualitative moments *p* to the *analytic* logical unity of the judgment "*S* is *p*". He says :

> “In the most simple predicative judgment a *double information* is processed” (p. 247).

Underlying the syntactic “subject / predicate” information concerning what he calls the “functional forms” of the terms of the proposition, there exists another information concerning the “kernel forms” “substrate and spatial extension” and “quality and filling-in”. This underlying information is presupposed by the syntactic one. Predication is a process based on

> “the covering of the kernel forms as syntactic material by the functional forms” (p. 248).

This logical *typing* via syntactic categories of the synthetic perceptive dependence relations between spatial extensions and qualities must be correctly formalized.

Exactly as it was emphasized by Husserl in *Erfahrung und Urteil*, we can say that, at least in the case of perceptive judgments, *language is a categorial typing of low level perceptive algorithms and a reflexive decompilation of their neural implementation*. We can therefore give now a precise meaning to the difference between phenomenal perceptive structures and predicative perceptive judgments.

The fundamental difference between the λ-calculus logical case and the perceptive case is that the neural calculus processing visual information is not a λ-calculus. But we can nevertheless go quite far in linking its geometrical structures to logical typing procedures. I tried to achieve this task using topos theory.

From geometry to logic : the perceptive analog of the Curry-Howard correspondance

When we look precisely at the processes underlying perception, we note that fillings-in result of *gluing* together *local* fillings-in. The fovea is a sort of local spot which glance through the ambiant space M and which is controled by the kinesthesic movements of the body, the head, and the eyes. This means that the map $g_{v,S} : W_{v,S} \to Q$ must be viewed as a *section* on $W_{v,S}$ of the *fibration* $\pi : M \times Q \to M$. It can be shown experimentally that such fibrations are neurally implemented in the functional architecture of the retinotopic areas of the primary visual cortex.

But the sections of a fibration (the filling-in of spatial extensions by qualities) satisfy two fundamental properties already identified by Husserl :

(i) the *restriction* of sections defined on W to subdomains V of W.

(ii) the *gluing* of compatible sections defined on a covering W_i of W in a global section defined on W.

The axiomatization of these properties yields the fundamental concept of a *sheaf*.

Let us suppose that the extensions W we consider are open subsets of the global space M (this hypothesis is false, we will return to this point). Let $\mathfrak{Q}$ be the sheaf of the sections g and $\mathfrak{Q}_i$ the subsheaves corresponding to the qualitative categories Q_i. Let $\mathfrak{p}$ be the subsheaf $\mathfrak{Q}_i$ corresponding to the quality p. The perceptive truth conditions of the perceptive state of affairs $<S, p>$ are

$$\text{“}S \text{ is } p\text{” is realized iff } g_{v,S} \in \Gamma\mathfrak{p}(W_{v,S})$$

($g_{v,S}$ is a section of the subsheaf $\mathfrak{p}$ over the spatial extension $W_{v,S}$).

Now, the category **Sh**(M) of sheaves on a base space M constitutes a *topos*, and, as was shown by Bill Lawvere in the 70s, every topos is canonically endowed with *an internal logic* where sheaves are identified to *types*. All these technicalities are necessary *to localize the notion of truth*.

The sub-object $\mathfrak{p}$ of $\mathfrak{O}$ corresponds to a "predicate" on $\mathfrak{O}$ in the topos sense, that is to a morphism $\varphi_{\mathfrak{p}} : \mathfrak{O} \to \Omega$, where Ω is the subobject classifier of the category **Sh**(M) of sheaves on M. We have $\Omega(U) := \{W \subset U\}$ and the True map True : $1 \to \Omega$ (where 1 is the terminal object of **Sh**(M)) is defined by True(U) : $1 \to U \in \Omega(U)$ that is by the *maximal* element of $\Omega(U)$: to be true over U is to be true "everywhere" over U. If $j : \mathfrak{p} \to \mathfrak{O}$ is the monomorphism defining $\mathfrak{p}$, its characteristic map $\varphi_{\mathfrak{p}} : \mathfrak{O} \to \Omega$ is given by the map $\varphi_{\mathfrak{p}}(U) : \mathfrak{O}(U) \to \Omega(U)$ which associates to each section g of $\mathfrak{O}(U)$ the maximal subextension V of U s.t. $g_{|V}$ belongs to $\mathfrak{p}(V)$. $\varphi_{\mathfrak{p}}$ is given by the pull-back

$$\begin{array}{ccc} \mathfrak{p} & \to & 1 \\ j \downarrow & & \downarrow \text{True} \\ \mathfrak{O} & \to & \Omega \\ & \varphi_{\mathfrak{p}} & \end{array}$$

We have then

$$\text{“}S \text{ is } p\text{” is realized iff } \varphi_{\boldsymbol{p}}(W_{v,S})(g_{v,S}) = \text{True.}$$

If we compare with the classical interpretation:

$$\text{“}S \text{ is } p\text{” is realized iff } p(S) = \text{True,}$$

we see that to take into account the geometrical structures of perception we have to substitute the “sheaf” or “topos” predicate $\varphi_{\boldsymbol{p}}(W_{v,S})(g_{v,S})$ for the classical predicate p. But this exactly means that the predicate *p types* the filling-in of spatial domains with some quality.

A remark to conclude. In perceptive statements, *boundaries* play a fundamental role. And one knows since Brentano that boundaries are somehow paradoxical entities. To tackle this point, we can use Lawvere's idea of *co-Heyting* algebras. In a Heyting algebra of open sets, the negation $\neg U$ of U is the *interior* of its complementary set, that is the largest open set V such that $U \cap V = \varnothing$. In a co-Heyting algebra of closed sets, the negation $\neg F$ of F is dually the smallest element H such that $F \cup H = 1$ (that is the closure of its complementary open set). One has

$$\neg(F \cap H) = \neg F \cup \neg H, \text{ but only } \neg(F \cup H) \subseteq \neg F \cap \neg G.$$

One defines then the *boundary* ∂F of F as the intersection $F \cap \neg F$. ∂F is therefore defined by logical "contradiction". The boundary operator satisfies the Leibniz rule :

$$\partial(F \cap H) = (\partial F \cap H) \cup (F \cap \partial H).$$

Boundaries are characterized by $\partial B = \varnothing$ that is by $\neg B = 1$ or $\neg\neg B = 0$. In general, the double negation $\neg\neg F \subset F$ is the "regular core" of F (the closure of its interior). One has of course $F = (\neg\neg F) \cup \partial F$.