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A neo-transcendental approach to the continuum problem

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Abstract

One of the main epistemological difficulties concerning the continuum problem is the conflict existing between, on the one hand, the mathematical need of "Platonist" higher order infinites (large cardinal axioms) for elaborating a "good" set theory of the continuum, and, on the other hand, traditional "antiplatonist" (nominalist) philosophical requisites. Even if there exist alternative approaches (e.g. categorical) to the continuum, the "Platonist" challenge in set theory must be taken up. For that, we must understand mathematical Platonism in mathematics in a non ontological "transcendent" way. We have proposed a "neo-transcendental" Platonism. It has something to do with what Hugh Woodin recently called "conditional" Platonism.

1 Introduction

In mathematics, Platonism concerns the legitimacy of non constructive axioms positing the existence of higher infinities. It is in general denied because of its apparently too strong ontological commitment.

Classical dominant antiplatonism is nominalist and ontologically "deflationary". It argues that mathematics must be constructive for metaphysical reasons. But ontological considerations are irrelevant for mathematics, because there is no ontology of mathematical structures. Mathematics are *objective* in the strongest sense, but "objectivity" does not mean "ontology". And as far as transcendental philosophy is the philosophical thematization of the difference between scientific objectivity and metaphysical ontology, we will call *transcendental Platonism* a Platonism rooted in a transcendantal conception of objectivity.

2 The philosophical aporia of platonism

The classical question of Platonist realism in philosophy is twofold:

1. that of the existence of mathematical idealities, that is of the acceptability of an ontology of independent (transcendent) abstract entities (ontological realism),

2. that of the acceptance of truth conditions transcending our cognitive abilities, and our epistemic capacities of accessing objects (semantic realism).

Classical Platonism takes for granted that ideal subsistent, transcendent, and abstract objects can serve as "truth makers" for mathematical statements. As was emphasized by Crispin Wright:

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> "The traditional platonist answer is that the truth-conditions of pure mathematical statements are constituted by the properties of certain mind-independent abstract objects, the proper objects of mathematical reflection and study." [1]

This type of realism is classically rejected because *reference* is considered as the converse relation of a *causal* action of an external state of affairs on the mind and ideal abstract entities cannot possess any causal efficiency. It is argued that, as far as "to exist" means "to exist physically in the external world as a material spatio-temporal thing", it is impossible to get an epistemic access (any learning, any belief, moreover any true belief, moreover any rationally justified true belief, any knowledge) to abstract non causal entities. As was emphasized by Michael Resnik:

> "If we have no physical traffic with the most basic mathematical entities and they are not literally the products of our own minds either, how can we learn any mathematics? How could it even be possible for us to acquire beliefs about mathematical objects? Since Platonic mathematical objects do not exist in space or time the very possibility of our acquiring knowledge and beliefs about them comes into question." [2]

In other words, the causal theory of reference forbids, according to Philip Kitcher, that symbolic constructions

> "provide any type of access to abstract reality".[3]

This classical conception of Platonism runs into an antinomy:

Thesis. *Mathematics are descriptive and true. They describe abstract transcendent entities which exist in an ideal external world and serve as truth-makers.*

Antithesis. *Mathematics are prescriptive and not descriptive. They are analytic and conventional, and concern grammatical rules for the use of abstract concepts. They don't have external truth-makers.*

The spectrum of the philosophical replies to this antinomy is very large indeed.

• For radical Platonists such as Gödel, we do have an intuitive access to idealities and theses intuitions exceed the ressources of a Turing machine.

• For empirist Platonists such as John Burgess, the possibility of a causal interaction with abstracta is a purely scientific (and not philosophical) problem pertaining to cognitive sciences:

> "A philosopher's confession that knowledge in pure and applied mathematics perplexes him constitutes no sort of argument for nominalism, but merely an indication that the philosopher's approach to cognition is inadequate." [4]

According to Burgess, the only acceptable mathematical epistemology must be *internal* to mathematics and an externalist argument such as the causal theory of reference is therefore irrelevant.

• Penelope Maddy has also developed such an idea.

• According to Michael Resnik, one of the most influential advocates of a *structural* Platonism, the main error of the nominalist conception of mathematics is to believe that any well defined objectivity has to refer to *things*. But in mathematics, quantification acts on elements of structures defined through a network of relations. As was clearly explained by Chihara,

[1] Wright [1988], p.426.
[2] Resnik [1988], p.403.
[3] Kitcher [1988], p.527.
[4] Burgess [1983].

> "Resnik is a platonist of sorts: he believes in the existence of abstract mathematical objects. By characterizing these objects as mere positions in a structure (...), he thought he could avoid the chief philosophical problems that have plagued the traditional platonic views of mathematics." [5]

At the other extremity of the epistemological spectrum, antiplatonists often reduce mathematical contents to mental representations. Such a "psychological" perspective subordinates the epistemology of mathematics to some cognitive psychology

• A good exemple is that of Philip Kitcher.[6] According to him, mathematics constitute a symbolic activity structuring our experience using idealities. But these symbolic constructions has nothing to do with objects. They only specify

> "the constructive power of an ideal subject" (subjective idealism)[7].

• At the nominalist extremity of the epistemological spectrum we find radical antiplatonists such as Hartry Field in *Science without Numbers*.[8] Field gives up the problem of "truth-makers" in mathematics and, inspired by the well known procedures of elimination of theoretical terms (Mach, Hempel), he tries to show that scientific theories using mathematics are in some sense "conservative" on theories *without* mathematics.

We see that the rejection of Platonism is justified mainly by a parallelism between the objectivity of mathematics and an ontology of things. For instance, when Feferman evaluates Gödel's Platonism saying:

> "I am convinced that the platonism which underlies Cantorian set theory is utterly unsatisfactory. (...) To echo Weyl, platonism is the medieval metaphysics of mathematics; surely we can do better." [9]

he interprets Platonism as the ontological thesis stating that ZFC bears on a well defined world where everything would be determined and decidable.[10] As this is false, the mathematical non constructive concepts are inherently vague and platonism comes therefore into question. But we can also consider that, in mathematics, vericonditional semantics possesses neither ontological nor cognitive contents, that set theoretical "ontology" is a *semantic* which "mimics" an ontology, that it is only a "quasi-ontology" and that the very question is not the ontology of mathematics but the link between mathematics and objectivity.

Between, on the one hand, a transcendent ontological reality (transcendent realism) and, on the other hand, an immanent reality reduced to its epistemic accessibility (subjective idealism) – which are both irrelevant for mathematics –, we need an alternative perspective. Our option is to correlate mathematics to the deep structures of objectivity, and, as far as the transcendental conception of objectivity is up to now the only known philosophy not confusing objectivity with ontology, we have to correlate mathematics with transcendental elements of objectivity. Now, one of the key elements is the *continuum* as a "pure intuition" (Kant's transcendental aesthetics). Therefore, in what concerns the continuum, we think that a philosophically sound strategy is to try to *model* mathematically, for instance in the framework of set theory, the properties of the continuum as a background structure of objectivity.

I think we can consider Peirce as the founder of such a philosophical strategy. His philosophy of the continuum is one of the best. It is a *non compositional* conception based on a logic of *vagueness*, where the principles of non contradiction and excluded middle are no longer valid. Peirce defended the thesis of an "inexhaustibility" of the continuum.[11] He criticized nominalism and logical atomism, and developed the idea that a logic adequate to the continuum had to be a logic of vagueness. He had a clear

[5] Chihara [1990], p.145.
[6] See e.g. Kitcher [1983] and [1988].
[7] Kitcher [1983], p.160.
[8] Field [1980].
[9] Feferman [1989].
[10] ZFC means Zermelo-Fraenkel set theory with the axiom of choice.
[11] See Claudine Engel-Tiercelin *La pensée-signe*. See also Machuco [1993].

consciousness of the limits of the Weierstrass-Cantor-Dedekind arithmetization of the continuum and of what he called “the distrust of intuition” . For him, it was possible to insert in every Dedekind cut full models of $\mathbb{R}$ of incommensurable scales. As far as I know, he was the first to think of the continuum as a non archimedean field and he initiated the perspective going to Veronese, non standard Analysis and surreal numbers. In fact, he defined the cardinal of $\mathbb{R}$ as an inaccessible cardinal

$$|\mathbb{R}| > 2^{2^{2^{\cdots \aleph_0 \text{ times}}}}.$$

Our main philosophical problem will be to justify these anti-nominalist approaches of the intuitive continuum and to present models of its non constructive properties in the framework of set theory. What must be the structure of such models is a *mathematical* open question which must be tackled avoiding any philosophical prejudice. Now, a fantastic accumulation of deep and difficult results show that this is possible if we accept a very rich – maximal and not minimal – quasi-ontology for sets. They justify strong platonist commitments.

We will now recall some of these classical results.

3 Projective determinacy and large cardinals

3.1 The Borel and projective hierarchies

In descriptive set theory one works on $\mathbb{R}$ or on $\mathcal{N} = \mathbb{N}^{\mathbb{N}}$ ($\mathcal{N} = \omega^{\omega}$), or on $\mathcal{C} = 2^{\mathbb{N}}$, and, more generally, on metric, separable, complete, perfect (closed without isolated points) spaces $\mathcal{X}$. One consider in $\mathcal{X}$ different definable classes of subsets Γ. The first is the Borel hierarchy constructed from the open sets by iterating the operations of complementation and of “projection” $\mathcal{X} \times \mathbb{N} \to \mathcal{X}$. If $P \subseteq \mathcal{X} \times \mathbb{N}$ (that is, if P is a countable family of subsets $P_n \subseteq \mathcal{X}$), one considers the subset of $\mathcal{X}$ defined by $\exists^{\omega} P = \exists^{\mathbb{N}} P := \{x \in \mathcal{X} \mid \exists n \; P(x,n)\}$. It is the union $\bigcup_n P_n$ of the P_n.

The open subsets are noted Σ^0_1 , and the closed subsets $\Pi^0_1 = \neg\Sigma^0_1$. The Borel hierarchy B is then defined recursively by:

$$\begin{aligned} \Pi^0_n &= \left\{\neg\varphi \mid \varphi \in \Sigma^0_n\right\} = \neg\Sigma^0_n \\ \Sigma^0_{n+1} &= \exists^{\omega}\neg\Sigma^0_n = \exists^{\omega}\Pi^0_n \\ \Delta^0_n &= \Pi^0_n \cap \Sigma^0_n. \end{aligned}$$

This hierarchy is *strict*:

$$\begin{array}{ccccc} & & \Sigma^0_n & & \\ & \nearrow & & \searrow & \\ \Delta^0_n & & & & \Delta^0_{n+1} \\ & \searrow & & \nearrow & \\ & & \Pi^0_n & & \end{array}$$

One defines then the hierarchy of *projective* sets using a supplementary principle of construction, namely projections by *continuous* projections $\mathcal{X} \times \mathcal{N} \to \mathcal{X}$, written $\exists^{\mathcal{N}}$. One gets that way a new hierarchy beginning with the class $\Sigma^1_1 = \exists^{\mathcal{N}}\Pi^0_1$ – the so called *analytic* subsets – and continuing with the classes:

$$\begin{aligned} \Pi^1_n &= \left\{\neg\varphi \mid \varphi \in \Sigma^1_n\right\} = \neg\Sigma^1_n \\ \Sigma^1_{n+1} &= \exists^{\mathcal{N}}\neg\Sigma^1_n = \exists^{\mathcal{N}}\Pi^1_n \\ \Delta^1_n &= \Pi^1_n \cap \Sigma^1_n. \end{aligned}$$

- For instance, $P \subseteq \mathcal{X}$ is Σ^1_1 if there exists a *closed* subset $F \subseteq \mathcal{X} \times \mathcal{N}$ such that:

$$P(x) \Leftrightarrow \exists\alpha\ F(x,\alpha).$$

- In the same way, $P \subseteq \mathcal{X}$ is Σ_2^1 if there exists an *open* subset $G \subseteq \mathcal{X} \times \mathcal{N} \times \mathcal{N}$ such that:

$$P(x) \Leftrightarrow \exists\alpha\ \forall\beta\ G(x,\alpha,\beta), \text{ etc.}$$

As the Borel hierarchy, the projective hierarchy is *strict*. Moreover, it is a continuation of the Borel hierarchy according to:

Suslin theorem. $B = \Delta_1^1$.

This beautiful theorem can be interpreted as a *construction principle*: it asserts that the complex operation of continuous projection can be reduced to an iteration of simpler operations of union and complementation.

There exist Π_n^1 and Σ_n^1 sets who are very natural in classical Analysis. For instance in the functional space $C[0,1]$ of real continuous functions on $[0,1]$ endowed with the topology of uniform convergence, the subset:

$$\{f \in C\,[0,1] \mid f \text{ smooth}\}$$

is Π_1^1 (and not Δ_1^1).

In the space $C[0,1]^\omega$ of sequences (f_i) of functions, the subset:

$$\left\{ (f_i) \in C\,[0,1]^\omega \;\middle|\; \begin{array}{c} (f_i) \text{ converges for the topology} \\ \text{of simple convergence} \end{array} \right\}$$

is Π_1^1 and the subset:

$$\left\{ (f_i) \in C\,[0,1]^\omega \;\middle|\; \begin{array}{c} \text{a sub-sequence converges for the topology} \\ \text{of simple convergence} \end{array} \right\}$$

is Σ_2^1 and every Σ_2^1 can be represented that way (Howard Becker [12]):

Becker representation theorem. *For every* Σ_2^1 *set* $S \subseteq C[0,1]$ *there exists a sequence* (f_i) *such that*

$$S = A_{(f_i)} = \left\{ g \in C\,[0,1] \;\middle|\; \begin{array}{c} \text{a sub-sequence of } (f_i) \text{ converges towards } g \\ \text{for the topology of simple convergence} \end{array} \right\}.$$

Other examples are given by the compact subsets $K \in \mathcal{K}(\mathbb{R}^n)$ of $\mathbb{R}^n$: for $n \geq 3$,

$$\{K \in \mathcal{K}(\mathbb{R}^n) \mid K \text{ is arc connected}\}$$

is Π_2^1 and for $n \geq 4$,

$$\{K \in \mathcal{K}(\mathbb{R}^n) \mid K \text{ is simply connected}\}$$

is also Π_2^1.

3.2 The regularity of projective sets

The French school (Borel, Baire, Lebesgue) and the Polish school (Suslin, Lusin, Sierpinski) initiated the study of the Borel and projective classes and achieved deep results concerning there *regularity* and their *representation* where “regularity” means Lebesgue measurability, or the perfect set property (to be countable or to contain a perfect subset), or the Baire property (to be approximated, in the sense of a meager symmetric difference, by an open subset).

The first regularity theorem is the celebrated:

Cantor-Bendixson theorem. *If* $A \subseteq \mathbb{R}$ *is closed, then* A *can be decomposed in a unique way as* $A = P + S$ *where* P *is* perfect *and* S countable.

[12] See Becker [1992].

As a perfect set P is of cardinality $|P| = 2^{\aleph_0}$, the *continuum hypothesis CH holds for the closed sets* Π^0_1.

Another early great classical theorem of regularity is the:

Suslin theorem. *The analytic subsets* Σ^1_1 *shares the perfect subset property and CH is therefore true for the* Σ^1_1.

To prove this theorem one uses the representation of the Σ^1_1 subsets as $\aleph_0$-Suslin sets:

Definition. $P \subseteq \mathcal{X}$ *is* χ*-Suslin (where* χ *is an infinite cardinal) if there exists a closed subset* $F \subseteq \mathcal{X} \times \chi^{\mathbb{N}}$ *such that* P *is the projection of* F $(P = \exists^{\chi^{\mathbb{N}}} F)$.

If $\chi = \aleph_0$, then $P = \exists^{\mathcal{N}} F$ and $P \in \Sigma^1_1$.

A theorem of Martin says that $P \subseteq \mathcal{X}$ is $\aleph_n$-Suslin iff $P = \bigcup_{\xi < \aleph_n} P_\xi$ is a union of Borel sets P_ξ.[13]

Using this representation as projections of closed sets allows to use the representation of closed sets $F \subseteq X^{\mathbb{N}}$ by means of *trees* on X.[14]

Definition. *A tree on* X *is a set* T *of finite sequences of elements of* X *such that if* $u \in T$ *and* $v \leq u$ *is an initial segment of* u *then* $v \in T$. *A path in* T *is an infinite branch, namely a function* $f \in X^{\mathbb{N}}$ *such that* $\forall n\ f|_n \in T$. $[T]$ *is the set of paths of* T.

Theorem. $F \subseteq X^{\mathbb{N}}$ *is closed iff there exists a tree* T *on* X *such that* $F = [T]$.

Using this representation theorem one proves the regularity result:

Theorem. *If* P *is* χ*-Suslin of cardinality* $|P| > \chi$, *then* P *contains a perfect subset.*[15]

Suslin theorem is the simplest case: if P is Σ^1_1, then P is $\aleph_0$-Suslin. If it is uncountable, then $|P| > \aleph_0$ and therefore P contains a perfect subset.

In the same way, one can show that the Σ^1_1 sets share the Baire property and that the Σ^1_1 and Π^1_1 are Lebesgue measurable. But it is *impossible* to show in ZF that the Δ^1_1 and Σ^1_2 share the perfect set property and to show in ZFC that the Δ^1_2 share the Baire property. In fact many of the "natural" properties of the projective sets go far beyond the demonstrative strength of ZF and ZFC. *It is therefore justified to look for additional axioms.*

3.3 The underdetermination of cardinal arithmetic in ZFC

In fact, ZFC determines quite nothing of the cardinal arithmetic of a universe of sets V.[16]

In a model V of ZFC, let $F(\alpha)$ be the function defined by $2^{\aleph_\alpha} = \aleph_{F(\alpha)}$. One can show that:

(i) F is a monotonous increasing function: if $\alpha \leq \beta$ then $F(\alpha) \leq F(\beta)$;

(ii) König's law: $\mathrm{cf}\left(\aleph_{F(\alpha)}\right) > \aleph_\alpha$, where the *cofinality* $\mathrm{cf}(\alpha)$ of an ordinal α is defined as the smallest cardinal χ such that there exists a subset X of cardinality χ which is cofinal in α (i.e. $\mathrm{Sup}\, X = \alpha$). The cardinal χ is called *regular* if $\mathrm{cf}(\chi) = \chi$.

If GCH holds, König's law is trivial because $F(\alpha) = \alpha+1$, every cardinal $\aleph_{\alpha+1}$ is regular, and therefore $\mathrm{cf}\left(\aleph_{\alpha+1}\right) = \aleph_{\alpha+1} > \aleph_\alpha$.

The fact that ZFC underdetermines cardinal arithmetic is particularly striking in the following result:

Easton theorem. *For the* regular $\aleph_\alpha$ *one can impose by forcing in ZFC the law* $2^{\aleph_\alpha} = \aleph_{F(\alpha)}$ *for quite every function* F *satisfying (i) and (ii).*

Contrary to first order aritmetic which is ZF-absolute (invariant with respect to extensions of the universe), the structures and notions such as $\mathcal{N}$, $\mathbb{R}$, $\mathrm{Card}(\chi)$, $x \to \mathcal{P}(x)$, $x \to |x|$, second order arithmetic, *are not ZF-absolute.* They can vary widely from one universe V to another and can't have absolute truth value in ZF. This *vagueness* has been emphasized by Hugh Woodin in his recent paper "Set theory after Russell. The journey back to Eden" (2003). It is used as a major argument by antiplatonists. But

[13] See Moschovakis [1980], p.97.
[14] See Grigorieff [1976].
[15] See Moschovakis [1980], p.79.
[16] See Jacques Stern [1976].

vagueness is not a so dramatic argument against Platonism. It shows mainly that it is necessary to *classify* the different models V of ZF and ZFC.

Two opposed strategies are possible, both introduced by Gödel, one "minimalist" and the other "maximalist".

3.4 The "minimalist" strategy of the constructible universe

The first strategy consists in *restricting* the universe V. It is Gödel's strategy $V = L$ of constructible sets (Gödel 1938).

To define L one substitute, in the construction of the cumulative hierarchy of V by means of the power operation $x \to \mathcal{P}(x)$, the sets $\mathcal{P}(x)$ – which are not ZF-absolute – by smaller sets $\mathcal{D}(x) = \{y \subseteq x \mid y$ elementary (definable by a first order formula of the structure $< x, \in, \{s | s \in x\} >)\}$ – which are ZF-absolute. L is then defined using a transfinite induction on ordinals: $L_0 = \emptyset$, $L_{\xi+1} = \mathcal{D}(L_\xi)$, $L_\lambda = \bigcup_{\xi<\lambda} L_\xi$ if λ is a limit ordinal, and $L = \bigcup_{\xi \in On} L_\xi$.

Gödel (1938-1940) has shown that if $V = L$ it is possible to define a *global well ordering* on L, which is a very strong form of AC. He proved also that $ZF + (V = L) \vdash HCG$.

L is in fact the *smallest* inner model of V, that is:

(i) $On \subset L$,

(ii) L is transitive: if $y \in_V x \in_L L$, then $y \in_L L$,

(iii) $(L, \in |_L)$ is a model of ZF.

L can be defined in V by a statement $L(x) =$ "x is constructible" which is *independent* of V (ZF-absolute). In that sense, it is a *canonical* model of ZFC.

In the constructible universe L we have $\mathcal{N} \subseteq L$ (every subset of $\mathcal{N}$ is constructible).[17] This implies the existence of a Δ^1_2 well-ordering $<$ of $\mathcal{N}$. According to a theorem due to Fubini such a well ordering cannot be measurable and there exist therefore in L Δ^1_2 sets which are not measurable.

In what concerns CH, one uses the fact that the Δ^1_2 well ordering $<$ on $\mathcal{N}$ is of course Σ^1_2, and that the Σ^1_2 are the $\aleph_1$-Suslin sets. But, according to a theorem due to Schönfield, this implies that the ordinal of the well ordering $<$ is $< \aleph_2$ (that is $\leq \aleph_1$) and CH is therefore valid.

The philosophical problem raised by such results is that they are in some sense counterintuitive. They result from the fact that the AC, which implies the existence of very complicated and irregular sets, remains valid in L and the axiom $V = L$ forces some of these wildly irregular sets to exist inside the projective hierarchy which should be composed only of relatively regular sets.

But in spite of these intrinsic limitations, L is a very interesting model of ZFC, a model possessing a fascinating "fine structure" and very rich combinatorial properties investigated by Jensen.

3.5 The "maximalist" strategy of large cardinals.

It is therefore justified to *reverse* the constructive strategy and to look for *additional* axioms which could be considered as "natural" for ZF and ZFC, and to try to *generalize* to such enlarged axiomatics the search for canonical models and fine combinatorial structures.

Different strategies can be considered:

(i) to iterate transfinitely theories $T_{\alpha+1} = T_\alpha +$ "consistency of T_α" starting from ZF or ZFC;

(ii) to postulate "good" regularity properties of projective sets, and therefore of the continuum;

(iii) to make the theory of the continuum "*rigid*", that is define at what conditions the properties of $\mathbb{R}$ cannot be further modified by forcing.

[17] See Moschovakis [1980], pp.486 sq.

Reminder. Cohen's forcing allows to construct "generic" extensions M' of models M of ZF or ZFC. For that, one suppose that a partially ordered set of forcing conditions P is given. A set of conditions $C \subseteq P$ is *dense* if for every $p \in P$ there is a smaller $c \in C$. One defines then *generic* classes G of conditions. $G \notin M$ is generic over M if (i) $p \in G$ and $p \leq q \in P$ imply $q \in G$, (ii) for every $p, q \in G$, there exists a common lower $r \leq p$, $r \leq q$ with $r \in G$, (iii) for every dense set C of conditions $C \in M$, there exists $p \in C$ such that $p \in G$. The main theorem says that there exists then a ZF-model $\mathcal{A} = M[G]$ such that (1) M is an inner model of $\mathcal{A} = M[G]$, (2) G is not a set in M but is a set in $\mathcal{A} = M[G]$, (3) if $\mathcal{A}'$ is another model statisfying (1) et (2), then there exists an elementary embedding $j : \mathcal{A} \prec \mathcal{A}'$ such that $j(\mathcal{A})$ is an inner model of $\mathcal{A}'$ and $j|_M = \mathrm{Id}(M)$, (4) $\mathcal{A}$ is essentially unique.

A very good strategy is to try to *reduce* – and even to neutralize – the variability induced by forcing. The ideal aim would be a *forcing invariance* making the theories of $\mathbb{R}$ and $\mathcal{P}(\mathbb{R})$ in some sense as "rigid" as first order arithmetic. It is an extremely difficult program and we will only evoke some classical results concerning $\mathbb{R}$. CH concerns $\mathcal{P}(\mathbb{R})$ whose forcing invariance is the object of Woodin's works.

The three strategies (i), (ii), (iii) converge towards the introduction of large cardinal axioms.

Philosophically, the confusion between a strong "quasi-ontology" for sets and a realist "true" ontology of abstract idealities seems to disqualify such axioms. But we think that it is a mistake. Indeed, we think that one of the best *philosophical* formulation of incompleteness is precisely to say that a "good" theory of the continuum requires a very strong "quasi-ontology" for sets, a *maximal one*, and not a minimal one. A "good" regularity of the continuum entails, *for objective reasons*, a strong "Platonist" commitment.

This key point has been perfectly emphasized by Patrick Dehornoy:

> "les propriétés mettant en jeu des objets aussi "petits" que les ensembles de réels (du point de vue de la cardinalité et de celui du nombre minimal d'itérations de l'opération "passer à l'ensemble des parties" nécessaires à leur construction à partir de l'ensemble vide) sont liées à d'autres propriétés mettant en jeu des ensembles "immenses" qui paraissent très éloignés de ces mêmes points de vue." [18]

There exist many theorems showing that the Platonist "cost" of a "good" theory is very high. Let us for instance mention a striking theorem due to Robert Solovay.

Let CM be the axiom of existence of a measurable cardinal (see below).

Solovay theorem (1969). *$ZFC + CM \vdash$ every Σ^1_2 is "regular" (Baire property, Lebesgue measurability, perfect set property).*[19]

3.6 Projective determinacy and the "regularity" of the continuum.

A very interesting regularity hypothesis is the so called *determinacy* property. One considers infinite games on sets X. Each player (I and II) plays in turn an element a of X:

$$\begin{array}{cccccccccc} I & a_0 & & & a_2 & & & & ... \\ & & \searrow & \nearrow & & \searrow & & \nearrow & \\ II & & & a_1 & & & a_3 & & \end{array}$$

At the end of the game we get a sequence $f \in X^{\mathbb{N}}$. Let $A \subset X^{\mathbb{N}}$. The player I (resp. II) wins the play f of the game $G = G_X(A)$ associated to A if $f \in A$ (resp. if $f \notin A$).

Definition. *A is called* determined *(written* $\mathrm{Det}(A)$ *or* $\mathrm{Det}\, G_X(A)$*) if one player has a winning strategy. Therefore A is determined iff*

$$\exists a_0 \forall a_1 \exists a_2 ... (a_0, a_1, a_2, ...) \in A.$$

[18] Dehornoy [1988].

[19] See Moschovakis [1980], p.284.

Determinacy is a strong property of "regularity". Indeed, for every $B \subset \mathbb{R}$, there exists $A \subset \mathcal{N}$ s.t. $\mathrm{Det}(A) \Leftrightarrow$"$B$ satisfies the Baire property and the perfect subset property and is measurable".

The first theorem linking determinacy with the projective hierarchy has been the key result:[20]

Theorem of Gale and Stewart (1953). *In ZFC, closed subsets A of $X^{\mathbb{N}}$ (the Π^0_1 sets) are determined.*

After long efforts, Donald Martin proved a fundamental theorem which ends a first stage of the story:

Martin theorem (1975). $ZFC \vdash$ *"Borel sets (the Δ^1_1 sets) are determined".*

This celebrated result shows that ZFC is a "good" axiomatics for the Borel subsets of $\mathbb{R}$. But, it is the *limit* of what is accessible in ZFC. Indeed, ZFC *cannot* entail the determinacy of Σ^1_1 sets since in the constructible model L of ZFC there exist Σ^1_1 sets which dont share the perfect set property. As for Π^1_1 sets, their determinacy implies the measurability of the Σ^1_2 sets but there exists in L a Δ^1_2 well ordering of $\mathcal{N}$ which cannot be measurable.

3.7 The necessity of large cardinals

To prove determinacy results for projective sets beyond Δ^1_1, one must introduce *additional axioms* and many converging results show that the best are *large cardinal axioms.* The first example was introduced by Stan Ulam:

Definition. *A cardinal $\chi > \omega$ is* measurable *if it bears a free (i.e. non principal) ultrafilter $\mathcal{U}$ which is χ*-complete *(that is stable w.r.t. intersections $\bigcap_{\lambda<\chi} X_\lambda$ with $\lambda < \chi$). It is equivalent to say that χ bears a* measure *μ with range $\{0,1\}$ (with $\mu(\chi) = 1$), diffuse (without atoms: $\forall \xi \in \chi(\mu(\{\xi\}) = 0)$) and χ-additive. The equivalence is given by $\mu(A) = 1 \Leftrightarrow A \in \mathcal{U}$ and $\mu(A) = 0 \Leftrightarrow \chi - A \in \mathcal{U}$.*

A first typical result was another theorem due to Donald Martin:

Martin theorem (1970). $ZFC + CM \vdash \mathrm{Det}\left(\Sigma^1_1\right)$.

Corollary. Solovay theorem (1969): $ZFC + CM \vdash \Sigma^1_2$ "regular".

Scott theorem (1961). *CM is false in $V = L$ and therefore $ZFC \nvdash CM$.*

Measurable cardinals χ are large. Such a χ is *regular* (there exists no unbounded $f : \lambda \to \chi$ with $\lambda < \chi$), *strongly inaccessible* ($\forall \lambda < \chi$, $2^\lambda < \chi$), and with χ strongly inaccessible smaller cardinals. It shares a fundamental *combinatorial* partition property of *Ramsey* type.[21] Let $\chi^{[n]}$ be the set of finite subsets of χ of cardinal n and $\chi^{<\omega} = \bigcup_n \chi^{[n]}$ the set of all finite subsets of χ.

Definition. *A* partition *of $\chi^{[n]}$ is an application $F : \chi^{[n]} \to \lambda$ (the classes of equivalence are the fibers of F). A subset $I \subseteq \chi$ is called F*-homogeneous *if $I^{[n]}$ is completely included in a fiber of F ($\forall A, B \in I^{[n]}$ $(F(A) = F(B))$). If $F : \chi^{<\omega} \to \lambda$ is a partition of the finite subsets of χ, I is said F-homogeneous if $\forall n\, \forall A, B \in I^{[n]}$ $(F(A) = F(B))$.*

Definition. *An ultrafilter $\mathcal{U}$ on χ is said* normal *if it satisfies the following property: for every $f \in \chi^\chi$ if $\{\xi < \chi \mid f(\xi) < \xi\} \in \mathcal{U}$ then there exists $\lambda_0 < \chi$ s.t. $\{\xi < \chi \mid f(\xi) = \lambda_0\} \in \mathcal{U}$, that is*

$$\forall f \in \chi^\chi \left[f(\xi) < \xi \ \mathcal{U}\text{-a.e.} \Rightarrow \exists \lambda_0 < \chi \ \ (f(\xi) = \lambda_0 \ \mathcal{U}\text{-a.e.})\right]$$

.

A measurable cardinal bears a normal ultrafilter. The Ramsey property is expressed by the following generalization of Ramsey theorem:

Rowbottom theorem (1971). *If χ is a measurable cardinal, $\mathcal{U}$ a normal ultrafilter on χ and if $F : \chi^{<\omega} \to \lambda$ is a partition of χ with $\lambda < \chi$, then there exists an* homogeneous *subset I of χ with $I \in \mathcal{U}$ (i.e. I is very large).*

The (difficult) proof of Martin theorem[22] uses a *representation lemma* for Σ^1_1 and Π^1_1 sets which says essentially that if $A \subseteq \mathcal{N}$ is Π^1_1 then there exists a map $f : \mathcal{N} \to \mathcal{N}$ of complexity Δ^1_1 s.t., if we identify

[20] See. Grigorieff [1976] and Moschovakis [1980], p.288.

[21] See Moschovakis [1980], p.368.

[22] Moschovakis [1980], pp.370 sq.

any element $\gamma \in \mathcal{N}$ with the binary relation on $\mathbb{N}$: $n \leq_\gamma m \Leftrightarrow \gamma\left(2^{n+1}3^{m+1}\right) = 1$, then $\forall \alpha \in \mathcal{N}\ f(\alpha)$ (i.e. $\leq_{f(\alpha)}$) is a total order and $\alpha \in A \Leftrightarrow$"$f(\alpha)$ is a well ordering".

One constructs then an auxiliary game A^* where the player II plays by selecting ordinals *in the cardinal* χ. At the end of the game, one gets an $\alpha = (a_0, a_1, ...) \in \mathcal{N}$ and a sequence of ordinals $\xi_i < \chi$. The rules of A^* use the representation lemma. One then shows that A^* is *open*, and therefore determined and that a winning strategy for A^* enables to construct a winning strategy for A. It is for proving that point that one needs the Ramsey property.

3.8 Determination and reflection phenomena

To "measure" the size of large cardinals, the best is to use the associated *reflection* phenomena which are of a very deep philosophcal value.[23] Intuitively, reflection means that the properties of the whole universe V are reflected in sub-universes. As was emphasized by Matthew Foreman:

> "Any property that holds in the mathematical universe should hold of many set-approximations of the mathematical universe."

Definition. *A cardinal* χ reflects *a relation* $\Phi(x, y)$ *if for ordinals*

$$\forall \alpha < \chi\ \left[\exists \beta \geq \chi\ \Phi(\alpha, \beta) \Rightarrow \exists \beta^* < \chi\ \Phi(\alpha, \beta^*)\right].$$

Let j be an elementary embedding $j : M \prec M^*$ of models of ZFC where M^* is an *inner model of* M. Such a reflection of M into one of its inner model can be interpreted as a sort of *symmetry*. It is equivalent to a large cardinal hypothesis. Indeed, if $\alpha \in On(M)$ is an ordinal in M, one has $j(\alpha) \in On(M^*) \subset On(M)$ and, because of the elementarity of j , $\alpha < \beta \Leftrightarrow j(\alpha) < j(\beta)$. This implies $j(\alpha) \geq \alpha$. One shows that there exists necessarily an α s.t. $j(\alpha) > \alpha$. Let χ be the smallest of these α. It is called the *critical ordinal* of j. It is a large – in fact measurable – cardinal, which increases indefinitely as M^* draws nearer to M, the limit $M^* = M$ being inconsistent according to a theorem of Kunen.

To see that χ defines a reflection phenomenon, let $\Phi(\alpha, \chi)$ be a relation true in M for $\alpha < \chi$. If M^* is sufficiently close to M for $\Phi(\alpha, \chi)$ to remain true in M^*, then $M^* \models \exists (x < j(\chi)) \Phi(\alpha, x)$ (it is sufficient to take $x = \chi$). But according to the elementariness of the embedding j this is equivalent to $M \models \exists (x < \chi) \Phi(\alpha, x)$.

To go beyond measurable cardinals, specialists use the following technique. Let V_ξ be the cumulative hierarchy of sets up to level ξ. For χ critical (and therefore measurable), one has $V_\chi^{M^*} = V_\chi^M$ (that is M and M^* are equal up to level χ).

Definition. *The cardinal* χ *is called* superstrong *in* M *if there exists an elementary inner embedding* j *s.t.* $V_{j(\chi)}^{M^*} = V_{j(\chi)}^M$ (that is $M = M^*$ up to $j(\chi)$ and not only up to χ).

Between measurable and superstrong cardinals, Hugh Woodin introduced another important class of large cardinals.

Definition. *A cardinal* δ *is called a Woodin cardinal if for every map* $F : \delta \to \delta$, *there exist* $\kappa < \delta$ *and an elementary embedding* j *of critical ordinal* κ *s.t.* $F|_\kappa : \kappa \to \kappa$ *and* $V_{j(F(\kappa))}^{M^*} = V_{j(F(\kappa))}^M$ *(that is* $M = M^*$ *up to* $j(F(\kappa))$*).*

Woodin has shown that:

(i) if δ is a Woodin cardinal, there exist infinitely many smaller measurable cardinal $\chi < \delta$,

(ii) if λ is a superstrong cardinal, there exist infinetely smaller Woodin cardinals $\delta < \lambda$.

A key result is the:

Martin-Steel theorem (1985). *Let* W_n *be the axiom: there exist* n *Woodin cardinals* δ_i *and a measurable cardinal* $\kappa > \delta_i\ \forall i$, *then* $ZFC + W_n \vdash \text{Det}\left(\Pi^1_{n+1}\right)$.

The reciprocal is due to Woodin.

[23] See Martin-Steel [1989] and Patrick Dehornoy [1989].

Corollary. *If there exists a superstrong cardinal λ then $ZFC + Superstrong \vdash PD$ (projective determinacy: all the projective subsets of $\mathbb{R}$ are determined).*

It is for this reason that specialists consider that $ZFC + PD$ is a "good" axiomatics for $\mathbb{R}$. We must also emphasize the

Martin-Steel-Woodin theorem (1987). *If there exists a superstrong cardinal λ then $L(\mathbb{R})$ (the smallest inner model of ZF containing $\mathbb{R}$) satisfies the axiom of complete determinacy AD: every $A \subseteq \mathbb{R}$ is determined. (This result is stronger than the previous one since $\mathcal{P}(\mathbb{R}) \cap L(\mathbb{R})$ is a larger class than the projective class.)*

AD is incompatible with AC since AC enables the construction of a non determined well ordering on $\mathbb{R}$ (see above).

But the most significative results concern perhaps the situation where no property of $\mathcal{N}$ can be further modified in a forcing extension. In that case, the theory of the continuum becomes "*rigid*". Woodin and Shelah has shown that it is the case if there exists a *supercompact* cardinal κ. κ is γ-supercompact if there exists an elementary embedding $j : V \prec M$ s.t. $\mathrm{crit}(j) = \kappa$, $\gamma < j(\kappa)$ and $M^\gamma \subseteq M$. κ is supercompact if it is γ-supercompact for every $\gamma \geq \kappa$ (κ is κ-supercompact iff it is measurable). Such a deep result makes clear the nature of the "Platonist" axioms which are needed for a "good" theory of the continuum.

But the main problem with large cardinal axioms is that they cannot settle CH because CH can be violated by a "small" forcing of size $\aleph_1$ and a theorem due to Levy and Solovay shows that small forcings don't affect large cardinals.

In 1984 Matthew Foreman proved that *generalized* elementary embeddings $j : V \prec M$ defined not in V but in a generic extension $V[G]$ can have very small critical cardinals, as small as $\aleph_1$.

> "These embeddings can be viewed as *virtual* versions of large cardinal embeddings, whose specifics are revealed by forcing with the appropriate partial ordering."

But the deepest approach is given by Woodin's extraordinary recent works on Ω-logic and the negation of CH.

4 Conclusion

All these convergent results show what are the conditions for a "good" set theoretical determination of the continuum as pure intuition in the sense of Kant and Peirce. They justify Gödel's Platonism conceiving of additional axioms as some kind of "physical hypotheses". The nominalist antiplatonist philosophy of mathematics criticizing them as ontological naive beliefs must be reconsidered and substituted with a "conditional" Platonism in Woodin's sense, a Platonism "conditional" to axioms which "rigidify" the continuum and make its properties forcing invariant. What, for my part, I interpret as a "neo-transcendental" platonism.

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