Les manifestations de la transcendance du continu en théorie des ensembles

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1 The underdetermination of cardinal arithmetic in ZFC.

We work in $\mathbb{R}$ or in $\mathcal{N} = \omega^\omega$ or in $\{0,1\}^\omega$.

1.1 Easton theorem

ZFC determines quite nothing of the cardinal arithmetic of a universe V.[1]

In a model V of ZFC, let $F(\alpha)$ be the function defined by $2^{\aleph_\alpha} = \aleph_{F(\alpha)}$. One can show that:

(i) F is a monotonous increasing function: if $\alpha \leq \beta$ then $F(\alpha) \leq F(\beta)$;

(ii) *König's law*: $\operatorname{cf}\left(\aleph_{F(\alpha)}\right) > \aleph_\alpha$, where the *cofinality* $\operatorname{cf}(\alpha)$ of an ordinal α is defined as the smallest cardinality χ of a cofinal (i.e. unbounded) subset X of α (i.e. $\operatorname{Sup} X = \alpha$). The cardinal χ is called *regular* if $\operatorname{cf}(\chi) = \chi$.

If GCH holds, König's law is trivial because $F(\alpha) = \alpha + 1$, every cardinal $\aleph_{\alpha+1}$ is regular and therefore $\operatorname{cf}(\aleph_{\alpha+1}) = \aleph_{\alpha+1} > \aleph_\alpha$.

The fact that ZFC underdetermines cardinal arithmetic is particularly evident in the following striking result:

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[1]Cf. Jacques Stern [1976].

Easton theorem. *For the* regular $\aleph_\alpha$ *one can impose by forcing in ZFC the law* $2^{\aleph_\alpha} = \aleph_{F(\alpha)}$ *for quite every function* F *satisfying (i) and (ii).*$\square$

The proof uses iterated Cohen forcing.

1.2 Cohen forcing

Cohen forcing (1963) allows to construct in a very systematic way "generic" extensions N of inner models M of ZF or ZFC where some properties become valid.

Suppose for instance that, starting with a ground inner model M of ZFC in V, you want to construct an inner model N where ω_1^M *collapses* and becomes *countable*. You need to have in N a surjection $f : \omega \to \omega_1^M$. Suppose that such an f exists. Then $\forall n$ the restriction $f\mid_n = (f(0), \ldots, f(n-1))$ exists and is an element of the ground model M. Let us therefore consider the set $P = \{p\}$ of finite sequences $p = (\alpha_0, \ldots, \alpha_{n-1})$ of countable ordinals of M ($\alpha_i < \omega_1^M$). Such p are called *forcing conditions* and must be interpreted as forcing $f\mid_n = p$. P exists, is well defined in M with a natural partial order $q < p$ iff $p \subset q$. If f exists we can consider $G = \{f\mid_n\}$ which is a subset of P in V s.t. $\cup G = f$. But as $f \notin M$, G *is not a subset of* P *in* M.

If f exists then it is trivial to verify that G satisfies the following properties:

1. *Gluing and restriction conditions* (see topos theory): if $p, q \in G$, then p and q are compatible in the sense that $p \leq q$ or $q \leq p$ and therefore $\exists$ a common smaller $r \leq p$, $r \leq q$ with $r \in G$.

2. $\forall n \exists p \in G$ s.t. $n \in dom(p)$ (i.e. $dom(f) = \omega$).

3. $\forall \alpha < \omega_1^M \exists p \in G$ s.t. $\alpha \in range(p)$ (i.e. $range(f) = \omega_1^M$, it is the fundamental condition of surjectivity).

Cohen's idea is to construct Gs satisfying these properties and to show that extending the ground inner model M by such a G yields an appropriate inner model $N = M[G]$ which is the smaller inner model containing M and G.

So, one supposes that a partially ordered set of forcing conditions P is given. A set of conditions $D \subseteq P$ is called *dense* if for every $p \in P$ there is a smaller $d \leq p$, $d \in D$. One defines then *generic* classes G of conditions. $G \notin M$ is generic over M if

1. $p \in G$ and $p \leq q \in P$ imply $q \in G$,
2. $\forall p, q \in G$, $\exists$ a common smaller $r \leq p$, $r \leq q$ with $r \in G$,
3. $\forall$ dense set D of conditions $D \in M$, $\exists p \in D$ such that $p \in G$ (i.e. $G \cap D \neq \emptyset$).

If G is generic, the properties (2) and (3) are automatically satisfied since the sets of conditions $D_n = \{p \in P : n \in dom(p)\}$ for $n \in \omega$ and $E_\alpha = \{p \in P : \alpha \in range(p)\}$ for $\alpha < \omega_1^M$ are dense: (1) means $G \cap D_n \neq \emptyset$ and (2) means $G \cap E_\alpha \neq \emptyset$.

Cohen's main theorem. There exists a $ZFC-$model $\mathcal{A} = M[G]$ such that (1) M is an inner model of $\mathcal{A} = M[G]$, (2) G is not a set in M but is a set in $\mathcal{A} = M[G]$, (3) if $\mathcal{A}'$ is another model statisfying (1) et (2), then there exists an elementary embedding $j : \mathcal{A} \prec \mathcal{A}'$ such that $j(\mathcal{A})$ is an inner model of $\mathcal{A}'$ and $j|_M = \mathrm{Id}(M)$, (4) $\mathcal{A}$ is essentially unique.

An essential feature of forcing extensions is that it is possible to describe $M[G]$ using the language $\mathcal{L}_G$ which is the language $\mathcal{L}$ of M extended by a new symbol constant for G. As was emphasized by Patrick Dehornoy, forcing is "comme une extension de corps dont les éléments sont décrits par des polynômes du corps de base".

In particular the validity of a formula φ in $M[G]$ can be coded by a *forcing relation* $p \Vdash \varphi$ defined in M. The definition of $p \Vdash \varphi$ is rather technical but an excellent intuition is given by the idea of "localizing" truth, p being interpreted as a local domain (an open set of some topological space), $q < p$ meaning $q \subset p$ and $p \Vdash \varphi$ meaning that φ is "locally true everywhere" on p.

Forcing theorem. For every generic $G \subseteq P$, $M[G] \models \varphi$ iff $\exists p \in G$ s.t. $p \Vdash \varphi$.

Using forcing we can add to $\mathbb{R} \sim \mathcal{P}(\omega)$ new elements called *generic reals*. Let P the partial order of binary finite sequences $p = (p(0), \ldots, p(n-1))$. If G is generic, $f = \cup G$ is a map $f : \omega \to \{0, 1\}$ which is the characteristic function $f = 1_A$ of a new subset $A \subseteq \omega$ and $A \notin M$. Indeed, if $g : \omega \to \{0, 1\}$ defines a subset $B \subseteq \omega$ which belongs to M, then the set of conditions $D_g = \{p \in P : p \nsubseteq g\} \in M$ is dense (if p is any finite sequence it can be extended to a sequence long enough to be different from g) and therefore $G \cap D_g \neq \emptyset$. But this means $f \neq g$.

To prove *nonCH* one adds to M a great number of generic reals. More precisely, one embedds ω_2^M into $\{0, 1\}^\omega$ (equivalent to $\mathbb{R}$) using as forcing conditions the set P of finite binary sequences of $\omega_2^M \times \omega$. If G is generic, then $f = \cup G$ is a map $f : \omega_2^M \times \omega \to \{0, 1\}$ that is a family $f = \{f_\alpha\}_{\alpha < \omega_2^M}$ of generic reals $f_\alpha : \omega \to \{0, 1\}$. Using density arguments one shows that f yields an embedding $\omega_2^M \hookrightarrow \{0, 1\}^\omega$ in $M[G]$ and that ω_2^M *doesn't collapse* in $M[G]$ (because P is $\omega-$saturated, i.e. there doesn't exist in P any infinite countable subset of incompatible elements). This implies immediately *nonCH*.

Easton theorem is proved by *iterating* such constructions and adding to every regular $\aleph_\alpha$ as many new subsets as it is necessary to have $2^{\aleph_\alpha} = \aleph_{F(\alpha)}$.

1.3 Absoluteness

Contrary to first order aritmetic which is ZF-absolute (invariant relative to extensions of the universe, Schönfield theorem), the structures and notions $\mathcal{N}$, $\mathbb{R}$, Card(χ), $x \to \mathcal{P}(x)$, $x \to |x|$, second order arithmetic, *are not ZF-absolute.* They can vary widely from one model to another and can't have absolute truth value in ZF. This "vagueness" is one of the main arguments of antiplatonists such as Feferman. It has been emphasized by Hugh Woodin in his recent paper *Set theory after Russell. The journey back to Eden* (2003) that vagueness is not an admissible argument against platonism. It shows only that it is necessary to *classify* the different models of ZF and ZFC.

In fact, the strong variability of the possible models of ZFC is for me an argument in favor of the set theoretical approach of the *irreducibility* of the continuum to the discrete. The reducibility would mean that the continuum would be made of points which can be "*individuated*". But this is not the case in the set theoretical construction where "individuation" means "definability".

To tackle this problem, two opposed strategies are possible, both introduced by Gödel, one "minimalist" and the other "maximalist". To explain that, we must first introduce some classes of sets of reals in descriptive set theory.

2 Borel and projective hierarchies

In descriptive set theory one works in $\mathbb{R}$ or in $\mathcal{N} = \omega^\omega$ or in $\{0,1\}^\omega$, and, more generally, on metric, separable, complete, perfect (closed without isolated points) spaces $\mathcal{X}$ (Polish spaces). One considers in $\mathcal{X}$ different definable classes of subsets Γ. The first is the *Borel hierarchy* constructed from the open sets by iterating the operations of complementation and of "projection" $\mathcal{X} \times \omega \to \mathcal{X}$. If $P \subseteq \mathcal{X} \times \ \omega$ (that is, if P is a countable family of subsets $P_n \subseteq \mathcal{X}$), one considers the subset of $\mathcal{X}$ defined by $\exists^\omega P = \{x \in \mathcal{X} \mid \exists n \ P(x,n)\}$. It is the union $\bigcup_{n\in\omega} P_n$.

The Σ^0_1 are the open subsets, the $\Pi^0_1 = \neg\Sigma^0_1$ are the closed subsets and the Borel hierarchy B is defined by:

$\Pi^0_n = \{\neg\varphi \mid \varphi \in \Sigma^0_n\} = \neg\Sigma^0_n,\ \Sigma^0_{n+1} = \exists^\omega\neg\Sigma^0_n = \exists^\omega\Pi^0_n,\ \Delta^0_n = \Pi^0_n \cap \Sigma^0_n$.

This hierarchy is *strict*:

$$\begin{array}{ccccc} & & \Sigma^0_n & & \\ & \nearrow & & \searrow & \\ \Delta^0_n & & & & \Delta^0_{n+1} \\ & \searrow & & \nearrow & \\ & & \Pi^0_n & & \end{array}$$

One defines then the hierarchy of *projective* sets using a supplementary principle of construction, namely projections by continuous projections $\mathcal{X} \times \mathcal{N} \to \mathcal{X}$, written $\exists^{\mathcal{N}}$. One gets that way a new hierarchy beginning with the class $\Sigma^1_1 = \exists^{\mathcal{N}}\Pi^0_1$ – the so called *analytic* subsets – and continuing with the classes:

$\Pi^1_n = \{\neg\varphi \mid \varphi \in \Sigma^1_n\} = \neg\Sigma^1_n,\ \Sigma^1_{n+1} = \exists^{\mathcal{N}}\neg\Sigma^1_n = \exists^{\mathcal{N}}\Pi^1_n,\ \Delta^1_n = \Pi^1_n \cap \Sigma^1_n$.

For instance, $P \subseteq \mathcal{X}$ is Σ^1_1 if there exists a *closed* subset $F \subseteq \mathcal{X} \times \mathcal{N}$ such that:

$P(x) \Leftrightarrow \exists\alpha \ F(x,\alpha)$.

In the same way, $P \subseteq \mathcal{X}$ is Σ^1_2 if there exists an *open* subset $G \subseteq \mathcal{X} \times \mathcal{N} \times \mathcal{N}$ such that:

$P(x) \Leftrightarrow \exists\alpha \ \forall\beta \ G(x,\alpha,\beta)$, etc.

As the Borel hierarchy, the projective hierarchy is *strict*.

The projective hierarchy is a continuation of the Borel hierarchy according to:

Suslin theorem. $B = \Delta^1_1$.$\square$

This theorem can be interpreted as a *construction principle*: it asserts that the complex operation of continuous projection can be reduced to an iteration of simpler operations of union and complementation.

There exist strict Π_n^1 and Σ_n^1 sets who are very natural in classical analysis. For instance, in the functional space $C[0,1]$ of real continuous functions on $[0,1]$ endowed with the topology of uniform convergence, the subset:

$$\{f \in C\,[0,1] \mid f \text{ smooth}\}$$

is Π_1^1 (and not Δ_1^1).

In the space $C[0,1]^\omega$ of sequences (f_i) of functions, the subset:

$$\left\{\begin{array}{c}(f_i) \in C\,[0,1]^\omega \mid (f_i) \text{ converges for the topology} \\ \text{of simple convergence}\end{array}\right\}$$

is Π_1^1 and the subset:

$$\left\{\begin{array}{c}(f_i) \in C\,[0,1]^\omega \mid \text{a sub-sequence converges for the topology} \\ \text{of simple convergence}\end{array}\right\}$$

is Σ_2^1 and every Σ_2^1 can be represented that way (Howard Becker [2]):

Becker representation theorem. *For every Σ_2^1 set $S \subseteq C[0,1]$ there exists a sequence (f_i) such that*

$$S = A_{(f_i)} = \left\{\begin{array}{c} g \in C\,[0,1] \mid \text{a sub-sequence of } (f_i) \\ \text{converges towards } g \text{ for the topology} \\ \text{of simple convergence}\end{array}\right\}.\square$$

Another examples are given by the compact subsets $K \in \mathcal{K}(\mathbb{R}^n)$ of $\mathbb{R}^n$: for $n \geq 3$,

$$\{K \in \mathcal{K}(\mathbb{R}^n) \mid K \text{ arc connected}\}$$

is Π_2^1 and for $n \geq 4$,

$$\{K \in \mathcal{K}(\mathbb{R}^n) \mid K \text{ simply connected}\}$$

is also Π_2^1.

[2] Cf. Becker [1992].

3 The "minimalist" strategy of the constructible universe.

The first strategy for constraining the structure of ZF-universes consists in *restricting* the universe V. It is Gödel's strategy $V = L$ of *constructible* sets (Gödel 1938).

To define L one substitute, in the construction of the cumulative hierarchy of V by means of the $x \to \mathcal{P}(x)$ operation, the sets $\mathcal{P}(x)$ — which are not ZF-absolute — by smaller sets $\mathcal{D}(x) = \{y \subseteq x \mid y$ elementary (definable by a first order formula of the structure $< x, \in, \{s|s \in x\} >)\}$ — which are $ZF-$ absolute. L is then defined using a transfinite induction on ordinals: $L_0 = \emptyset$, $L_{\xi+1} = \mathcal{D}(L_\xi)$, $L_\lambda = \bigcup_{\xi<\lambda} L_\xi$ if λ is a limit ordinal, and $L = \bigcup_{\xi \in On} L_\xi$.

Gödel (1938-1940) has shown that if $V = L$ it is possible to define a *global well ordering* on L, which is a very strong form of AC. He proved also that in ZF $(V = L) \vdash HCG$.

L is in fact the *smallest* inner model of V, that is:

(i) $On \subset L$,

(ii) L is transitive: if $y \underset{V}{\in} x \underset{L}{\in} L$,then $y \underset{L}{\in} L$,

(iii) $(L, \in|_L)$ is a model of ZF.

L can be defined in V by a statement $L(x) =$ "x is constructible" which is *independent* of V ($ZF-$absolute). In that sense, it is a *canonical* model of ZFC.

In the constructible universe L we have $\mathcal{N} \subseteq L$ (every subset of $\mathcal{N}$ is constructible).[3] This implies the existence of a Δ^1_2 well-ordering $<$ of $\mathcal{N}$. According to a theorem due to Fubini, such a well ordering

[3] Cf. Moschovakis [1980], pp. 486 sq.

cannot be measurable and there exist therefore in L Δ^1_2 sets which are not measurable.

In what concerns CH, one uses the fact that the Δ^1_2 well ordering $<$ on $\mathcal{N}$ is a fortiori Σ^1_2, and that the Σ^1_2 are the $\aleph_1$-Suslin sets. But, according to a theorem due to Schönfield, this implies that the ordinal of $<$ is $< \aleph_2$ and CH is therefore valid.

These results are in some sense counterintuitive. They result from the fact that the AC, which implies the existence of very complicated and irregular sets, remains valid in L and the axiom $V = L$ forces some of them to exist *inside* the projective hierarchy which should be composed only of relatively simple and regular sets.

But in spite of these intrinsic limitations, L is a very interesting model of ZFC, a model possessing a "fine structure" and very rich combinatorial properties investigated by Jensen.

4 The "maximalist" strategy of large cardinals.

It is therefore justified to reverse the strategy and to look for *additional* axioms which could be considered as "natural" for ZF and ZFC, and to try to *generalize* to such augmented axiomatics the search of canonical models and fine combinatorial structures. As was emphasized by John Steel:

> "In extending ZFC, we are attempting to *maximize interpretative power*".

And there is place for philosophy in such a maximizing strategy since the problem is not to find a solution to the continuum problem but to understand what means "to be a solution".

Different strategies have been considered:

(i) Iterate transfinitely theories $T_{\alpha+1} = T_\alpha +$ "consistency of T_α" starting from ZF or ZFC.

(ii) Postulate "good" regularit y properties of projective sets, and therefore of the continuum.

(iii) Make the theory of the continuum "*rigid*", that is define *at what conditions the properties of* $\mathbb{R}$ *cannot be further modified by forcing.*

A very good strategy is then to try to *reduce* – and even to neutralize – the variability induced by forcing. The ideal aim would be forcing invariance making the theories of $\mathbb{R}$ and $\mathcal{P}(\mathbb{R})$ in some sense as "rigid" as first order arithmetic. It is an extremely difficult program and we will first evoke some classical results concerning $\mathbb{R}$. CH concerns $\mathcal{P}(\mathbb{R})$ whose forcing invariance is the object of more recent Woodin's works.

The two strategies (i), (ii) converge towards the introduction of large cardinal axioms.

Philosophically, the nominalist confusion between a strong "quasi-ontology" for sets and a realist "true" ontology of abstract idealities has disqualified such axioms. But we think that it has been a great mistake. Indeed, we think that one of the best *philosophical* formulation of incompleteness is precisely to say that a "good" theory of the continuum requires a very strong "quasi-ontology" for sets, a *maximal one* (and not a minimal one). A "good" regularity of the continuum entails *for objective reasons* a strong "platonist" commitment.

This key point has been perfectly emphasized by Patrick Dehornoy:

> « properties which put into play objects as "small" as sets of reals (...) are related to other properties which put into

play very "huge" objects which seem very far from them. »[4]

There are many theorems showing that the platonist "cost" of a "good" theory is very high. Let us for instance mention a striking theorem due to Robert Solovay.

Let CM be the axiom of existence of a measurable cardinal (see below).

Solovay theorem (1969). $ZFC + CM \vdash \forall\Sigma^1_2$ *is "regular" (Baire property, Lebesgue measurability, perfect set property).*[5]$\square$

5 The regularity of projective sets.

5.1 The regularity of analytic sets

The French school (Borel, Baire, Lebesgue) and the Russian and Polish schools (Suslin, Luzin, Sierpinski) initiated the study of the Borel and projective classes and achieved deep results concerning there *regularity* and their *representation* where "regularity" means Lebesgue measurability, or the perfect set property (to be countable or to contain a perfect, i.e. closed without isolated point, subset), or the Baire property (to be approximated by an open subset up to a meager set, i.e. a countable union of nowhere dense sets).

The first regularity theorem is the celebrated:

Cantor-Bendixson theorem. *If* $A \subseteq \mathbb{R}$ *is* closed, *then* A *can be decomposed in a unique way as* $A = P + S$ *where* P *is* perfect *and* S countable.$\square$

As a perfect set P is of cardinality $|P| = 2^{\aleph_0}$, the *continuum hypothesis CH holds for the closed sets* Π^0_1.

Another early great classical theorem of regularity is the:

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[4]Dehornoy [1988].

[5]Cf. Moschovakis [1980], p. 284.

Suslin theorem. *The analytic subsets* Σ^1_1 *shares the perfect subset property and CH is therefore true for the* Σ^1_1.□

In the same way, one can show that the Σ^1_1 share the Baire property and that the Σ^1_1 and Π^1_1 are Lebesgue measurable. But it is *impossible* to show in ZF that the Δ^1_1 and Σ^1_2 share the perfect set property and to show in ZFC that the Δ^1_2 share the Baire property. In fact many of the "natural" properties of the projective sets go far beyond the demonstrative strength of ZF and ZFC. *It is therefore justified to look for additional axioms.*

5.2 Projective determinacy and the "regularity" of the continuum.

A very interesting regularity hypothesis is the so called *determinacy* property. One considers infinite games on sets X. Each player (I and II) plays in turn an element a of X:

$$\begin{array}{ccccccccc} I & a_0 & & & a_2 & & & & ... \\ & & \searrow & \nearrow & & \searrow & \nearrow & & \\ II & & a_1 & & & a_3 & & & \end{array}$$

At the end of the game we get a sequence $f \in X^{\mathbb{N}}$. Let $A \subset X^{\mathbb{N}}$. The player I (resp. II) wins the play f of the game $G = G_X(A)$ associated to A if $f \in A$ (resp. if $f \notin A$).

Definition. *A is called* determined *(written* $\mathrm{Det}(A)$ *or* $\mathrm{Det}\, G_X(A)$*) if one player has a winning strategy. Therefore A is determined iff*

$$\exists a_0 \forall a_1 \exists a_2 ... (a_0, a_1, a_2, ...) \in A. \square$$

Determinacy is a strong property of "regularity". Indeed, $\forall B \subset \mathbb{R}$, $\exists A \subset \mathcal{N}$ s.t. $\mathrm{Det}(A) \Leftrightarrow B$ satisfies the Baire property and the perfect subset property and is measurable.

The first theorem linking determinacy with the projective hierarchy has been the key result:[6]

[6] Cf. Grigorieff [1976] and Moschovakis [1980], p. 288.

Theorem of Gale and Stewart (1953). *In ZFC, closed subsets A of $X^{\mathbb{N}}$ (the Π^0_1) are determined.*$\square$

After many efforts, Donald Martin proved a fundamental theorem which ends a first stage of the story:

Martin theorem (1975). *$ZFC \vdash$ Borel sets (the Δ^1_1) are determined.*$\square$

This celebrated result shows that ZFC is a "good" axiomatic for the Borel subsets of $\mathbb{R}$. But, it is the *limit* of what is provable in ZFC. Indeed, ZFC cannot imply the determinacy of Σ^1_1 sets since in the constructible model L of ZFC there exist Σ^1_1 which dont share the perfect set property. As for Π^1_1 sets, their determinacy implies the measurability of the Σ^1_2, but in L there exists a well ordering Δ^1_2 of $\mathbb{R}$ which cannot be measurable.$\square$

6 The necessity of large cardinals.

To prove determinacy results for projective sets beyond Δ^1_1, one must introduce additional axioms and many converging results show that the best are large cardinal axioms. The first example was introduced by Stan Ulam. If X is a set, a filter $\mathcal{U}$ over X is a set of subsets $\mathcal{U} \subseteq \mathcal{P}(X)$ s.t. $\emptyset \notin \mathcal{U}$, if $U \in \mathcal{U}$ and $U \subseteq V$ then $V \in \mathcal{U}$, if $U, V \in \mathcal{U}$ then $U \cap V \in \mathcal{U}$ (i.e.the complementary set is an ideal of the Boolean algebra $\mathcal{P}(X)$). $\mathcal{U}$ is an *ultrafilter* if it is maximal, namely if for every $U \subseteq X$, either $U \in \mathcal{U}$ or $X - U \in \mathcal{U}$. For every $x \in X$ $\mathcal{U}_x = \{U \subseteq X : x \in U\}$ is an ultrafilter called principal.

Definition. *A cardinal $\chi > \omega$ is* measurable *if it bears a free (i.e. non principal) ultrafilter $\mathcal{U}$ which is χ-*complete *(that is stable w.r.t. $\bigcap_{\lambda<\chi} X_\lambda$ with $\lambda < \chi$). It is equivalent to say that χ bears a* measure *μ with range $\{0, 1\}$ (with $\mu(\chi) = 1$), diffuse (without atoms: $\forall \xi \in \chi(\mu(\{\xi\}) = 0)$) and χ-additive. The equivalence is given by $\mu(A) = 1 \Leftrightarrow A \in \mathcal{U}$ and*

$\mu(A) = 0 \Leftrightarrow \chi - A \in \mathcal{U}$.□

A first typical result was another theorem due to Donald Martin:

Martin theorem (1970). $ZFC + MC \vdash \text{Det}\left(\Sigma_1^1\right)$.□

Corollary: Solovay theorem (1969): $ZFC + MC \vdash \Sigma_2^1$ "regular".

Scott theorem (1961). *MC is false in $V = L$ and therefore $ZFC \nvdash CM$.*□

The proof of Scott's theorem uses the concept of *ultrapower* $V^{\mathcal{U}}$ where $\mathcal{U}$ is an ultrafilter on a set S, the elements of $V^{\mathcal{U}}$ being maps $f : S \to V$, f and g being equivalent if they are equal a.e. that is if $\{s \in S : f(s) = g(s)\} \in \mathcal{U}$ (any element x of V is represented by the constant map $f(s) = x$ and this defines a canonical embedding $j : V \hookrightarrow V^{\mathcal{U}}$). If $\varphi(x_1, \ldots, x_n)$ is a formula of the language of V, $\varphi(f_1, \ldots, f_n)$ is valid in $V^{\mathcal{U}}$ $\left(V^{\mathcal{U}} \models \varphi(f_1, \ldots, f_n)\right)$ iff $\varphi(f_1, \ldots, f_n)$ is valid a.e., that is if $\{s \in S : V \models \varphi(f_1(s), \ldots, f_n(s))\} \in \mathcal{U}$. One shows that $V^{\mathcal{U}}$ is well founded if the ultrafilter $\mathcal{U}$ is ω_1-complete and that there exists an isomorphism between $\left\langle V^{\mathcal{U}}, \in_{\mathcal{U}}\right\rangle$ and $\langle M_{\mathcal{U}}, \in\rangle$ where $M_{\mathcal{U}}$ is an inner model (collapsing lemma). A fundamental theorem of Łoš says that $j : V \prec M_{\mathcal{U}}$ is an elementary embedding. If j is an elementary embedding $j : M \prec M^*$ of models of ZFC where M^* is an inner model of M and if $\alpha \in On(M)$ is an ordinal in M, one has $j(\alpha) \in On(M^*) \subset On(M)$ and, because of the elementarity of j, $\alpha < \beta \Leftrightarrow j(\alpha) < j(\beta)$. This implies $j(\alpha) \geq \alpha$. One shows that there exists necessarily an α s.t. $j(\alpha) > \alpha$. Let χ be the smallest of these α. It is called the *critical ordinal* $crit(j)$ of j.

Theorem. If the free ultrafilter $\mathcal{U}$ on the measurable cardinal χ is $\chi-$complete, then $crit(j) = \chi$ and therefore $j(\chi) > \chi$.

Corollary: Scott theorem.

Indeed, if there exists a MC let χ be the least MC and suppose that $V = L$. Elementarity implies $M_{\mathcal{U}} = L$ since $M_{\mathcal{U}}$ is an inner model satisfying the axiom of constructibility, therefore $L \subseteq M_{\mathcal{U}} \subseteq V = L$, and, in $M_{\mathcal{U}}$, $j(\chi)$ is the least MC, which contradicts $j(\chi) > \chi$.

Measurable cardinals χ are large; such a χ is *regular* (there exists no unbounded $f : \lambda \to \chi$ with $\lambda < \chi$), *strongly inaccessible* ($\forall \lambda < \chi$, $2^{\lambda} < \chi$), and with χ strongly inaccessible smaller cardinals.

7 The transcendance of $\mathbb{R}$ over L: $0^{\#}$

7.1 Indiscernible ordinals

The transcendence of V over L can be measured through *indiscernible* ordinals (Silver, 1966). If $\mathcal{M} = \langle M, E \rangle$ is a structure with a binary relation E which looks like $\in$, a set I of ordinals of M (in the sense of E) is called a set of indiscernibles if for every $n-$ary formula $\varphi(x_1, \ldots, x_n)$ the validity of φ on I is independent of the choice of the x_i: that is for every sequences $c_1 < \ldots < c_n$ and $d_1 < \ldots < d_n$

$$\mathcal{M} \models \varphi(c_1, \ldots, c_n) \text{ iff } \mathcal{M} \models \varphi(d_1, \ldots, d_n)$$

We look at structures $\mathcal{M} = \langle M, E \rangle$ which are elementary equivalent to some $\langle L_{\lambda}, \in \rangle$ with λ a limit ordinal. Let $\Sigma(\mathcal{M}, I)$ be the set of formulae φ which can be satisfied by $\mathcal{M}$ on I. This defines particular sets of formulae called EM-sets (from Ehrenfeucht-Mostowski, 1956). The E-M theorem says that if Σ is a theory having infinite models and if $\langle I, < \rangle$ is any total order, then there exists a model $\mathcal{M}$ of Σ containing I for which I is a set of indiscernibles. Moreover, $\mathcal{M}$ can be chosen in such a way that $\mathcal{M}$ is the *Skolem hull* of I. The Skolem hull is constructed by adding for every $(n+1)-$ary formula $\varphi(y, x_1, \ldots, x_n)$ with $x_i \in I$ a Skolem term $t_{\varphi}(x_1, \ldots, x_n)$ which is the smallest (for the well ordering of L) y s.t. $\varphi(y, x_1, \ldots, x_n)$ if such an y exists or 0 otherwise. Such an $\mathcal{M}$ is essentially unique and its transitive collapse (isorphism with a structure where E becomes $\in$) is writen $\mathcal{M}(\Sigma, \alpha)$ and $I(\Sigma, \alpha)$ where α is the order type of I ($I(\Sigma, \alpha)$ is therefore a set of true $\in -$ordinals).

One can develop a theory of EM-sets and of their well-foundedness.

When it exists, the set S of *Silver indiscernibles* is characterized by the following properties for all *uncountable* cardinals κ:

1. $\kappa \in S$.

2. $S \cap \kappa$ is of order-type κ.

3. $S \cap \kappa$ is closed and unbounded in κ if κ is rregular ("closed" means to contain limit ordinals $< \kappa$).

4. $S \cap \kappa$ is a set of indiscernibles for $\langle L_\kappa, \in \rangle$.

5. $\text{Hull}^{L_\kappa}(S \cap \kappa) = L_\kappa$, in other words every constructible element $a \in L_\kappa$ is *definable* by a definite description with parameters in the indiscernibles $S \cap \kappa$.

Theorem. If there exists a MC then S exists and moreover $L_\kappa \prec L_\lambda$ for every uncountable $\kappa < \lambda$.

The deep meaning of S is that (if it exists) it enables to define in V the *truth* in L. Indeed, let $\varphi(x_1, \ldots, x_n)$ be a formula. There exists an uncountable cardinal κ s.t.

$$\text{for all } (x_i) \in L_\kappa,\ L \models \varphi(x_i) \text{ iff } L_\kappa \models \varphi(x_i)$$

As $L_\kappa \prec L_\lambda$ if $\kappa < \lambda$, we have

$$L \models \varphi(x_i) \text{ iff } L_\lambda \models \varphi(x_i) \text{ for all } \lambda \geq \kappa$$

Now, we arithmetize. Let $T = \{\ulcorner\varphi\urcorner : L_{\aleph_1} \models \varphi\}$ be the set of Gödel numbers of the φ valid in $L_{\aleph_1}$ and therefore in all the L_κ (κ uncountable) by elementarity. Then

$$L \models \varphi \text{ iff } \ulcorner\varphi\urcorner \in T$$

defines the truth in L. This is not in contradiction with Gödel-Tarski uncompleteness theorems since $\aleph_1$ and T are *not definable* in L and therefore the truth of L is not definable in L. As $L_{\aleph_\omega} \prec L$ and the $\aleph_i \in S$, $L \models \varphi(x_i)$ for $x_i \in S$ iff $L_{\aleph_\omega} \models \varphi(\aleph_i)$.

7.2 The set $0^\#$

Solovay called $0^\#$ (0 sharp) the set (if it exists) defined by

$$0^\# = \{\varphi : L_{\aleph_\omega} \models \varphi(\aleph_i)\}$$

which codes the formulae which are true on the indiscernibles of L.

We must emphasize the fact that as $L_{\aleph_1} \prec L$ every constructible set $x \in L$ which is *definable* in L is *countable* since its definite description is valid in $L_{\aleph_1}$ by elementarity and therefore $x \in L_{\aleph_1}$.

Corollary. If there exists a MC, the constructible continuum $\mathbb{R}^L$ is countable.

Via arithmetization through Gödel numbers, the non constructible set $0^\#$ can be considered as a very special subset $\notin L$ of $\omega = \mathbb{N}$ or as a special real coding the truth in L. Its existence implies that every uncountable cardinal κ of V is an indiscernible of L and shares *all* large cardinal axioms verified by L.

A property equivalent to the existence of $0^\#$ is the *non rigidity* of L: there exists a non trivial elementary embedding $j : L \prec L$. Indeed, as $Hull^L(S) = L$, for every $x \in L$ there exists a Skolem term s.t. $x = t(i_{\alpha_1}, \ldots, i_{\alpha_n})$, i_α being the $\alpha-$th element of S. j is then simply defined by *the shift* on indiscernibles

$$j(x) = j(t(i_{\alpha_1}, \ldots, i_{\alpha_n})) = t(i_{\alpha_1+1}, \ldots, i_{\alpha_n+1})$$

One shows that it is an elementary embedding and as $j(i_0) = i_1$, j is non trivial. This shows that $V \neq L$ since a theorem of Kunen proves that

$$ZFC \vdash \text{ there exists no } j : V \prec V$$

Another characterization of the existence of $0^{\#}$ is that there is a level of L with *uncountably* many indiscernibles.

The existence of $0^{\#}$ is a principle of transcendence of V over L. If $0^{\#}$ doesn't exist, then V looks like L according to the result:

Covering lemma. If $0^{\#}$ doesn't exist, then if x is an uncountable set of ordinals there exists a constructible set $y \supseteq x$ of the same cardinality as x.

8 Determination and reflection phenomena.

To "measure" the size of large cardinals, the best is to use the associated *reflection* phenomena which are of a very deep philosophcal value.[7] Intuitively, reflection means that the properties of the whole universe V are reflected in sub-universes. As was emphasized by Matthew Foreman:

> « Any property that holds in the mathematical universe should hold of many set-approximations of the mathematical universe. »

Definition. *A cardinal* χ reflects *a relation* $\Phi(x, y)$ *if:*
$\forall\alpha(\in On) < \chi \;\; [\exists\beta \geq \chi \; \Phi(\alpha, \beta) \Rightarrow \exists\beta^* < \chi \; \Phi(\alpha, \beta^*)]$.□

Let j be an elementary embedding $j : M \prec M^*$ of models of ZFC where M^* is an inner model of M. Such a reflection of M into one of its inner model can be interpreted as a sort of *symmetry.* It is equivalent to a large cardinal hypothesis. Indeed, if $\alpha \in On(M)$ is an ordinal in M, one has $j(\alpha) \in On(M^*) \subset On(M)$ and, because of the elementarity of j, $\alpha < \beta \Leftrightarrow j(\alpha) < j(\beta)$. This implies $j(\alpha) \geq \alpha$. One shows that there

[7]Cf. Martin-Steel [1989], Patrick Dehornoy [1989].

exists necessarily an α s.t. $j(\alpha) > \alpha$. Let χ be the smallest of these α. It is called the *critical ordinal* of j. It is a large – in fact measurable – cardinal, which increases indefinitely when M^* comes near to M, the limit $M^* = M$ being inconsistent according to a theorem of Kunen.

To see that it is a reflection phenomenon, let $\Phi(\alpha, \chi)$ be a relation true in M for $\alpha < \chi$. If M^* is sufficiently close to M for $\Phi(\alpha, \chi)$ to remain true in M^*, then $M^* \models \exists(x < j(\chi))\Phi(\alpha, x)$ (it is sufficient to take $x = \chi$). But according to the elementarity of the embedding j this is equivalent to $M \models \exists(x < \chi)\Phi(\alpha, x)$.

To go beyond measurable cardinals, specialists use the following technique. Let V_α be the cumulative hierarchy of sets up to level α. For χ critical (and therefore measurable), one has $V_\chi^{M^*} = V_\chi^M$ (that is the equality of M and M^* up to level χ).

Definition. *The cardinal χ is called superstrong in M if there exists an elementary inner embedding j s.t. $V_{j(\chi)}^{M^*} = V_{j(\chi)}^M$* (that is $M = M^*$ up to $j(\chi)$ and not only up to χ).□

Between measurable and superstrong cardinals, Hugh Woodin introduced another class of large cardinals.

Definition. *A cardinal δ is called a Woodin cardinal if for every map $F : \delta \to \delta$, there exists $\kappa < \delta$ and an elementary embedding j of critical ordinal κ s.t. $F|_\kappa : \kappa \to \kappa$ and $V_{j(F(\kappa))}^{M^*} = V_{j(F(\kappa))}^M$ (that is $M = M^*$ up to $j(F(\kappa))$).*□

Woodin has shown that:

(i) if δ is a Woodin cardinal, there exist infinitely many smaller measurable cardinal $\chi < \delta$,

(ii) if λ is a superstrong cardinal, there exist infinetely many smaller Woodin cardinals $\delta < \lambda$.

A key result is the:

Martin-Steel theorem (1985). *If there exist n Woodin cardinals δ_i and a measurable cardinal $\kappa > \delta_i\ \forall i$, then $ZFC \vdash \text{Det}\left(\Pi^1_{n+1}\right)$.*□

The reciprocal is due to Woodin.

Corollary. *If there exists a superstrong cardinal λ then $ZFC+$* Superstrong $\vdash$ Projective Determinacy *(all the projective subsets of $\mathbb{R}$ are determined).*□

It is for this reason that specialists consider that $ZFC+$ Projective Determinacy is a "good" axiomatics for $\mathbb{R}$. We must also emphasize the:

Martin-Steel-Woodin theorem (1987). *If there exists a superstrong cardinal λ then $L(\mathbb{R})$ (the smallest inner model of ZF containing $\mathbb{R}$) satisfies the axiom of complete determinacy AD: $\forall A \subseteq \mathbb{R}$ is determined. (This result is stronger than the previous one since $\mathcal{P}(\mathbb{R}) \cap L(\mathbb{R})$ is a larger class than the projective class.)*□

AD is incompatible with AC since AC enables the construction of a non determined well ordering on $\mathbb{R}$ (see above).

But the most significative results concern perhaps the situation where no property of $\mathbb{R}$ can be further modified in a forcing extension. In that case, the theory of the continuum becomes "*rigid*". Woodin and Shelah has shown that it is the case if there exists a *supercompact* cardinal κ. κ is γ-supercompact if there exists an elementary embedding $j : V \prec M$ s.t. $\text{crit}(j) = \kappa$, $\gamma < j(\kappa)$ and $M^\gamma \subseteq M$. κ is supercompact if it is γ-supercompact $\forall \gamma \geq \kappa$ (κ is κ-supercompact iff it is measurable). Such a deep result makes clear the nature of the "platonist" axioms which are needed for a "good" theory of the continuum.

But the main problem with large cardinal axioms is that they cannot settle CH because CH can be violated by a "small" forcing of size $\aleph_2$

and a theorem due to Levy and Solovay shows that small forcings don't affect large cardinals.

9 Woodin's Ω-logic

The deepest contemporary approach to the continuum problem is given by Woodin's extraordinary recent works on Ω-logic and the negation of CH.

Large cardinal axioms (LCA) can decide some properties of regularity of $\mathbb{R}$ but they cannot decide CH since a "small" forcing (to add $\aleph_2$ new subsets to ω) is sufficient to force $\neg CH$ from a CH-model and such a small forcing remains possible whatever LCA are introduced (Levy-Solovay theorem). We need therefore a new strategy. The most natural one is to try to make the properties of the continuum *immune* relatively to forcing, that is to make the continuum in some sense "*rigid*".

So we look for theories sharing *some absoluteness properties relatively to forcing*. This is called "*conditional generic absoluteness*" (see John Steel, "Generic absoluteness and the continuum problem", 2004).

The fragment of V where CH "lives" is H_2 where H_k is the set of the sets x which are hereditary of cardinal $|x| < \aleph_k$.

The fragment $H_0 = V_\omega$ is the set of hereditary finite sets and is equivalent to first order arithmetic $\langle \omega = \mathbb{N}, +, ., \in \rangle$. In one direction $\mathbb{N}$ can be retrieved from H_0 using von Neumann's construction of ordinals and, conversely, H_0 can be retrieved from $\mathbb{N}$ via Ackermann's trick: if p, q are integers, $p \in q$ if the p-th digit in the binary extension of q is 1.

Schönfield theorem. H_0 is absolute.

The fragment $H_1 = \langle \mathcal{P}(\omega) = \mathbb{R}, \omega, +, ., \in \rangle$ of countable sets of finite ordinals correspond to second order arithmetic. We have seen that to settle most of its high order properties (regularity of $\mathbb{R}$), we need LCA. The strategy for deciding $ZFC-$undecidable properties φ in a fragment

H of V is the following (P. Dehornoy):

> "toute axiomatisation gelant les propriétés de H vis-à-vis du forcing (i.e. neutralisant le forcing au niveau de H) entraîne φ".

We have seen that under the LCA "there exists a proper class of Woodin cardinals" (PCW) we have:

Theorem. $ZFC + PCW \vdash H_1$ is immune relatively to forcing.

Martin-Steel theorem. Projective Determinacy rigidifies H_1 and is the "good" theory for H_1.

The main problem tackled by Woodin is to generalize these results to H_2 associated to $\mathcal{P}(\omega_1)$. $\mathcal{P}(\omega_1)$ is not $\mathcal{P}(\mathbb{R})$ if $\neg CH$, but nevertheless it is possible to interpret CH by an H_2-formula φ_{CH}. The problem with H_2 is that "small" forcings preserve LCA and therefore H_2 *cannot* be rigidified by LCA.

See Patrick Dehornoy: *Progrès récents sur l'hypothèse du continu*, Séminaire Bourbaki, Mars 2003.

A first result is that, under LCA, CH rigidifies V at the Σ_1^2 level (Σ_1 formulae for $V_{\omega+2}$).

Theorem (Woodin). If PCW_{meas}, all generic extensions $V[G]$ satisfyng CH are Σ_1^2-equivalent, that is CH is *generically complete* at the Σ_1^2 level.

As was emphasized by Woodin, the metamathematical meaning of this result is that if a problem is expressed by a Σ_1^2-formula φ then it is "settled by CH" and immunized against forcing under appropriate LCA. But:

Theorem (Abraham, Shelah). This is false at the Σ_2^2 level.

Woodin's fundamental idea to overcome the dramatic difficulty of the problem at the H_2 level is to *strengthen logic* by restricting the admissible models. In a first step he introduced the notion of Ω^*-derivability.

Definition. T being a theory (ZFC), we have $T \vdash_{\Omega^*} \varphi$ iff for every generic extension $V[G]$ and every level α, $(V_\alpha)^{V[G]} \models T$ implies $(V_\alpha)^{V[G]} \models \varphi$.

Woodin investigated deeply this new "strong logic". In particular he was able to show:

Theorem (Woodin). Under PCW_{meas} and CH the Ω^*-logic is complete at the Σ_1^2 level: for every φ Σ_1^2 either $ZFC + CH \vdash_{\Omega^*} \varphi$ or $ZFC + CH \vdash_{\Omega^*} \neg\varphi$.

In a second step, Woodin interpreted $T \vdash_{\Omega^*} \varphi$ as the semantic validity $T \models_\Omega \varphi$ of an $\Omega-$logic whose syntactic derivation $T \vdash_\Omega \varphi$ had to be defined. It is the most difficult part of his work. The definition (under PCW) is the following:

Definition. $T \vdash_\Omega \varphi$ iff there exists a *universally Baire* (UB) set $A \subseteq \mathbb{R}$ s.t. for every $A-$closed countable transitive model (*ctm*) M of T we have $M \models \varphi$ (in other words $M \models$ "$T \models_\Omega \varphi$").

$A \subseteq \mathbb{R}$ is UB if for every continuous map $f : K \to \mathbb{R}$, K compact Hausdorff, $f^{-1}(A)$ has the Baire property (there exists an open set U s.t. the symetric difference $f^{-1}(A)\,\Delta U$ is meager). A *ctm* M is $A-$closed if, for every *ctm* $N \supseteq M$, $A \cap N \in N$.

As, in the definition of $T \vdash_\Omega \varphi$, the class of admissible models is *restricted* to $A-$closed *ctm*, logic becomes strengthened.

$\Omega-$logic is *sound*: if $T \vdash_\Omega \varphi$ then $T \models_\Omega \varphi$. Woodin's main conjecture is the

$\Omega-$**conjecture**. $\Omega-$logic is *complete*: if $\models_\Omega \varphi$ then $\vdash_\Omega \varphi$.

As was emphasized by Woodin:

> "If the $\Omega-$conjecture is true, then generic absoluteness

is equivalent to absoluteness in Ω−logic and this in turn has significant metamathematical implications".

Now, the point is that when H_2 is rigidified, CH is *false*. It is in that sense Woodin can claim (*The CH II*, p.690):

> "Thus, I now believe the Continuum Hypothesis is solvable, which is a fundamental change in my view of set theory".

10 Conclusion

All these convergent results show what are the conditions for a "good" set theoretical determination of the continuum. They justify Gödel's platonism conceiving of additional axioms as some kind of "physical hypotheses". The nominalist antiplatonist philosophy of mathematics criticizing them as ontological naive beliefs must be reconsidered and substituted with a "conditional" platonism in Woodin's sense, a platonism "conditional" to axioms which "rigidify" the continuum and make its properties forcing invariant.